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Generalized Orthonormal Basis Functions in System Identification

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Over the last years a general theory has been developed for the construction and analysis of rational orthonormal basis functions, often called Generalized Orthonormal Basis Functions (GOBF) in the engineering literature. These investigations motivate the interest in the examination of the approximation properties of the rational orthonormal systems generated by a given set of poles. These basis can be viewed as an extension of the trigonometric system on the unit circle, that corresponds to the special choice when all of the poles are located at the origin. This paper provides a generalization of certain classical $\mathcal{L}_p$ norm convergence and summation theorems of the partial sums of Fourier series using GOBF expansions. Using the so called Hambo-domain techniques the paper considers the construction of minimal state space models of linear time-invariant (LTI) systems on the basis of systems representations in terms of GOBF expansions.

1 Introduction

It has been known that control design for dynamic systems usually requires the knowledge of an appropriate model of the system. These models can, in many cases, be derived from first principles but in more realistic situations from data that are the measured input/output signals of a system. Model construction from measured data is usually called system identification.

Traditionally, the main approach to system identification has been based on stochastic assumptions explaining the errors between the actual system and its models. Properly parameterizing the models, the elaborated results provide a point estimate for the parameters of a nominal model and additional statistical properties characterizing the estimated parameters and the goodness of fit. The control design was based on the nominal model (applying the certainty equivalence principle) disregarding most of the statistical information provided by the identification.

The appearance of the robust control paradigm in the late past decade accompanied by the formal $\mathcal{H}_\infty$ analysis and design theory incorporated the modeling uncertainties into control design. This started as a completely deterministic approach with the design based on a family of models given by a nominal model and an uncertainty model describing, e.g., the modeling error (or the bound on the magnitude of the error) in the frequency domain of interest. The design has been usually formulated as an $\mathcal{H}_\infty$ optimization (e.g. minimization of certain operator norms) over a set of stabilization controllers.

It was soon realized that existing methods of system identification are not capable to provide initial data for robust control and inspired intensive research on both fields. The first concepts for a solution were published in the early nineties by [19, 20]. This nonstochastic approach, usually called as worst-case identification for robust control, was proposed to identify a nominal LTI model from frequency response data.

There are various approaches to the identification problem both in time and frequency domain. Recent overviews using information based complexity and set membership approach to modeling and identification can be found in [32, 34, 33] and [47]. Concerning the choice of identification criteria, there appeared the worst case identification in l_1, see e.g. [45],[6],[13],[17] and others. Worst case identification under

$\mathcal{H}_\infty$ criterion appeared in a large number of papers included [19, 20], [15, 14], [36, 37],[29], [30]. Time domain approach to this problem appears in [5] and [53]. Closed–loop issues of identification for robust control were initiated by [11] using LQG/LTR approach and was further discussed by [46], [3] and [26, 27] introduced a generic scheme for the joint identification/control design by showing that the identification and control errors are identical in this scheme. This allows to elaborate very powerful iterative tools to obtain high closed loop performance. For nonlinear identification issues see [16].

In linear systems, control, and signal processing, rational approximation has always been an important issue and it has given rise to specific problems and insights in approximation theory, it has revived forgotten methods and initiated new directions of research. It is the intention of this paper to illustrate some of these innovating ideas that were born from this interaction of system theory, linear algebra and approximation theory.

The first mention of rational orthonormal systems seems to have occurred in the mid 20^{th} in the work of [43] and [31]. The context of this early work was application to approximation via interpolation, with the ensuing implications for generalizes quadrature formula's considered. The wide ranging work of [51] studied further the application of these bases for approximation on the unit disk and on the half plane.

For a given – possible infinite – set of zeros $\{\alpha_j\}$ let us consider the finite Blaschke products B_n of order $n \in \mathbb{N} := \{1, 2, \dots\}$ written under the form

$$B_n := \prod_{j=1}^{n} b_j, \quad b_j(z) := \frac{z - \alpha_j}{1 - \overline{\alpha}_j z},$$

where $|\alpha_j| < 1$, $(j = 1, 2, \dots, n)$ are given complex numbers. The functions ϕ_j defined inductively by $\phi_1(z) := \frac{d_1}{1-\overline{\alpha}_1 z}$ and

$$\phi_j(z) := \frac{d_j}{1 - \overline{\alpha}_j z} \prod_{i=1}^{j-1} b_{\alpha_i}(z), \quad j > 1,$$

where $d_j := \sqrt{1 - |\alpha_j|^2}$, form an orthonormal system in $\mathcal{H}_2$, the so–called Takenaka–Malmquist system. This system is also complete in $\mathcal{H}_p$, see e.g. [42]. These basis can be viewed as an extension of the trigonometric system on the unit circle.

The idea of decomposing representations of linear time-invariant dynamical systems and related input/output signals in terms of orthogonal components other than the standard Fourier series, dates back to the work of Lee and Wiener in the thirties, as reviewed in [28]. Laguerre functions have been very popular in this respect, mainly because of the fact that their frequency response is rational. In an attempt to find more general classes of orthogonal basis functions with this same property, [25] formulated a general class of functions, composed of damped exponentials, to be used for signal decomposition. In [48],[50]and [49], Laguerre functions and so-called two-parameter Kautz functions have been used in the identification of the expansion coefficients of approximate models by simple linear regression methods. Extending this work further, [21] has developed a theory on the construction of orthogonal basis functions, based on balanced realizations of inner (all-pass) transfer functions. A further generalization of this situation is presented in [35], where concatenations of freely chosen all-pass sections are considered as basis-generators.

The use of GOBF as linear model parameterizations in system identification problems has been shown to be attractive; this is due to the fact that smartly chosen basis functions can provide a fast rate of convergence of the corresponding series expansion, thus leading to linear model parameterizations with a limited number of parameters.

By introducing a special argument transform in the inner functions representing the generalization of the shift operator, these basis constructions can simple be related to the trigonometric bases. This was shown in [40]. The advantage coming from this property is that one can use the well known FFT or DFT to compute the coefficients of the models.

The first part of this paper provides a generalization of certain classical $\mathcal{L}_p$ norm convergence theorems of the partial sum operators of the Fourier series to the case when the partial sum operators are defined through orthonormal rational expansions. It is also known from the classical theory that the partial sum operators fail to be convergent the uniform norm, i.e., for $p = \infty$, and that certain summations, e.g. Fejér summation, of the partial sums of the Fourier series provides a useful tool in providing convergence. After showing that the Lebesgue functions are also unbounded for the rational kernels, a Fejér type summation theorem is proved for the situation, when the rational basis is defined by a periodic sequence of poles.

Probably one of the most important contributions to linear system theory has been the discovery by R.E. Kalman, see [24], that the theory of linear systems can be naturally accommodated in classical module theory. This observation led to a completely satisfactory theory of realization, i.e., theory that links external input/output descriptions with internal state space descriptions of the system. From an external point of view regarding systems, an input/output map was defined by Kalman as a module homomorphism, over the ring of polynomials, between appropriately defined spaces of input and output functions. This implies a linear context and the module property implies time invariance. The causality property can be introduced implicitly by considering the input/output map from the past inputs to the space of future outputs.

The conventional realization problem of linear system theory is one of constructing from the infinite sequence of the Markov parameters the state space realization of the transfer function, i.e., the construction of the real constant matrices $\{A, B, C, D\}$ such that $G(z) = D + C(z\mathbb{I} - A)^{-1}B$. Generally the constraint that A is of least dimension is applied.

The study of the problem is greatly aided by the concept of Hankel matrix and the factorization of this matrix into two matrices with full column rank – infinite observability matrix – and full row rank – infinite controllability matrix is fully exploited. When the data contain only a finite sequence $\{G_i\}$, of Markov parameters then one is faced with a partial realization problem. It is interesting to note, that the partial realization problem of linear system theory can be interpreted as a multiple–point interpolation problem, for further details see [1]. Based on a transformation generated by the rational orthonormal expansion a conventional realization problem is associated to the original problem but in a different domain. An algorithm is given that provides the state space solution if the realization is known in the transformed domain.

2 Approximation by rational orthonormal functions on the unit circle

In what follows we will be concerned mainly with complex function theory on the unit disk. Therefore, to set the notation let us denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{C}$ the set of complex numbers and let $\mathbb{Z}$ be the set of integers. The open unit disc, its boundary and its exterior will be denoted by

$$\mathbb{D} := \{z \in \mathbb{C} \,|\, |z| < 1\}, \quad \mathbb{T} := \{z \in \mathbb{C} \,|\, |z| = 1\}, \quad \text{and} \quad \mathbb{E} := \{z \in \mathbb{C} \,|\, |z| > 1\}.$$

Let us denote by $\mathcal{I}$ the integral mean on $\mathbb{T}$, i.e.,

$$\mathcal{I}(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\omega})d\omega. \tag{1}$$

By $\mathcal{L}_p$, $1 \le p \le \infty$ will be denoted the classical $\mathcal{L}_p(\mathbb{T})$ Banach space endowed with the norm

$$||f||_p := \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\omega})|^p d\omega\right)^{\frac{1}{p}} \quad f \in \mathcal{L}_p, 1 \le p < \infty$$

and

$$||f||_\infty := ess.\sup_{\omega\in\mathbb{T}} |f(e^{i\omega})| \quad \text{for} \quad f \in \mathcal{L}_\infty, p = \infty.$$

The scalar product considered in $\mathcal{L}_2$ is the usual one, i.e.,

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\omega})\overline{g}(e^{i\omega})d\omega = \mathcal{I}(f\overline{g}) \quad f, g \in \mathcal{L}_2.$$

$\mathcal{H}_2$ will be the Hardy space of square integrable functions on $\mathbb{T}$ with analytic continuation on the unit disc, i.e., $\mathcal{H}_2$ can be identified with $\mathcal{H}_2(\mathbb{D})$. Its orthogonal complement in $\mathcal{L}_2$ will be denoted by $\mathcal{H}_{2,\perp}$, i.e., $\mathcal{H}_{2,\perp}$ can be identified with $\mathcal{H}_2(\mathbb{E})$. Thus, the stable transfer functions are elements of $\mathcal{H}_2$. A classical introduction in $\mathcal{H}_p$ theory is [23], [38] and [9]. A more advanced treatise of the topic is in [10].

As otherwise is not stated, it will be supposed $z \in \mathbb{T}$, i.e., $z := e^{i\omega}$, $\omega \in \mathbb{R}$.

The finite Blaschke product of order n with zeros that corresponds to $\boldsymbol{\alpha}_{n_b}$ will be denoted by G_b, i.e.,

$$G_b(z) = \prod_{j=1}^{n_b} b_j(z) \quad \text{where} \quad b_j(z) = \frac{1 - \overline{\alpha}_j z}{z - \alpha_j},\ |\alpha_j| < 1, \tag{2}$$

and the Blaschke product with poles that corresponds to $\boldsymbol{\alpha}_{n_b}$ will be denoted by G_b^*, i.e., $G_b^* = \prod_{j=1}^{n_b} \overline{b}_j$ where $\overline{b}_j(z) = \frac{z-\alpha_j}{1-\overline{\alpha}_j z}$, $|\alpha_j| < 1$, and it is clear that $G_b^*(e^{i\omega}) = \overline{G_b(e^{i\omega})}$, where the overbar denotes complex conjugation. Through this paper it will be supposed that the Blaschke condition, i.e. $\sum_{i=1}^{\infty}(1 - |\alpha_i|) = \infty$, is fulfilled.

If $\mathcal{P}_k$ denotes the space of polynomials of degree at most k, and we denote by $\eta(z) := \prod_{i=1}^{n}(1 - \overline{\alpha}_i z)$, and $\omega(z) := \prod_{i=1}^{n}(z - \alpha_i)$, then consider the sets

$$\mathcal{R}_n := \{\frac{p}{\omega} \,|\, p \in \mathcal{P}_{n-1}\}, \quad \mathcal{R}_{-n} := \{\frac{p}{\eta} \,|\, p \in \mathcal{P}_{n-1}\},$$

respectively. Accordingly, one can set $\mathcal{R}_{\pm n} := \{\frac{p}{\eta\omega} \,|\, p \in \mathcal{P}_{2n-1}\}$, i.e., the orthogonal sum of $\mathcal{R}_{-n}$ and $\mathcal{R}_n$. If $\mathcal{R}_n$ includes the constant functions then it will be denoted by $\mathcal{R}_n^0$ and accordingly, $\mathcal{R}_{\pm n}^0 := \mathcal{R}_{-n} \oplus \mathcal{R}_n^0$. Let us observe that $\mathcal{R}_n$ and $\mathcal{R}_n^0$ are the sets of stable strictly proper transfer functions and proper transfer functions, respectively, that corresponds to a fixed denominator structure, i.e., to a fixed set of poles.

In what follows a central role will be played, in engineering terms, by the phase function of the Blaschke product, introduced in [40], that will be called "β function" throughout this paper.

Let us denote the phase function of a single term by $\beta_j(\omega)$, i.e., $b_{\alpha_j}(e^{i\omega}) = e^{i\beta_j(\omega)}$. An explicit expression for this function can be given as follows: denote the poles (zeros) of the Blaschke product by $\alpha_j = \rho_j e^{i\theta_j}$, then $\beta_j(\omega) = \theta_j + \tau_{s(\rho_j)}(\omega - \theta)$, where $\tau_s(\omega) = 2\arctan(s \tan\frac{\omega}{2})$ and $s(\rho_j) = \frac{1+\rho_j}{1-\rho_j}$, with $\omega \in [-\pi, \pi]$, and it is extended periodically to $\mathbb{R}$ by $\tau_s(\omega + 2\pi) = \tau_s(\omega) + 2\pi$, see [39].

For the derivatives one has

$$\beta_j'(\omega) = \frac{1 - |\alpha_j|^2}{|1 - \overline{\alpha}_j e^{i\omega}|^2}. \tag{3}$$

Note that $\beta_j : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function with $\beta_j'(\omega) = |\varphi_j(e^{it})|^2 = |\phi_j(e^{i\omega})|^2$. Hence for a finite Blaschke product B_n of order n, there exist a monotone increasing, invertible and differentiable function $\beta_{(n)}(\omega)$ mapping the interval $\mathbb{R}$ onto itself, such that,

$$B_n(e^{i\omega}) = e^{in\beta_{(n)}(\omega)}, \tag{4}$$

where the function $\beta_{(n)}(\omega)$ can be expressed as

$$\beta_{(n)}(\omega) := \frac{1}{n}\sum_{k=1}^{n} \beta_k(\omega). \tag{5}$$

The derivative of the inverse is bounded moreover if there is a constant $0 < c < 1$ such that $|\alpha_k| < c$ for $k = 1, \cdots, n$ then one has the uniform bounds

$$\frac{1-c}{2} \le \beta_{(n)}'(\omega) \le \frac{2}{1-c}, \quad \text{and} \quad \frac{1-c}{2} \le \gamma_{(n)}'(\omega) \le \frac{2}{1-c}. \tag{6}$$

2.1 Reproducing kernels

A fundamental concept and a basic tool in investigating approximation results concerning subspaces in Hilbert spaces are the concept of reproducing property and the reproducing kernels. Moreover, they play a central role in the construction of the interpolation operators, that are very useful in building practical algorithms.

The reproducing kernel $K : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ of a subspace $\mathcal{V} \subset \mathcal{L}_2$ is defined by its reproducing property, i.e.,

$$\forall f \in \mathcal{V} \quad f(w) = \langle f, K(\cdot, w)\rangle, \qquad w \in \mathbb{T}.$$

If an orthonormal basis $\{\psi_j(z) \,|\, j = 1, \ldots, n\}$, defined by in the $n < \infty$ dimensional subspace $\mathcal{V}$ is considered, then the reproducing kernel, or the Dirichelet kernel of the system, is given by

$$K_n(z, w) = \sum_{k=1}^{n} \psi_k(z)\overline{\psi}_k(w), \qquad z, w \in \mathbb{T},$$

and it is independent of the choice of the orthonormal basis $\{\psi_j(z)\}$, see [2]. The orthogonal projection onto $\mathcal{V}$ is given by

$$\mathbf{P}_{\mathcal{V}} f(w) = \langle f, K_n(\cdot, w)\rangle, \quad f \in \mathcal{L}_2.$$

For any set of distinct points $\mathbf{w}_n := \{w_1, w_2, \cdots, w_n\}$ on $\mathbb{T}$,

$$\langle K_n(\cdot, w_i), K_n(\cdot, w_j)\rangle = K_n(w_i, w_j) \quad \text{and the matrix} \quad \mathbf{K}_n(\mathbf{w}_n) := [K_n(w_i, w_j)]_{i,j=1}^{n}$$

is positive semidefinite.

Applying this to the subspace $\boldsymbol{\mathcal{R}}_n$ one can obtain $K_n(z, w) = \sum_{k=1}^{n} \phi_k(z)\overline{\phi}_k(w)$, as a reproducing kernel that can be expressed in a compact form by the the following Christoffel– Darboux formula, see [8, 12]:

Lemma 2.1

$$K_n(z, w) := \sum_{k=1}^{n} \phi_k(z)\overline{\phi}_k(w) = \frac{B_n(z)\overline{B}_n(w) - 1}{1 - z\overline{w}}, \quad z, \mu \in \mathbb{D} \cup \mathbb{T}, \tag{7}$$

where B_n is given in (4).

Using the expression for the derivative of the function $\beta_{(n)}$ one has

$$K_n(e^{i\omega}, e^{i\omega}) := \sum_{k=1}^{n} |\phi_k(e^{i\omega})|^2 = n\beta^{'}_{(n)}(\omega). \tag{8}$$

Lemma 2.2 *For any set of distinct points $\mathbf{w}_n := \{w_1, w_2, \cdots, w_n\}$ on $\mathbb{T}$ the system defined by $\boldsymbol{\kappa}_n := \{\kappa_j(z) := K_n(z, w_j) \,|\, j = 1, \cdots, n$ forms a basis for $\boldsymbol{\mathcal{R}}_n$.*

2.2 $\mathcal{L}_p$ norm convergence of the partial sums

As it was seen in the analysis of the asymptotical bias a central question is the approximation property of the system that is used for parametrization. This section gives a generalization of the classical $\mathcal{L}_p$ norm convergence theorem of the partial sums of the Fourier series to the partial sums in the rational orthonormal system generated by the sequence of poles $(\alpha_k \,|\, k \in \mathbb{Z})$.

Let us recall the completeness property of the Takenaka–Malmquist system in the $\mathcal{H}_p$ spaces, that follows from [23],pp. 64 and [10],pp. 53:

Theorem 2.1 *The system $\Phi = \{\phi_k \,|\, k \in \mathbb{N}\}$ is complete in $\mathcal{H}_p$ for $0 < p < \infty$ if and only if the Blaschke condition holds, i.e.,*

$$\sum_{k=1}^{\infty}(1 - |\alpha_k|) = \infty.$$

The system $\Phi = \{\phi_k, \, \phi_{-k} \,|\, k \in \mathbb{N}\}$ is complete in $\mathcal{L}_p$ for $0 < p < \infty$ if and only if the Blaschke condition holds. Let us denote the partial sums of the expansion of a function f in the orthonormal system $\{\phi_k \,|\, k \in \mathbb{Z}\}$ by $\mathcal{S}_n f$ i.e., $\mathcal{S}_n f(w) = \langle f, S_n(\cdot, w)\rangle$.

Proposition 1 *For $f \in \mathcal{L}_p, \quad 1 < p < \infty$ one has*

$$||\mathcal{S}_n f||_p \leq C_p ||f||_p \tag{9}$$

and if $\sum_{k=1}^{\infty}(1 - |\alpha_k|) = \infty$ then

$$\lim_{n \to \infty} ||f - \mathcal{S}_n f||_p = 0. \tag{10}$$

2.3 Summation theorems

Let us consider the situation, when the set of poles that generate the orthonormal system is formed by a periodic repetition of the same finite sequence $\alpha_d = (\alpha_k \,|\, k = 1, \dots, d)$ and consider for these systems the "block" analogous of the Fejér summation, $\mathcal{F}_n = \frac{1}{n}\sum_{k=1}^{n} \mathcal{S}_{kd}$, i.e, the operator with the kernel $F_n := \frac{1}{n}\sum_{k=1}^{n} S_{kd}$.

Proposition 2 *For* $f \in \mathcal{L}_\infty$ *then* $||\mathcal{F}_n f||_\infty \le C||f||_\infty$ *and for all of the continuous* f *one has* $\lim_{n\to\infty} ||f - \mathcal{F}_n f||_\infty = 0$.

Using this theorem one can prove that the classical, i.e., not "block" , summation for the periodic case is also convergent, .i.e.,

Proposition 3 *For* $f \in \mathcal{L}_\infty$ *and* $\mathcal{F}_N^c := \frac{1}{N}\sum_{k=1}^{N} \mathcal{S}_k$ *then if* $1 - |\alpha_k| > \delta > 0,\ k \in \mathbb{N}$, *one has* $||\mathcal{F}_N^c f||_\infty \le C||f||_\infty$ *and for all of the continuous* f *one has* $\lim_{N\to\infty} ||f - \mathcal{F}_N^c f||_\infty = 0$.

3 A discrete rational orthonormal system on the unit circle

In practice one has to deal with a finite amount of data, therefore it is necessary to construct methods that gives sufficiently accurate approximations based only on the available finite information. Moreover, if one uses interpolation type methods, then it is possible to recover functions that are members of certain finite dimensional spaces, by using only a finite amount of data. A well known example for such methods is given by the operator that interpolates polynomials up to a certain degree, e.g., Lagrange type interpolation operators. The properties of these algorithms depend heavily on the set of the points where interpolation is required. For example, polynomial type interpolation defined on a uniform grid of the unit circle leads to FFT type algorithms. In what follows the theoretical background will be given for the construction of such discrete systems in the context of rational parameterizations.

Let us denote by $\mathbb{W}_n$ the set of the image of the roots of unity through the $\gamma_{(n)} = \beta_{(n)}^{-1}$ function, i.e.,

$$\mathbb{W}_n = \{\zeta_k = e^{i\gamma_k} \,|\, \gamma_k = \beta_{(n)}^{-1}(\eta_k),\ \eta_k \in \mathbb{U}_n\},$$

where $\mathbb{U}_n$ is the set of n equispaced points.

Considering as nodes the set $\mathbb{W}_n$, one can introduce the following rational interpolation operator:

$$(\mathcal{L}_n f)(z) := \sum_{\zeta \in \mathbb{W}_n} \frac{K_n(z,\zeta)}{K_n(\zeta,\zeta)} f(\zeta), \tag{11}$$

where f is a continuous function on $\mathbb{T}$ and $z \in \mathbb{T}$. Let us denote by

$$\mathbf{l}_{n,\zeta}(z) := \frac{K_n(z,\zeta)}{K_n(\zeta,\zeta)}, \quad \zeta \in \mathbb{W}_n.$$

From the definition of $\mathbb{W}_n$ and by (7) it follows that for $0 \le k, l < n,\ k \ne l$, one has:

$$\mathbf{l}_{n,\zeta_k}(\zeta_l) = \frac{1 - e^{in(\beta_n(\gamma_l) - \beta_n(\gamma_k))}}{K_n(\zeta_k,\zeta_k)(1 - \zeta_l\overline{\zeta}_k)} = \frac{1 - e^{2\pi(l-k)}}{K_n(\zeta_k,\zeta_k)(1 - \zeta_l\overline{\zeta}_k)} = 0.$$

Consequently, for $0 \le k, l < n$,

$$\mathbf{l}_{n,\zeta_k}(\zeta_l) = \delta_{k,l}, \tag{12}$$

i.e., $\mathbf{l}_{n,\zeta},\ \zeta \in \mathbb{W}_n$, are the Lagrange functions corresponding to the system $\{\phi_i \,|\, i = 1, \cdots, n\}$.

This implies that $\mathcal{L}_n f$ interpolates f at the points of $\mathbb{W}_n$, i.e., $\mathcal{L}_n f(\zeta) = f(\zeta),\ \zeta \in \mathbb{W}_n$. It is also clear that $\mathcal{L}_n f = f$ for $f \in \boldsymbol{\mathcal{R}}_n$, and $\{\mathbf{l}_{n,\zeta} \,|\, \zeta \in \mathbb{W}_n\}$ is a basis in $\boldsymbol{\mathcal{R}}_n$.

Let us define the discrete scalar product

$$[f,g]_n := \sum_{\zeta \in \mathbb{W}_n} \frac{f(\zeta)\overline{g}(\zeta)}{K_n(\zeta,\zeta)} = \sum_{\zeta \in \mathbb{W}_n} \frac{f(\zeta)\overline{g}(\zeta)}{n\beta_{(n)}'(\gamma)}, \tag{13}$$

where $\zeta = e^{i\gamma}$.

For the classical case, i.e., when $\alpha_1 = \cdots = \alpha_n = 0$, the β function is the identity and this scalar product is exactly the discrete Fourier scalar product defined by the trigonometric interpolation, i.e.,

$$[f,g]_n^f := \frac{1}{n}\sum_{\zeta\in\mathbb{U}_n} f(\zeta)\overline{g}(\zeta).$$

Using this discrete scalar product the interpolation operator can be written as:

$$(\mathcal{L}_n f)(z) = [f, K_n(\cdot, z)]_n,$$

for $f \in \mathbb{A}(\mathbb{D})$. Using this fact and by (12) it follows that for $\zeta, \xi \in \mathbb{W}_n$ one has

$$\mathcal{I}(\mathbf{l}_{n,\zeta}\overline{\mathbf{l}_{n,\xi}}) = \delta_{\zeta,\xi}, \tag{14}$$

where $\mathcal{I}$ is the integral operator.

It is easy to see using the reproducing property of the kernel that $\langle \mathcal{L}_n f, \mathcal{L}_n g\rangle = [f,g]_n$, and it follows that every orthonormal system $\{\psi_k \mid k = 1, \cdots, n\}$ on the subspace defined by the reproducing kernel is also discrete orthonormal, i.e., $[\psi_k, \psi_l]_n = \delta_{k,l} \quad \text{for} \quad 1 \le k, l \le n$.

Using the interpolation operator $\mathcal{L}_n$, see (11), one can introduce a quadrature formula as

$$\mathcal{I}_n(f) := \sum_{\zeta\in\mathbb{W}_n} \rho_\zeta^{(n)} f(\zeta), \quad \text{where} \quad \rho_\zeta^{(n)} := \mathcal{I}\Big(\frac{K_n(\cdot,\zeta)}{K_n(\zeta,\zeta)}\Big), \tag{15}$$

where $\mathcal{I}$ denotes the integral mean operator on $\mathbb{T}$. Then it is clear, that $\mathcal{I}_n(f) = \mathcal{I}(f)$ for all $f \in \boldsymbol{\mathcal{R}}_n$.

To get $\rho_\zeta^{(n)}$ it have been used the fact that for any $g \in \mathbb{A}(\mathbb{D})$ one has $\mathcal{I}(g) = g(0)$. Thus by (7) for $\zeta \in \mathbb{W}_n$ one has

$$\mathcal{I}(K_n(\cdot,\zeta)) = \sum_{k=1}^{n} \mathcal{I}(\phi_k)\overline{\phi}_k(\zeta) = \sum_{k=1}^{n} \phi_k(0)\overline{\phi}_k(\zeta) = 1 - B_n(0)\overline{B}_n(\zeta) = 1 - B_n(0).$$

Consequently by (15) the coefficients of the quadrature formula are of the form

$$\rho_\zeta^{(n)} := \frac{1 - B_n(0)}{K_n(\zeta,\zeta)} = \frac{1 - B_n(0)}{n\beta'_{(n)}(\gamma)}, \quad \text{where} \quad \zeta = e^{i\gamma}. \tag{16}$$

If one of the zeros of the Blaschke product is zero, say, $\alpha_n = 0$, hence $b_{\alpha_n}(z) = z$, one has $B_n(0) = 0$. It follows that in this case the coefficients of the quadrature formula

$$\rho_\zeta^{(n)} = \frac{1}{n\beta'_{(n)}(\gamma)} > 0, \quad \text{where} \quad \zeta = e^{i\gamma},$$

are positive.

For every $g \in \boldsymbol{\mathcal{R}}_{-n}$ one has $\mathcal{I}(g) = 0$, and $g = hB_{\overline{n}}$ for some $h \in \boldsymbol{\mathcal{R}}_n$. It follows, that $\mathcal{I}_n(g) = \mathcal{I}_n(h) = \mathcal{I}(h) = h(0)$, i.e., in general one cannot expect $\mathcal{I}(g) = \mathcal{I}_n(g)$. But $\mathcal{I}(g) = \mathcal{I}_n(g)$ if $g \in \boldsymbol{\mathcal{R}}_{-n} \cap z\boldsymbol{\mathcal{R}}_{-n}$.

One can obtain a completely analogous result as for the polynomial case, if one chose the quadrature formula induced by $\mathcal{L}_n^0$, based on the interpolation nodes $\mathbb{W}_n^0$.

Proposition 4 *Let us introduce the Gauss type quadrature formula*

$$\mathcal{I}_n^0(f) := \sum_{\zeta\in\mathbb{W}_n^0} \frac{f(\zeta)}{K_n^0(\zeta,\zeta)},$$

then $\mathcal{I}_n^0(f) = \mathcal{I}(f)$ *for all* $f \in \boldsymbol{\mathcal{R}}_{\pm n}^0$.

The asymptotic properties of the the quadrature formula induced by $\mathcal{L}_n$ can be summarized as follows:

Proposition 5 *For every* $n \in \mathbb{N}$, $n \geq 2$,

$$\sum_{\zeta \in \mathbb{W}_n} \frac{1}{K_n(\zeta,\zeta)} = \frac{1-|B_n(0)|^2}{|1-B_n(0)|^2} \tag{17}$$

and consequently for the norm of the functionals $\mathcal{I}_n$ *one has*

$$||\mathcal{I}_n|| = \sum_{\zeta \in \mathbb{W}_n} |\rho_\zeta^{(n)}| < \frac{2}{1-|\alpha_1|}. \tag{18}$$

Moreover, if $\sum_{k=1}^{\infty}(1-|\alpha_k|) = \infty$, *then*

$$\lim_{n\to\infty} \sum_{\zeta \in \mathbb{W}_n} \frac{1}{K_n(\zeta,\zeta)} = \lim_{n\to\infty} \sum_{\zeta \in \mathbb{W}_n} |\rho_\zeta^{(n)}| = 1. \tag{19}$$

As a corollary one has:

Proposition 6 *If* $\sum_{k=1}^{\infty}(1-|\alpha_k|) = \infty$, *then for every* $f \in \mathbb{A}(\mathbb{D})$, *one has*

$$\lim_{n\to\infty} \mathcal{I}_n(f) = \mathcal{I}(f).$$

Based on this result, one has the following generalization of the Erdős–Turán theorem for $\mathcal{L}_n$ on $\mathbb{A}(\mathbb{D})$:

Proposition 7 *Considering the interpolation operator*

$$(\mathcal{L}_n f)(z) = \sum_{\zeta \in \mathbb{W}_n} \frac{K_n(z,\zeta)}{K_n(\zeta,\zeta)} f(\zeta).$$

If $\sum_{k=1}^{\infty}(1-|\alpha_k|) = \infty$, *then for every* $f \in \mathbb{A}(\mathbb{D})$, *one has*

$$\lim_{n\to\infty} ||f - \mathcal{L}_n f||_2 = 0.$$

Let us conclude this section with an $\mathcal{L}_p$ norm convergence result of certain rational interpolation operators on the unit circle, based on an extension of the Marcinkiewicz–Zygmund type inequalities for the interpolation operator $\mathcal{L}_n$ on $\mathbb{A}(\mathbb{D})$.

Proposition 8 *Let* $f \in \mathcal{R}_n$, *and* $1 - |\alpha_k| > \delta > 0$, $k \in \mathbb{N}$. *Then there exist constants* $C_1, C_2 > 0$ *depending only on* p, *such that for* $1 < p < \infty$ *one has*

$$C_1||f||_p \leq [\mathcal{I}_n(|f|^p)]^{\frac{1}{p}} \leq C_2||f||_p.$$

Moreover, one has

$$||f - \mathcal{L}_n f||_p \leq C E_n(f),$$

and consequently,

$$\lim_{n\to\infty} ||f - \mathcal{L}_n f||_p = 0.$$

In a practical situation one would like to recover all rational transfer functions from a given finite dimensional subspace and, in the same time, to have an approximation property for the entire $\mathcal{H}_2$ space in order to cope with the possible unmodelled dynamics. This property are not granted by all interpolatory type approximation methods. Proposition 8. shows that the interpolation operator $\mathcal{L}_n$ defined in (11) has the required property necessary to have the asymptotic bias results.

4 The *Hambo*–transform

In what follows G_b will denote the finite Blaschke product that can be written under the form $G_b = \prod_{j=1}^{n_b} b_j$ where $b_j(z) = \frac{1-\overline{\alpha}_j z}{z-\alpha_j}, |\alpha_j| < 1$ and each pole of G_b is repeated according to its multiplicity. n_b is called the order of G_b. Denote by (A_b, B_b, C_b, D_b) the minimal balanced realization of G_b. Let us denote by

$$\psi_b(z) = B_b^T(z\mathbb{I} - A_b^T)^{-1} \tag{20}$$

and by

$$\nu_k^T(z) := V_k(z) := \psi_b^T {G_b}^{k-1}(z), \tag{21}$$

the components of the n_b-dimensional rational functions $V_k(z)$. Then the set $\{e_i^T V_k(z) \mid i = 1, \ldots, n_b, k \in \mathbb{N}\}$ will constitute an orthonormal basis for $\mathcal{H}_{2,\perp}$. In [42] this basis was extended to $\mathcal{L}_2$, i.e., one has that the set $\{e_i^T V_k(z) \mid i = 1, \ldots, n_b, k \in \mathbb{Z}\}$ constitutes an orthonormal basis for $\mathcal{L}_2$.

The *Hambo*-signal–transform on $\mathcal{L}_2$, induced by the GOBF expansion defined by G_b is defined as:

Definition 1 *Let us consider the expansion of a function $F \in \mathcal{L}_2$ in the generalized orthonormal basis, i.e.,*

$$F = \sum_{-\infty}^{\infty} \nu_k F_k, \quad F_k \in \mathbb{C}^{n_b}.$$

Then the Hambo-signal–transform $\mathcal{H}_\nu : \mathcal{L}_2 \to \mathcal{L}_2^{n_b}$, will be defined as

$$\mathcal{H}_\nu F(\lambda) = \tilde{F}(\lambda) := \sum_{-\infty}^{\infty} \lambda^{-k} F_k. \tag{22}$$

In what follows the subscript will be dropped in the notation of the signal transform. It is clear that this transform is a unitary transform between the two function spaces, i.e., $\langle F, G\rangle = \langle \mathcal{H}F, \mathcal{H}G\rangle$ and $\mathcal{H}\mathcal{H}^* = \mathbb{I}_{n_b}$, and $\mathcal{H}^*\mathcal{H} = \mathbb{I}$.

Denote by $\mathbf{J} : \mathcal{L}_2 \to \mathcal{L}_2$ the operator $(\mathbf{J}f)(z) = \overline{z}f(\overline{z})$ and by $\mathtt{S}$ the canonical shift operator on $\mathcal{H}_{2,\perp}$, i.e.,$(\mathtt{S}f)(z) = \overline{z}f(z)$. A Hankel operator will be meant as an operator $\mathbf{H} : \mathcal{H}_{2,\perp} \to \mathcal{H}_{2,\perp}$ such that $\mathtt{S}^*\mathbf{H} = \mathbf{H}\mathtt{S}$, i.e., $\mathbf{H} = \mathbf{P}_{\mathcal{H}_{2,\perp}} M_G \mathbf{J}|_{\mathcal{H}_{2,\perp}}$, where M_G is the multiplication (Laurent) operator defined by $G \in \mathcal{L}_\infty$.

If one has an operator $T : \mathcal{L}_2 \to \mathcal{L}_2, \quad y = Tu$ and consider the signal transforms $\mathcal{H}_\nu$ and $\mathcal{H}_\pi$ respectively, then the *Hambo*–image of the operator that acts on the transform domain will be $\tilde{T} := \mathcal{H}_\nu T \mathcal{H}_\pi^*$. It is also clear that the images of the orthogonal projection operators $\mathbf{P}_{\mathcal{H}_2}$ and $\mathbf{P}_{\mathcal{H}_{2,\perp}}$ will be the same type of projection operators $\tilde{P}_{\mathcal{H}_2} := \mathcal{H}_\nu \mathbf{P}_{\mathcal{H}_2} \mathcal{H}_\pi^*$ and $\tilde{P}_{\mathcal{H}_{2,\perp}} := \mathcal{H}_\nu \mathbf{P}_{\mathcal{H}_{2,\perp}} \mathcal{H}_\pi^*$, i.e, $\tilde{P}_{\mathcal{H}_2} = \mathbf{P}_{\mathcal{H}_2^{n_b}}$ and $\tilde{P}_{\mathcal{H}_{2,\perp}} = \mathbf{P}_{\mathcal{H}_{2,\perp}^{n_b}}$.

Lemma 4.1 *The Hambo–image of a Laurent operator* $\mathbf{L}$ *is also a Laurent operator* $\tilde{\mathbf{L}}$.

Making the usual identification of the Laurent operators on $\mathcal{L}_2^{n_b}$, with the functions $\mathcal{L}_\infty^{n_b \times n_b}$, one can introduce a *Hambo*-system–transform:

Definition 2 *Let us consider the function $G \in \mathcal{L}_\infty$. Then its Hambo-system–transform $\mathcal{H}_s : \mathcal{L}_\infty \to \mathbb{L}_\infty^{n_b \times n_b}$, will be defined as $\mathcal{H}_s G = \tilde{G}(\lambda)$ where $\tilde{G}(\lambda)$ is the symbol of the Laurent operator $\mathcal{H}G\mathcal{H}^*(\lambda)$.*

One can observe that the system *Hambo* image of a stable all pass(inner) function is a stable all pass(inner) function. Denote by Λ the system *Hambo* transform of $\mathtt{S}$. The *Hambo*-signal-transform maps a $\mathtt{S}^*$ invariant invariant subspace X to a Λ^* invariant subspace $\tilde{X}$ with the same dimension. The reverse statement is also true.

Lemma 4.2 *Consider the GOBF expansion coefficients of G, i.e., $G = \sum_{k=-\infty}^{\infty} \nu_k G_k$ and the Markov – Hambo Markov – parameters M_k of $\tilde{G}$, i.e, $\tilde{G} = \sum_{k=-\infty}^{\infty} M_k \overline{\lambda}^k$. Then*

$$M_k e_i = [\nu_{k+1}, G\psi_b e_i] = P_i G_{k+1} + Q_i G_k, \tag{23}$$

where

$$P_i e_j = P_j e_i, \qquad Q_i e_j = Q_j e_i, \tag{24}$$

$$P_i A_b = A_b P_i, \qquad Q_i(A_b D_b - B_b C_b) = (A_b D_b - B_b C_b) Q_i, \tag{25}$$

$$Q_i A_b - A_b Q_i = P_i(A_b D_b - B_b C_b) - (A_b D_b - B_b C_b) P_i. \tag{26}$$

and

$$P_i P_j = P_j P_i, \quad Q_i Q_j = Q_i Q_j, \quad Q_i P_j + P_i Q_j = P_j Q_i + Q_j P_i. \tag{27}$$

Denote the stable all pass function $\mathcal{H}_s \overline{z}$ by $N(\lambda)$. Since M_N is the it Hambo–system transform of s it is not surprising that the function $N(\lambda)$ will play a central role in the further investigations.

4.1 State space formulation and the *Hambo*–system–transform

Let us consider a state space representation $\{A, B, C, D\}$ of a proper rational transfer function and let us denote by $\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\}$ the realization of its *Hambo*–system–transform.

Proposition 9 *Given a system $G(z)$ having a minimal state space realization (A, B, C, D) and its Hambo–system transform $\tilde{G}$ with minimal state space representation $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$. Then the following identities hold:*

$$\begin{bmatrix} A^T & C^T C_b \\ 0 & A_b \end{bmatrix} \begin{bmatrix} X_o \tilde{A} & X_o \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} A & B B_b^T \\ 0 & A_b^T \end{bmatrix} + \begin{bmatrix} C^T D_b \\ B_b \end{bmatrix} \begin{bmatrix} C & D B_b^T \end{bmatrix} = \begin{bmatrix} X_o \tilde{A} & X_o \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix},$$

and

$$\begin{bmatrix} \tilde{A}^T & \tilde{C}^T C_b^T \\ 0 & D_b^T \end{bmatrix} \begin{bmatrix} \tilde{X}_o A & \tilde{X}_o B \\ C & D \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B} B_b \\ 0 & D_b \end{bmatrix} + \begin{bmatrix} \tilde{C}^T A_b^T \\ B_b^T \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} B_b \end{bmatrix} = \begin{bmatrix} \tilde{X}_o A & \tilde{X}_o B \\ C & D \end{bmatrix},$$

where X_o is the observability Gramian of the pair (A, C) and $\tilde{X}_o$ is the observability Gramian of the pair $(\tilde{A}, \tilde{C})$.

4.2 The question of the inversion

The question that the image of an operator is an operator of the same type or not is a fundamental issue in the *Hambo*–domain investigations. The reason for that is that the multiplication operators do not commute in general with each other on $\mathcal{L}_2^{n_b}$, but those on $\mathcal{L}_2$ does! As a consequence the inverse image of a multiplication operator, i.e., $\mathcal{H}^* \tilde{G} \mathcal{H}$ is not a multiplication operator in general, hence one cannot associate a function $G \in \mathcal{L}_\infty$ to it.

A fundamental observation that one can state is:

Proposition 10 *For a given $\tilde{G}$ there exist a $F \in \mathcal{L}_\infty$, such that $\tilde{G} = \tilde{F}$ if and only if $\tilde{G}(\lambda)$ commutes with $N(\lambda)$.*

For a characterization of the range space given in term of the *Hambo* Markov parameters see [41].

5 Minimal state space realization

In this section a method will be presented, as an application of the results conserning the *Hambo*–transform, to give a minimal state space realization for a transfer function represented in a generalized orthonormal basis. The outline of the procedure is the following:

- The first step of the algorithm is the computation of the *Hambo* Markov parameters using formula (23).
- In the *Hambo* domain one can perform a classical minimal state space realization algorithm, e.g. a Ho–Kalman algorithm.
- Having a minimal state space realization in the *Hambo* domain one can obtain a minimal state space realization in the original domain by applying the result of Proposition 9.

The classical Ho-Kalman algorithm, see [22], can be summarized as follows. Consider a MIMO system with q inputs and p outputs and denote:

$$H_r := \begin{pmatrix} G_1 & G_2 & \dots & G_r \\ G_2 & G_3 & \dots & G_{r+1} \\ \dots & & & \\ G_r & G_{r+1} & \dots & G_{2r} \end{pmatrix}$$

and by

$$\tau(H_r) := \begin{pmatrix} G_2 & G_3 & \dots & G_{r+1} \\ G_3 & G_4 & \dots & G_{r+2} \\ \dots & & & \\ G_{r+1} & G_{r+2} & \dots & G_{2r+1} \end{pmatrix},$$

where $\{G_k\}$ is the set of Markov parameters of the transfer function G, i.e. $G(z) = \sum_{k=1}^{\infty} G_k z^{-k}$, and r is greater than or equal to the MacMillan degree of the system. The operator τ corresponds to the adjoint shift operator $\mathtt{S}^*$. Then there exist matrices P and Q such that

$$PH_rQ = \begin{bmatrix} \mathbb{I}_s & 0 \\ 0 & 0 \end{bmatrix} = J.$$

Let us denote by

$$U_s = \begin{bmatrix} \mathbb{I}_s & 0 \end{bmatrix} \quad E_k = \begin{bmatrix} \mathbb{I}_k & 0_k & \dots & 0_k \end{bmatrix},$$

where the dimension of the matrix U_s may vary according to the dimensions of the expressions in which it appears. Then a minimal state space realization (A, B, C) is given by

$$A = U_s J P \tau(H_r) Q J U_s^*$$

$$B = U_s J P H_r E_q^* \quad C = E_p H_r Q J U_s^*.$$

As an observation, it has to be stated here that the realization algorithm presented above gives the desired result in the full information case. In the finite information case one has to deal with the partial realization problem, i.e., the problem of finding the minimal order[1] rational function that has as first Markov parameters the given ones. In view of the realization theory, the problem is equivalent to the one of giving an extension sequence such that the Ho–Kalman algorithm gives the unique rational function with the desired property. In general the set of extension sequences that leads to rational functions with the same degree that has the first Markov parameters identical to the given ones has more then one element, i.e., the solution of the minimal partial realization problem in general is not unique. In the classical case a characterization of the admissible extension sequences is given in [44]. It is interesting to note, that the partial realization problem of linear system theory can be interpreted as a multiple–point interpolation problem, for further details see [1].

Given a finite GOBF expansion, one can ask about the minimal order rational function that has the first expansion coefficients identical with the given ones. One tool for the study of this rational interpolation problem with minimal MacMillan degree is the so called Löwner matrix. This matrix encodes the information about the minimal admissible complexity as a simple function of its rank and the rank of its submatrices. This approach leads to a generalization of the classical realization theory, that can be considered as a special case of the rational interpolation problem with all the data provided at a simple point (infinity) of the complex plane. It can be shown that in such a case the Löwner matrix reduces to the familiar Hankel matrix. Moreover for an interpolation problem can be associated a Hankel matrix that allow the parametrization of all of the rational solutions of the interpolation problem, for details see e.g. [4].

By using the *Hambo*–system transform, the problem of minimal order rational interpolation can be set as a classical partial realization problem in the *Hambo*-domain. The partial realization problem using *Hambo*-domain techniques is a quite delicate question and it is beyond the scope of this work to answer it completely. As an idea, having a GOBF expansion one could easily obtain a *Hambo* domain description – through the *Hambo* Markov parameters – of the system and one could apply, then, a partial realization or even a model reduction algorithm in the *Hambo* domain. The question that makes this problem delicate is the fact that not every $\mathcal{L}_{\infty}^{n_b \times n_b}$ function is the *Hambo* image of a scalar rational function. Therefore it is an important question to characterize the range space of $\mathcal{H}_s$, in algorithmic terms, i.e., how to construct an algorithm that "select" among all the solutions the "true" ones – the solutions that are in the *Hambo* domain.

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[1] The largest among the numerator and denominator degree, called also MacMillan degree.

In [7] a solution is given for the partial realization problem if there are at least three known expansion coefficients. The main tool in obtaining the solution is the augmented Hankel matrix formed by the augmented *Hambo* Markov parameters $M_k^a := [\, M_k \mid L_k \mid L_{k+1} \,]$, where M_k are the *Hambo* Markov parameters and L_k are the expansion coefficients. For this setting one can apply directly any of the classical algorithms, the solution being given by $\{\tilde{A}, [\,\tilde{B} \mid B \mid \tilde{A}B\,], \tilde{C}\}$.

In a more general context to a minimal order rational interpolation problem one can associate an inner function G_b, such that a solution of the interpolation problem can be given as an element G of $H(G_b) = \mathcal{H}_2 \ominus G_b\mathcal{H}_2$, with coordinates g_i in a given orthonormal basis. The solution of the problem is to find all the extensions $G^e \in G_b\mathcal{H}_2$ of G with $G + G^e$ having minimal order. The *Hambo*–system–transform of G is of the form $M_0 + \overline{\lambda} M_1$, where $M_1 = \sum_{i=1}^{n_b} g_i Q_i$, and the extension problem can be formulated as a completion problem for M_1, i.e., to find coefficients $g_{2,i}^e$ such that

$$M_1^e = \sum_{i=1}^{n_b} g_i Q_i + g_{2,i}^e P_i \tag{28}$$

has the lowest possible rank. In the situation, when all the poles are placed at the origin, this is exactly the Hankel(Toeplitz) matrix completion problem, i.e., the classical minimal partial , realization problem. However whilst this classical completion problem is widely studied, see e.g. [52] or [18], almost nothing is known about the more general type completion problems (28).

6 Conclusion

This paper has discussed results in frequency domain related to approximate identification in $\mathcal{H}_2$ by using a rational orthonormal parametrization of the transfer function. The criteria for modeling and identification was formulated in terms of $\mathcal{L}_2$ norms. The presented method can be seen as a generalization of the results of the FIR modeling to the case of a generalized orthonormal rational basis.

It was shown that the approximation power of these rational basis functions concerning the entire $\mathcal{H}_2$ space is the same as for the FIR case, i.e., one has the asymptotical convergence of the partial sums formed using the expansion defined by a given set of generalized orthonormal rational basis functions.

From a practical point of view it is important to prove that the discrete approximation operators, i.e., the methods that use only a finite amount of data, can also provide the required convergence properties, necessary to establish the asymptotic bias results.

It was introduced an interpolation operator defined on a nonuniform grid, given by the image of the uniform grid through the inverse of a function determined only by the prescribed poles.

The properties of this interpolation method were investigated and it was shown that by using that operator one can obtain a method that is convergent on $\mathcal{H}_2$ and in the same time interpolates the rational transfer functions of a subspace determined by the given poles. Thus it was proved that the knowledge about the poles of the transfer function can be efficiently exploited in practice by using an identification method that is based on rational orthonormal expansions rather than a conventional FIR modeling.

Given the expansion coefficients of a rational transfer function G in a generalized orthonormal basis generated by an inner function G_b, one can construct a state space representation starting from the balanced realization of G_b, and following the rules known for the composition of the state space representation of the systems, but that representation is not minimal even in the SISO case, as was shown by an example, in general. However if pole-zero cancellation does not occur for the transfer function obtained forming the common denominator and doing all the computations, that representation is minimal for the SISO case, but not for MIMO systems, in general. Therefore one needs an algorithm to construct a minimal representation.

This paper gives a generalization of exact realization theory for expansions using generalized orthonormal basis functions. The resulting realization theory will be the same as the one obtained by application of the Ho–Kalman algorithm for the standard Fourier expansion.

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