Dutch Institute of Systems and Control

Course: System Identification
Spring 2018

Assignment number 1 (concerning Lectures 1, 2 and 3)

Issued: April 23, 2018 **Due: May 23, 2018**

Paul Van den Hof
Department of Electrical Engineering
Eindhoven University of Technology
email: p.m.j.vandenhof@tue.nl

Problem 1

Let $\mathcal{S}$ be a data generating system, defined by

$$G_0(z) = b_0 z^{-1} \quad \text{and} \quad H_0(z) = \frac{1}{1 + d_0 z^{-1}}$$

and let this system generate input/output data according to

$$y(t) = G_0(q)u(t) + H_0(q)e(t)$$

with $\{u(t)\}$ and $\{e(t)\}$ sequences of independent identically distributed zero mean stochastic processes (white noise) that are mutually uncorrelated, having a variance of resp. σ_u^2 and σ_e^2.

a. Consider a Box-Jenkins model set $\mathcal{M}$ containing two parameters such that $\mathcal{S} \in \mathcal{M}$. Derive an expression for the asymptotic covariance matrix of the estimated parameters, when applying a standard least squares prediction error identification criterion to data obtained from $\mathcal{S}$.

b. Derive a similar expression as under (a) for an Output Error model set that is chosen such that $G_0 \in \mathcal{G}$.

c. How do the asymptotic variances of the estimates of b_0 in both cases relate to eachother?

d. Can you improve on the smallest variance of the estimate of b_0 if d_0 is known a priori and does not need to be estimated?

Problem 2

Consider the following measurement problem:
We measure a particular physical variable θ_0 by using 5 different sensors, that each measure θ_0 under the influence of an additive noise disturbance e_i:

$$y_i = \theta_0 + e_i, \quad i = 1, \cdots 5$$

where e_i are independent zero-mean Gaussian random variables with variance σ_i^2.

a. Specify the maximum likelihood estimator for θ_0 on the basis of the five observations.

b. Calculate the variance of this estimator and compare it with the Cramér-Rao lower bound.

Problem 3

In Figure 1 a construction method is indicated for generating a sequence of basis functions $\{F_k(z)\}_{k=1,2,\cdots}$ that is complete in (spans) the space $\mathcal{H}_2$ of all stable systems. The sequence of scalar-valued functions $\{F_k(z)\}_{k=1,2,\cdots}$ is obtained by considering the transfers from u to the individual scalar components of $x_b^{(i)}(t)$, $i = 1, 2, \cdots$, and $x_b^{(i)}$ are balanced state vectors of the systems G_b as indicated in the Figure.

In this construction G_b refers to a stable all-pass system, characterized by the property that $|G_b(e^{i\omega})| = 1$ for all ω.

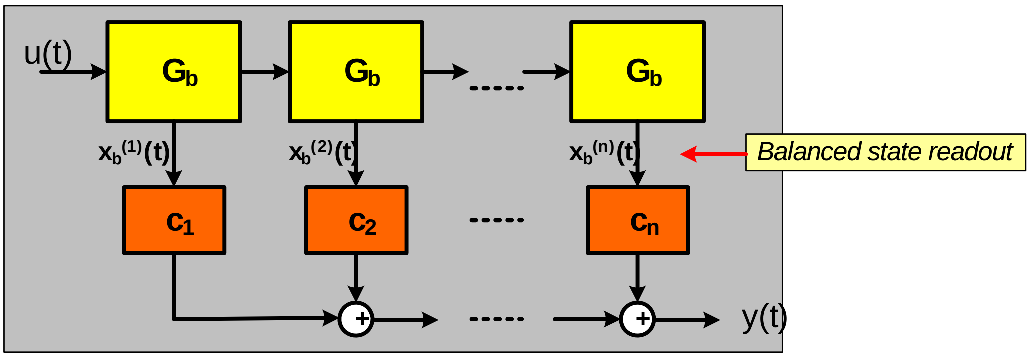

Figure 1: Construction of orthogonal basis functions

A state-space realization:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

with state (x) dimension n_x, has an observability matrix

$$\Gamma := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix}$$

and controllability matrix:

$$\Delta := [B \; AB \; A^2B \; \cdots].$$

A state-space representation is called balanced if it satisfies

$$\Gamma^T\Gamma = \Delta\Delta^T = \Sigma$$

where Σ is diagonal. If the underlying system is an all-pass system it can be shown that $\Sigma = I$ (identity matrix), while for an all-pass system a realization is balanced if and only if it has a ralization

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

that is unitary[1].

[1] A real-valued square matrix X is called unitary if it satisfies $XX^T = X^TX = I$.

- Show that the construction method for generating an infinite number of basis functions ($n \to \infty$) through input-to-state transfers as sketched in Figure 1 delivers functions that are all orthogonal in $\mathcal{H}_2$.

Problem 4

Consider the system

$$\mathcal{S}:\ y(t) = b_1 u(t-1) + w_0(t)\,,$$

where $w_0(t) = e_0(t) + k$, k being a constant and $e_0(t)$ a zero-mean white noise. Consider the model set

$$\mathcal{G} = \left\{ G(q,\theta) :\ G(q,\theta) = \beta_1 q^{-1} + \beta_2 q^{-2} \right\}$$

and assume that $u(t)$ is a zero-mean white noise input with variance σ_u^2.

a. Does $G_0(q) \in \mathcal{G}$?

b. Compute the asymptotic parameter estimates θ^*.

c. Does the fact that $E[w_0(t)] \neq 0$ affect the consistency of the identified parameters?

Problem 5

We have collected N input-output data on an unknown true system $\mathcal{S}$: $y(t) = G_0(q)u(t) + H_0(q)e(t)$ and identified the parameter vector $\hat{\theta}_N$ in the following model structure:

$$\mathcal{M} = \left\{ G(q,\theta) = \frac{bq^{-1}}{1 + fq^{-1}}\ ;\ H(q,\theta) = 1\ \mid\ \theta = \begin{pmatrix} b \\ f \end{pmatrix} \right\}$$

Using the data and $\hat{\theta}_N$, we have performed the model structure validation based on the residuals $\varepsilon(t, \hat{\theta}_N)$ and we have obtained the result presented in Figure 2, where the figure shows $\hat{R}_\varepsilon(\tau)$ (top) and $\hat{R}_{\varepsilon u}(\tau)$ (bottom) together with their 99% confidence bounds. We have also computed the variance $cov(G(e^{j\omega}, \hat{\theta}_N))$ of the identified model $G(q, \hat{\theta}_N)$ in the frequency domain using the formula of slide 2-43.

Based on the expression for $cov(G(e^{j\omega}, \hat{\theta}_N)$ we could derive a probabilistic uncertainty bound for $|G_0(e^{j\omega}) - G(e^{j\omega}, \hat{\theta}_N)|$ for each value of ω. Would this approach be valid in the given situation? Motivate your answer.

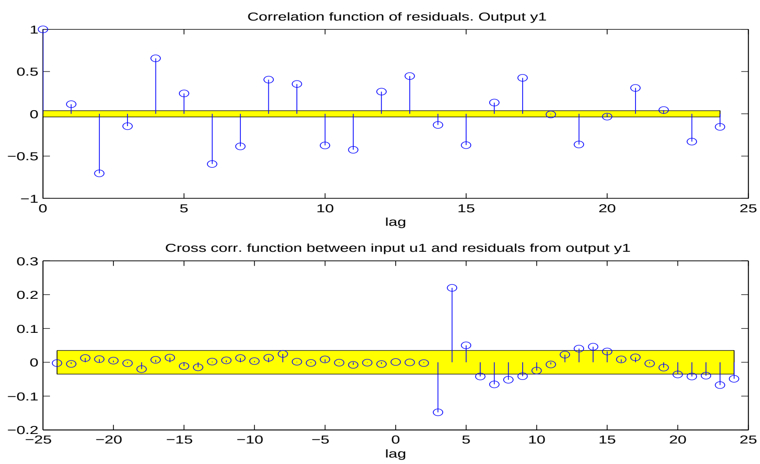

Figure 2: model structure validation