Dutch Institute of Systems and Control

Course: System Identification
Spring 2018

Assignment number 4 (concerning Lectures 7 and 8)

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Problem 1

Let us consider the following unknown true system[1]:

$$\mathcal{S}: \quad y(t) = G(z, \theta_0) u(t) + e(t) \tag{1}$$

where $e(t)$ is a white noise of variance σ_e^2 and $G(z, \theta_0)$ is parametrized as follows:

$$G(z, \theta_0) = \theta_0 \ \Lambda(z) \tag{2}$$

with $\theta_0 \in \mathbf{R}$ is an unknown parameter (θ_0 is a scalar) and $\Lambda(z)$ a known transfer function. The modulus $|\Lambda(e^{j\omega})|$ of $\Lambda(z)$ is represented in Figure 1. Let us also notice that the noise transfer function $H_0(z)$ in (1) is here given by $H_0(z) = 1$.

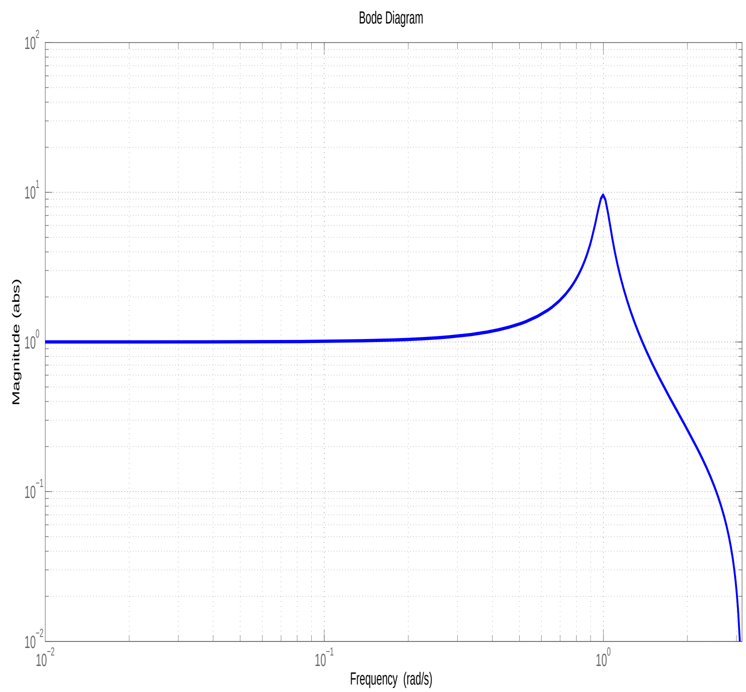

Figure 1: The modulus $|\Lambda(e^{j\omega})|$ of $\Lambda(z)$ as a function of the frequency ω

We choose the following full-order model structure $\mathcal{M}$ for $\mathcal{S}$:

$$\mathcal{M} = \{G(z, \theta) = \theta \ \Lambda(z) \mid \theta \in \mathbf{R}\}$$

Note that, since $H_0(z) = 1$, the noise model $H(z, \theta)$ is also chosen equal to 1.

[1]The sampling time T_s is here supposed equal to $T_s = 1$ second.

We apply an input signal $u(t)$ of power spectrum $\Phi_u(\omega)$ to the true system (1) for a duration N and the corresponding output $y(t)$ is collected. This yields the data set $Z^N = \{y(t)\ u(t) \mid t = 1...N\}$. Based on Z^N, we can identify an estimate $\hat{\theta}_N$ of θ_0 using the prediction error criterion i.e.

$$\hat{\theta}_N = arg\ \min_{\theta}\ \frac{1}{N}\sum_{t=1}^{N} \epsilon^2(t,\theta)$$

where $\epsilon(t,\theta) = y(t) - G(z,\theta)u(t)$. The variance σ_θ^2 of $\hat{\theta}_N$ is (asymptotically) given by:

$$\sigma_\theta^2 = \frac{\sigma_e^2}{N}\ \left(\bar{E}\psi(t,\theta_0)\psi^T(t,\theta_0)\right)^{-1} = \frac{\sigma_e^2}{N}\ \left(\bar{E}\psi^2(t,\theta_0)\right)^{-1} \tag{3}$$

with $\psi(t,\theta) = -\frac{\partial\varepsilon(t,\theta)}{\partial\theta}$. Notice that σ_θ^2 corresponds to the quantity $cov(\hat{\theta}_N) = \frac{P_\theta}{N}$ in the slides of lecture 7 and that σ_θ^2 is here a scalar and not a matrix.

a. Show that the variance σ_θ^2 of $\hat{\theta}_N$ given in (3) can also be expressed as:

$$\sigma_\theta^2 = \frac{\sigma_e^2}{N}\ \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |\Lambda(e^{j\omega})|^2\ \Phi_u(\omega)\ d\omega\right)^{-1} \tag{4}$$

We will now address the problem of optimally designing the spectrum $\Phi_u(\omega)$ of the input signal $u(t)$ in order to guarantee that the variance σ_θ^2 of $\hat{\theta}_N$ remains smaller than a given threshold γ. For this purpose, we consider the following optimization problem:

$$\begin{gathered} \min_{\Phi_u(\omega)}\quad \tfrac{1}{2\pi}\int_{-\pi}^{\pi}\ \Phi_u(\omega)\ d\omega \\ \text{subject to}\ \ \Phi_u(\omega) \geq 0\ \ \forall\omega \quad \text{and to} \\ \sigma_\theta^2 \leq \gamma \end{gathered} \tag{5}$$

Suppose first that we restrict attention to white noise input signals $u(t)$ having an arbitrary variance σ_u^2. In this case, the power spectrum $\Phi_u(\omega)$ is given by $\Phi_u(\omega) = \sigma_u^2\ \forall\omega$ and the decision variable of (5) is σ_u^2.

b. Denote by $\sigma_{u,opt}^2$ the value of σ_u^2 corresponding to the solution of (5) in this case. Give an expression of $\sigma_{u,opt}^2$ as a function of N, σ_e^2, $\Lambda(z)$ and γ.

Suppose now that we restrict attention to input signals having the following form:

$$u(t) = A\ sin(\omega_0\ t) \tag{6}$$

with an arbitrary amplitude A and a given frequency ω_0. In this case, the power spectrum $\Phi_u(\omega)$ is given by $\Phi_u(\omega) = \frac{\pi}{2}\ A^2\ \left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)$ and the decision variable of (5) is A.

c. Denote by A_{opt} the value of A corresponding to the solution of (5) in this case. Give an expression of A_{opt} as a function of N, σ_e^2, $\Lambda(z)$, ω_0 and γ.

Suppose finally that we restrict attention to input signals having the form (6) with an arbitrary amplitude A and an **arbitrary** frequency ω_0. In this case, the decision variable of (5) are both A and ω_0.

d. Denote by A_{opt} and $\omega_{0,opt}$ the value of A and ω_0 corresponding to the solution of (5) in this case. Give a numerical value for $\omega_{0,opt}$ as well as an expression for A_{opt} as a function of N, σ_e^2 and γ. Use for this purpose the answer to item (c.) and Figure 1.

Problem 2

Consider the first order system

$$y(t) = G_0(q)u(t) + H_0(q)e(t)$$

with

$$G_0(q) = \frac{b_0 q^{-1}}{1 + a_0 q^{-1}} \qquad H_0(q) = \frac{1}{1 + a_0 q^{-1}}$$

with a_0, b_0 given real-valued coefficients, and $\{e(t)\}$ a white noise process of zero mean and variance σ_e^2.
The input signal satisfies

$$u(t) = r(t) - cy(t)$$

with $c \in \mathbb{R}$ and $\{r(t)\}$ a quasi-stationary signal that is uncorrelated with $\{e(t)\}$.
Consider the situation that we want to identify this system with the direct identification method, using measurements of $\{u(t), y(t)\}$.

- What are the minimum excitation conditions on $\{r(t)\}$ in order to be able to arrive at a consistent estimate of $(G_0(q), H_0(q))$?
- Re-design the controller in such a way that it remains to have to property of having a finite impulse response (FIR), but additionally provides an experimental situation in which the direct method can yield a consistent estimate of the system (G_0, H_0) even when $\{r(t)\} = 0$, so on the basis of excitation of the loop through $\{e(t)\}$.