Orthonormal Basis Function Models

Beyond the LTI case

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Symposium: Four decades of data-driven modeling in systems and control – achievements and prospects

SZTAKI TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

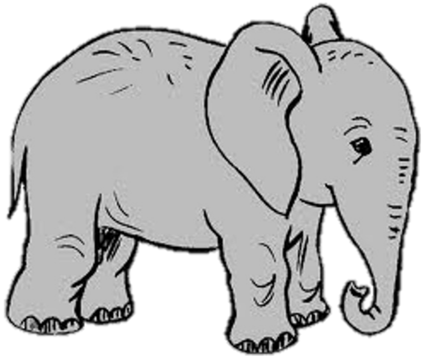
Two decades of motivation

- **Dynamic systems in engineering:**
  - Unprecedented growth of complexity
  - Increasing performance requirements
  - Nonlinear (NL), spatial, time-varying (TV) behaviors have been becoming important to address
- **Industrial practice:**
  - Linear Time-Invariant (LTI) framework
    - Powerful data-driven modelling methods
    - Systematic tools for shaping performance
    - Small operating range
  - Need for an NL/TV framework
    - Stability guarantees, but (in general) no performance shaping
    - Non-convex, cumbersome tools

The idea of surrogate linear modelling

The Engineers' Dream:

How to use "simple" linear control for NL systems with performance guarantees?



LTI Modelling

How to find the golden mean between simplicity and accuracy?

Can we embed or approximate NL/TV behaviors with linear structures?

LPV

PWA

LTV

SL

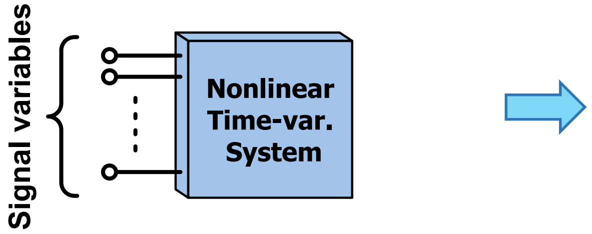
Koopman

Vast universe of Nonlinear and Time-Varying systems

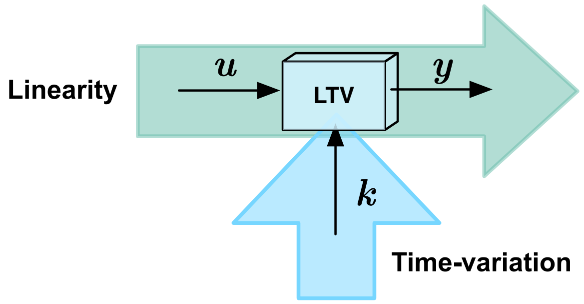
Linear time-varying systems

The Engineers' Dream:

How to use "simple" linear control for NL systems with performance guarantees?



$$x(k+1) = f(x(k), u(k))$$
$$y(k) = h(x(k), u(k))$$



$$x(k+1) = A(k)x(k) + B(k)u(k)$$
$$y(k) = C(k)x(k) + D(k)u(k)$$

Cumbersome

Surrogates are only applicable for a given trajectory/time variation

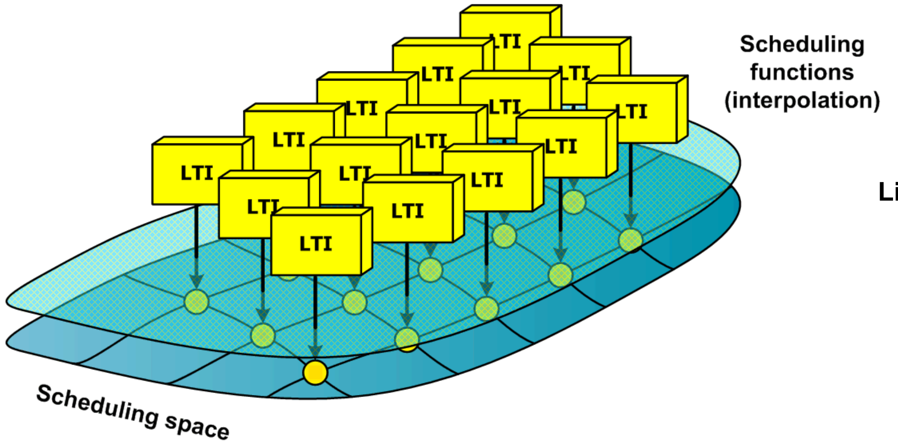
Transformation

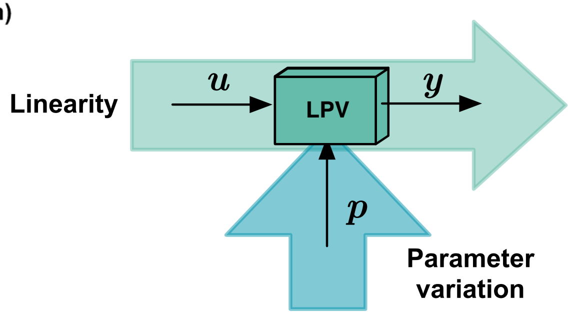
Linearization around a trajectory

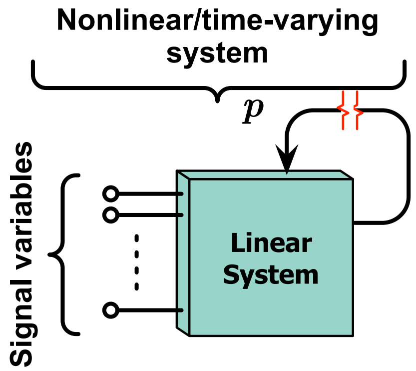
Linear parameter-varying systems

The Engineers' Dream:

How to use "simple" linear control for NL systems with performance guarantees?







Local approximation principle

$$x(k+1) = A(p(k))x(k) + B(p(k))u(k)$$
$$y(k) = C(p(k))x(k) + D(p(k))u(k)$$

Global embedding principle

Local synthesis
Gain scheduling
(blended LTI control)

Data-driven modelling?
2 decades ago:
immature approaches
(ARX, early subspace, local methods)

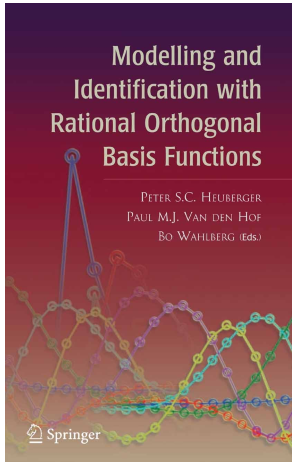
Global synthesis
with guarantees
(NL control)

Content

- Introduction
- Beyond LTI representations with OBFs
- Identification of LPV-OBF models
- A glimpse of control with OBF models
- Conclusions

The rise of the OBFs

- **OBFs-based system representations**
  - Roots: Takenaka & Malmquist 1925-1930, Wiener 1930
  - IIR-type of representations
  - Realization theory, transformation theory, etc.
  - Many important contributions
    (Heuberger, Van den Hof, Wahlberg, Bokor, Olivera e Silva, etc.)
- **OBFs based identification**
  - Linear-in-the-parameters model structure
  - FIR, ARX, OE variants
  - Efficient TD and FD methods
  - Powerful theory (consistency, approximation error and variance/uncertainty characterization)
  - Many important contributions
    (Van den Hof, Ninnes, Bokor, etc.)



Heuberger et. al.
Springer, 2005

Application example:

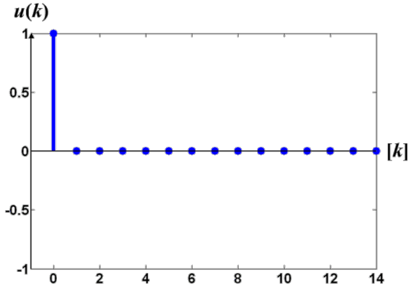
OBF identification of an FFC unit
(Donkelaar et. al. ACC 1998)

Series expansion representations

- **OBF representations (LTI-SISO case)**

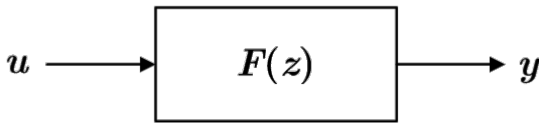
Advantages:

- Fast decay rate of the expansion
- Guaranteed stability: $\mathcal{H}_{2-}(\mathbb{E})$
- Unstable systems: co-prime form



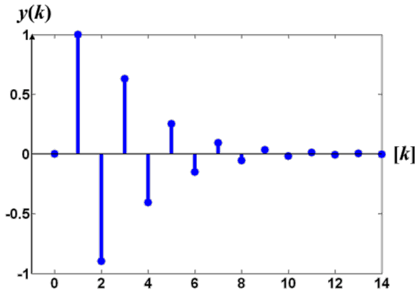
$$F(z) \approx \sum_{i=0}^{n_g} g_i z^{-i}$$

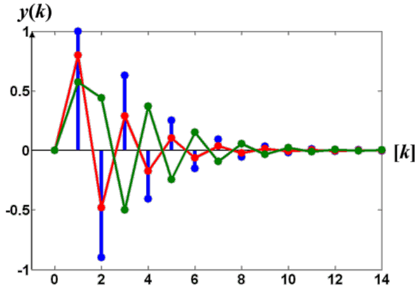
FIR



$$F(z) \approx \sum_{i=0}^{n_\phi} w_i \phi_i(z)$$

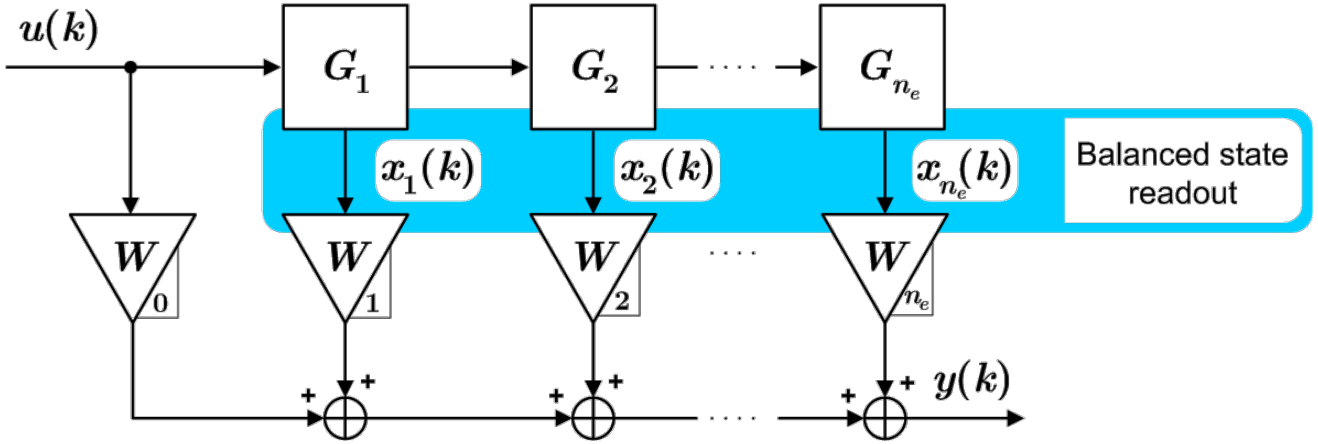
OBF





Series expansion representations

- **Generation of OBFs**
  - By a set of stable poles: $\Xi_n = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$
  - By a stable all-pass (inner) function: $G_\mathrm{b}(z) = \prod_{i=1}^{n} \frac{1 - z\lambda_i^*}{z - \lambda_i}$

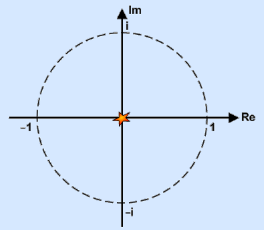


$$F(z) \approx \sum_{j=0}^{\infty} \sum_{i=1}^{n} w_{i+jn} \phi_i(z) G_\mathrm{b}^j(z)$$

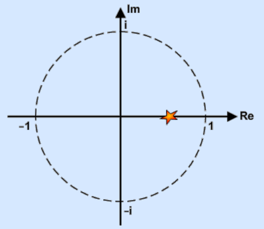
Question:
Limited to the LTI case?

Choice of poles determines the approximation error

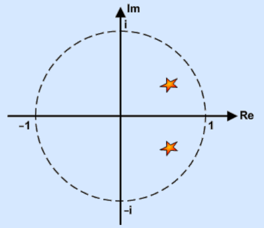
Type of basis functions



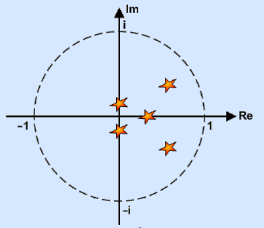
Pulse Basis
$\Xi_n = \{0\}$



Laguerre Basis
$\Xi_n = \{\lambda\} \in (-1, 1)$



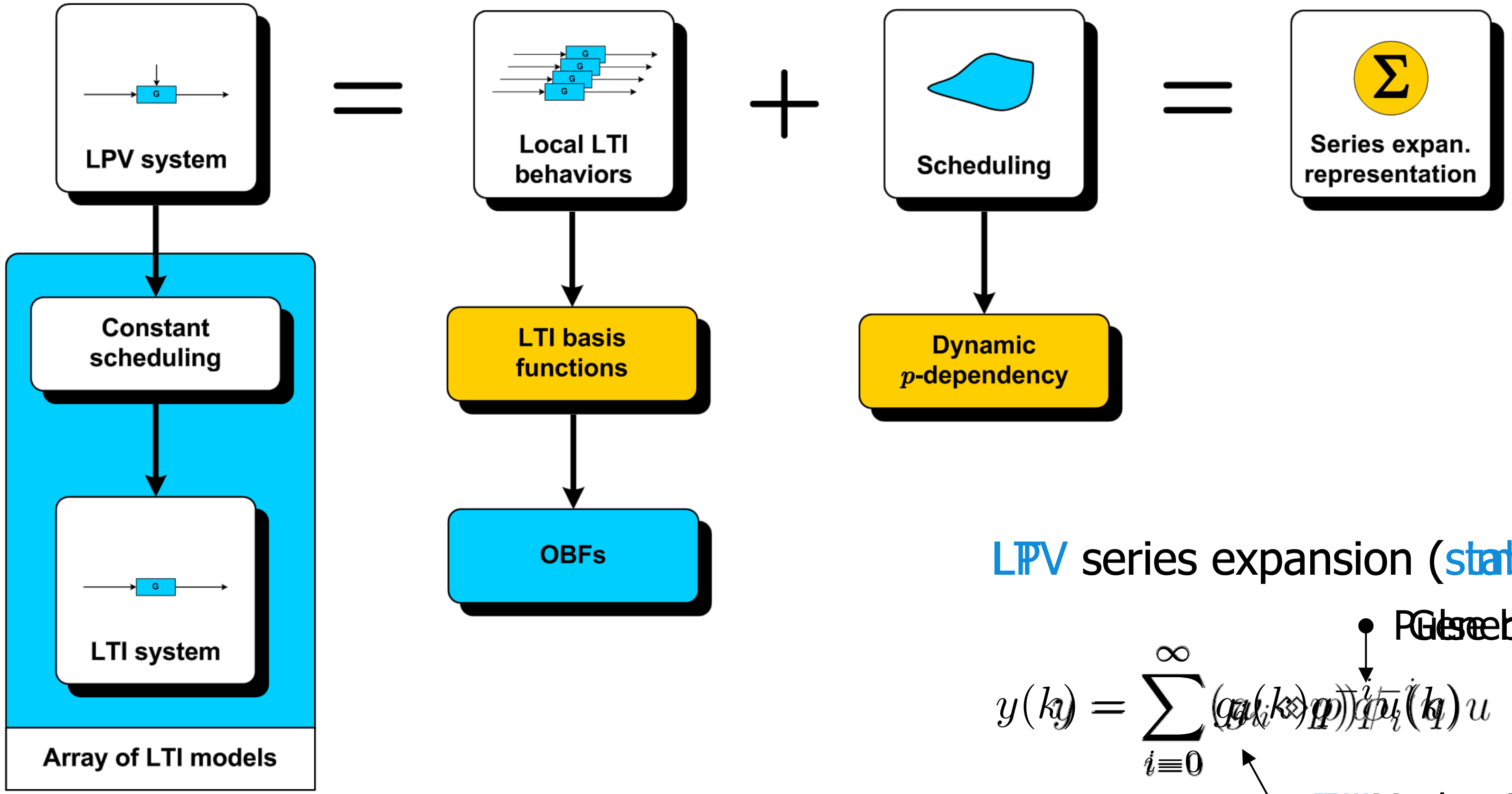
Kautz Basis
$\Xi_n = \{\lambda, \lambda^*\} \subset \mathbb{D}$



Hambo
$\Xi_n = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$

Series expansion representations

- **OBF representations (LPV-SISO case)**



LPV series expansion (simplification) [overlaid: LTI series expansion (stable)]

- Pulse basis / General basis

$$y(k)=\sum_{i=0}^{\infty}(g_i \diamond p)(k)\,\phi_i(q)\,u(k)$$

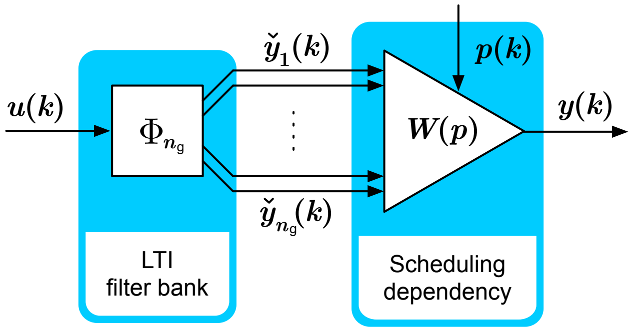
- Markov param. / Expansion coef.

Tóth et. al., CDC, 2009

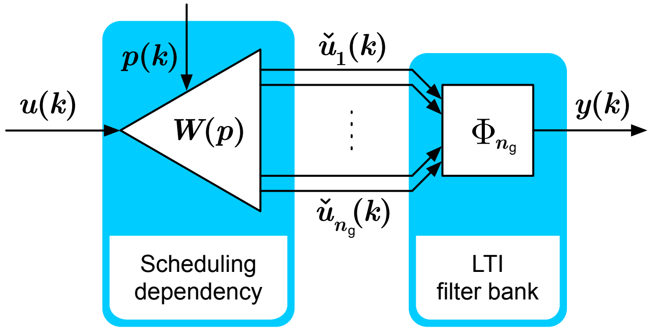
Tóth, Springer, 2010

Series expansion representations

- **OBF representations (LPV-SISO case)**
  - Resulting representation forms



Wiener LPV-OBF form



Hammerstein LPV-OBF form

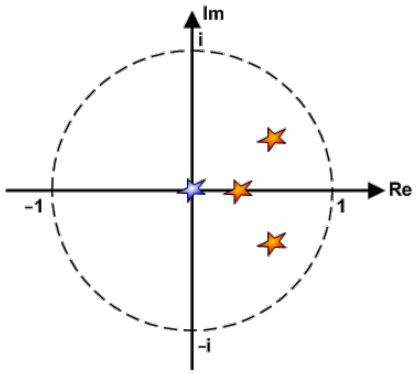
Question:
Which basis functions to take?

Basis selection for OBF models

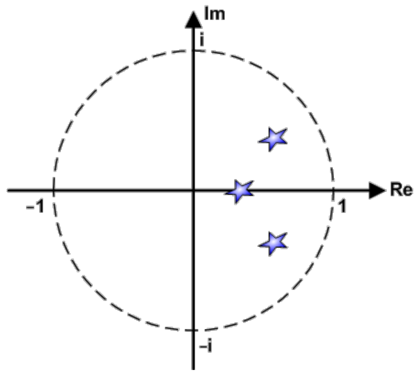
- **Selection of basis** (LTI case)

Approaches:

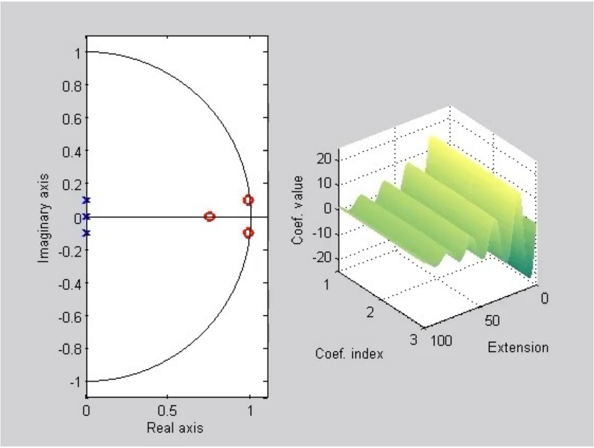
- Re-identification (Heuberger et. al. 1993)
- Iterative selective (Bodin, et. al. 1997)
- Kolmogrov n-width (Oliveira e Silva, 1996)



$$F(z) \approx \sum_{i=1}^{n_{\mathrm{g}}} g_i z^{-i}$$



$$F(z) = \sum_{i=1}^{3} w_i \phi_i(z)$$

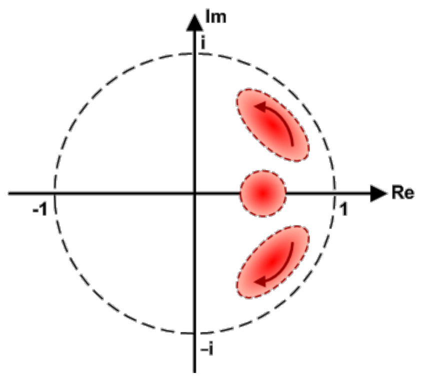


$$F(z) = \sum_{j=0}^{\infty} \sum_{i=1}^{3} w_{i+3j} \phi_i(z) G_{\mathrm{b}}^{j}(z)$$

Choice of the basis should be characterized in terms of a distance between basis poles and observed LTI system poles

Basis selection for OBF models

- **Selection of basis** (LPV case)



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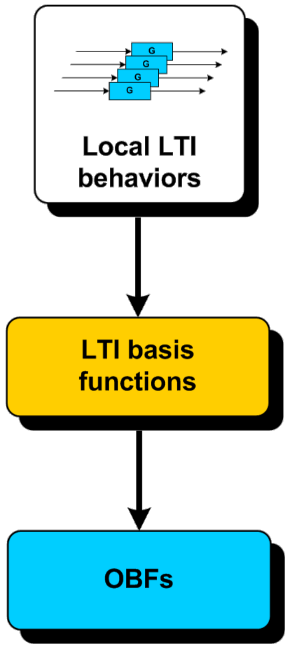


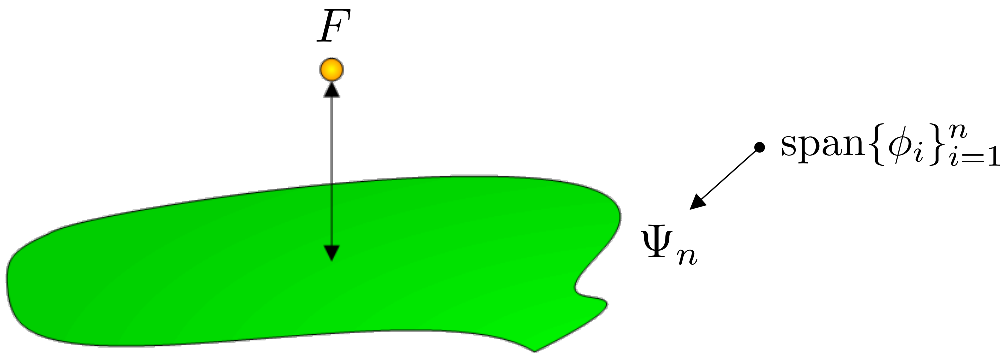
Question?
How to generalize to the case of pole regions?

Basis selection for OBF models

- **The Kolmogorov n-width theory**

  - The distance

Candidate system

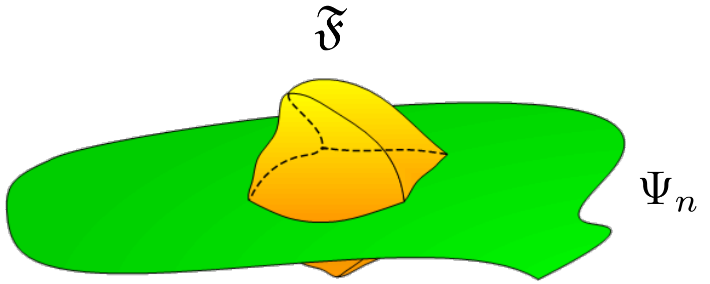


n-dim subspace in $\mathcal{H}_{2-}$

$$d_{\mathcal{H}_2}(F, \Psi_n) = \inf_{\hat{F} \in \Psi_n} \|F - \hat{F}\|_{\mathcal{H}_2}$$

  - The optimal subspace

Candidate system set



n-dim optimal subspace

$$\inf_{\Psi_n} \sup_{F \in \mathfrak{F}} d_{\mathcal{H}_2}(F, \Psi_n)$$

Basis selection for OBF models

- **The Kolmogorov n-width theory**

  Result (Oliveira e Silva):

  - Let $G_\mathrm{b}(z)$ be an inner function
  - Let $\mathfrak{F}$ have all its poles in the region:

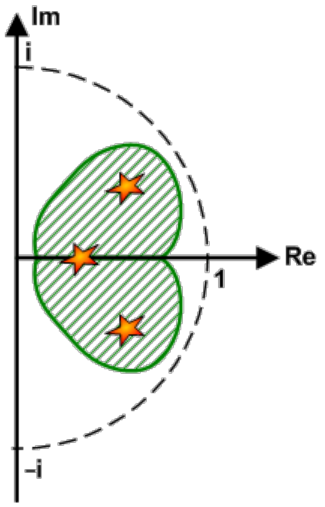
$$\{z \in \mathbb{D} \mid |G_\mathrm{b}(z^{-1})| > \rho\}$$

  - The OBFs, generated by $G_\mathrm{b}(z)$ are optimal in the n-width sense

$$F(z) = \sum_{j=0}^{\infty} \sum_{i=1}^{n} w_{i+jn} \phi_i(z) G_\mathrm{b}^j(z)$$

$$|w_{i+jn}| \leq c\rho^j$$

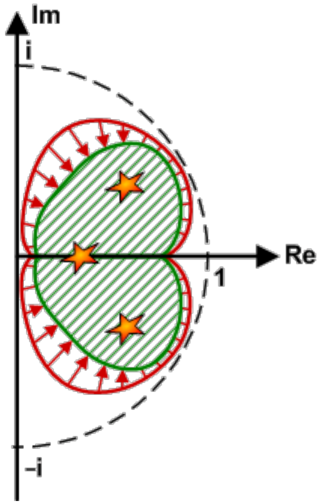
decay rate (minimal)



Basis selection for OBF models

- **The inverse Kolmogorov n-width theory**
  - Given a region of pole locations: Ω
  - Try to approximate it as

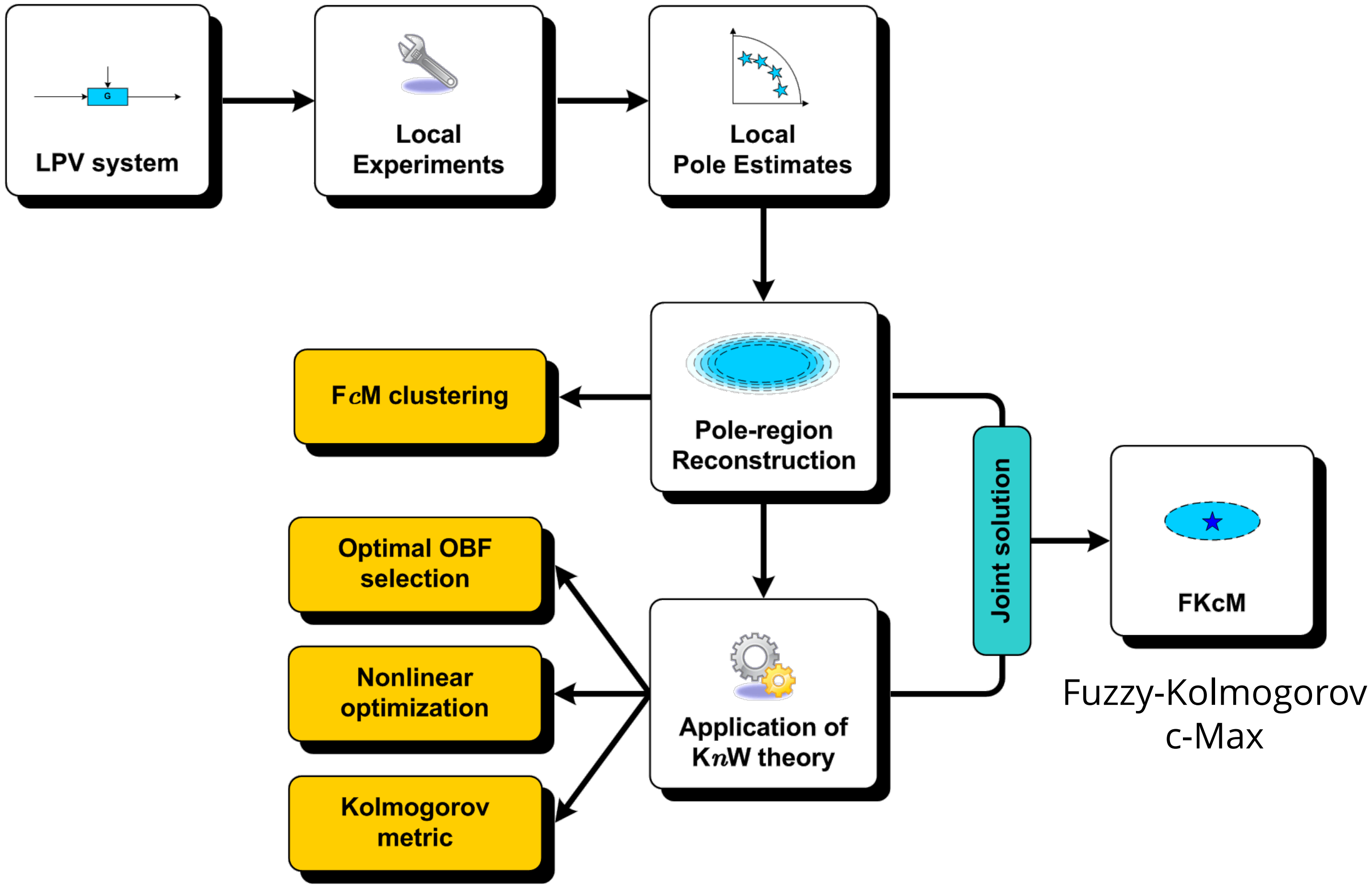
$$\Omega \approx \Omega(\Xi_n, \rho) = \left\{ z \in \mathbb{D} \mid |G_\mathrm{b}(z^{-1})| \leq \rho \right\}$$



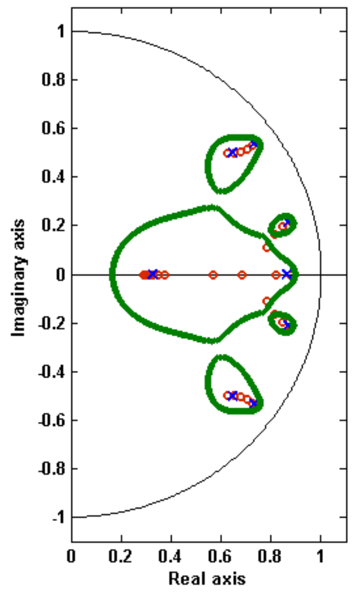
  - The n optimal OBF poles obtained through (Kolmogorov measure minimization):

$$\min_{\Xi_n \subset \mathbb{D}} \rho = \min_{\Xi_n \subset \mathbb{D}} \max_{z \in \Omega} |G_\mathrm{b}(z^{-1})| = \min_{\Xi_n \subset \mathbb{D}} \max_{z \in \Omega} \left| \prod_{k=1}^{n} \frac{z - \xi_k}{1 - z\xi_k^*} \right|$$

Basis selection for OBF models



Fuzzy-Kolmogorov c-Max



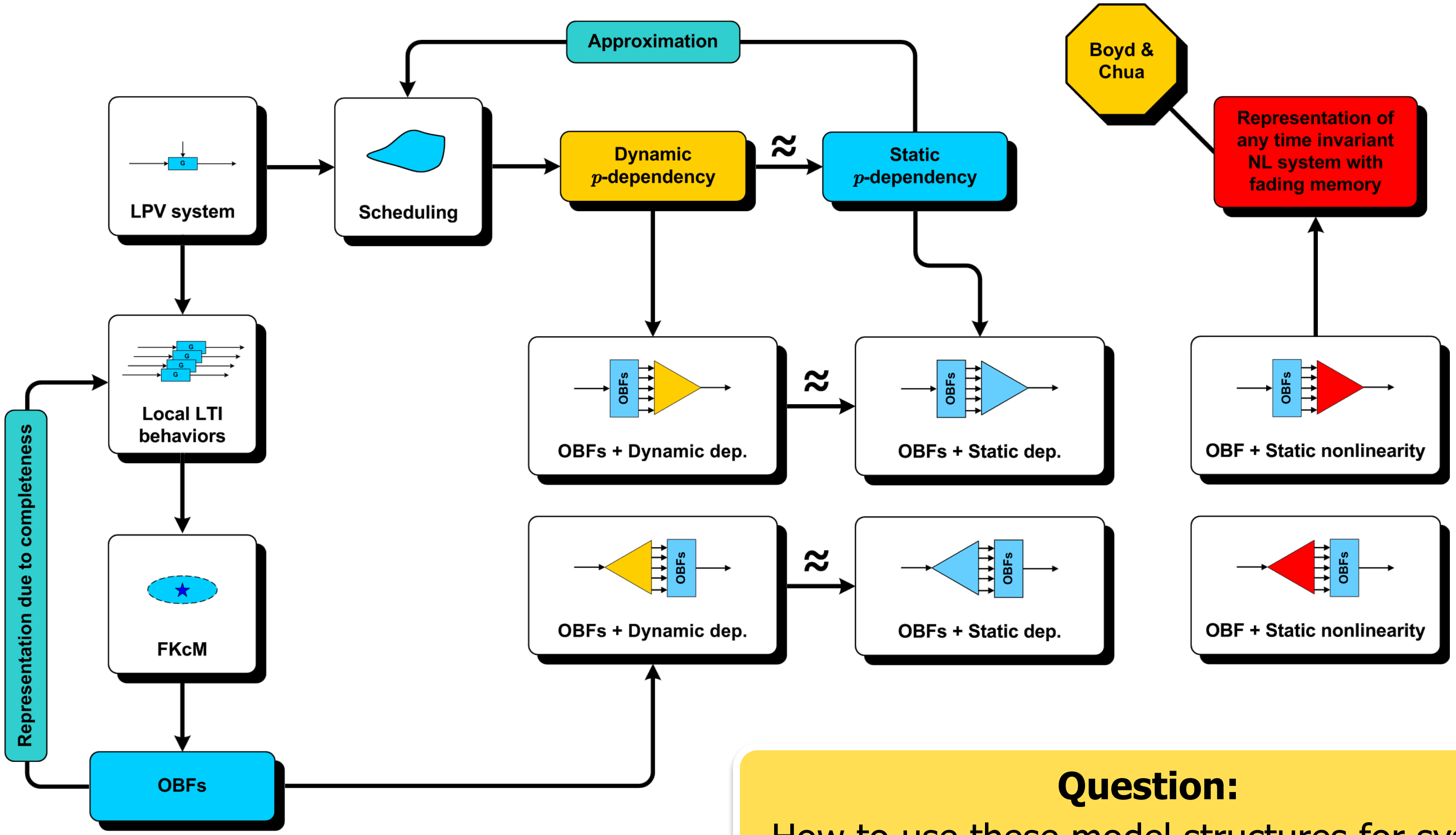
Tóth et. al., Automatica, 2009

Advanced methods:

- Min-max approach
- Randomized opt.
- SQP approaches
- Generalization for pole uncertainty

(Bachnas, PhD thesis, 2023)

Overview



Question:
How to use these model structures for sysID?

Content

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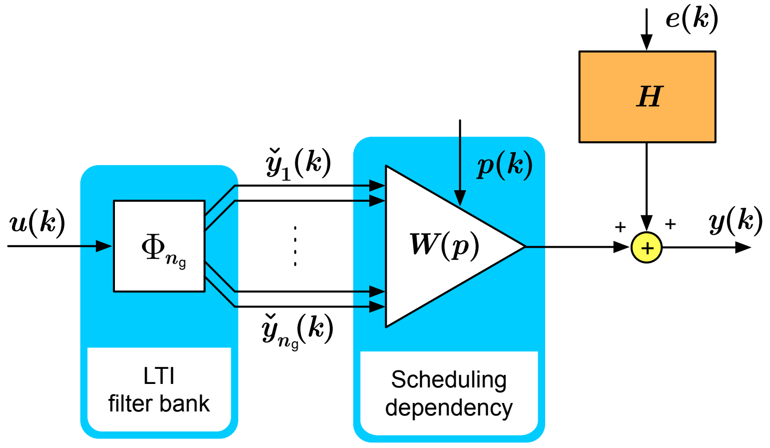
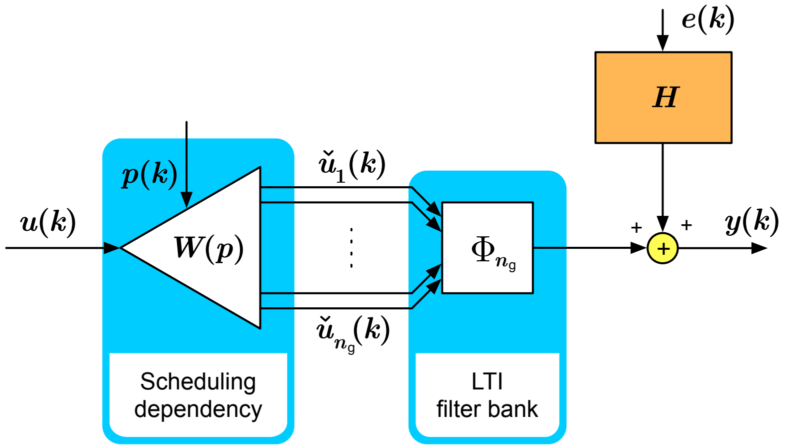
- Identification of LPV-OBF models
- A glimpse of control with OBF models
- Conclusions

Identification with LPV OBF models



Wiener LPV-OBF form

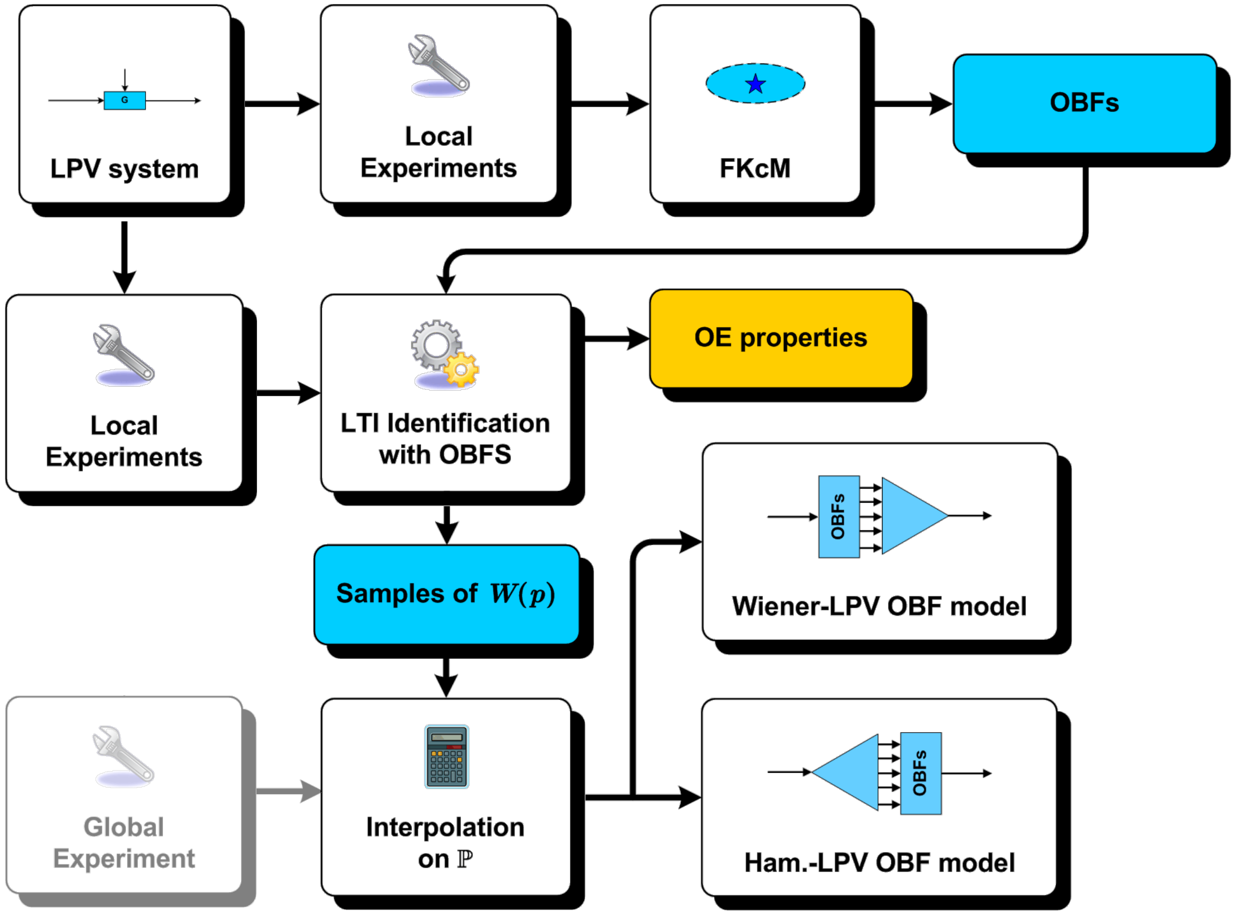


Hammerstein LPV-OBF form

- **Model structure properties**
  - FIR-like structure (linear regression)
  - Consistency on the truncated expansion
  - OE and BJ noise handling
  - No bias for noise uncorrelated to the input (OE structure)
  - This model class can deal with locally changing McMillian degree
  - No undermodeling asymptotically (completeness)

The LPV OBF approach

- **The local approach**



- Coefficient
- Behavioral $\mathcal{H}_\infty/\mathcal{H}_2$

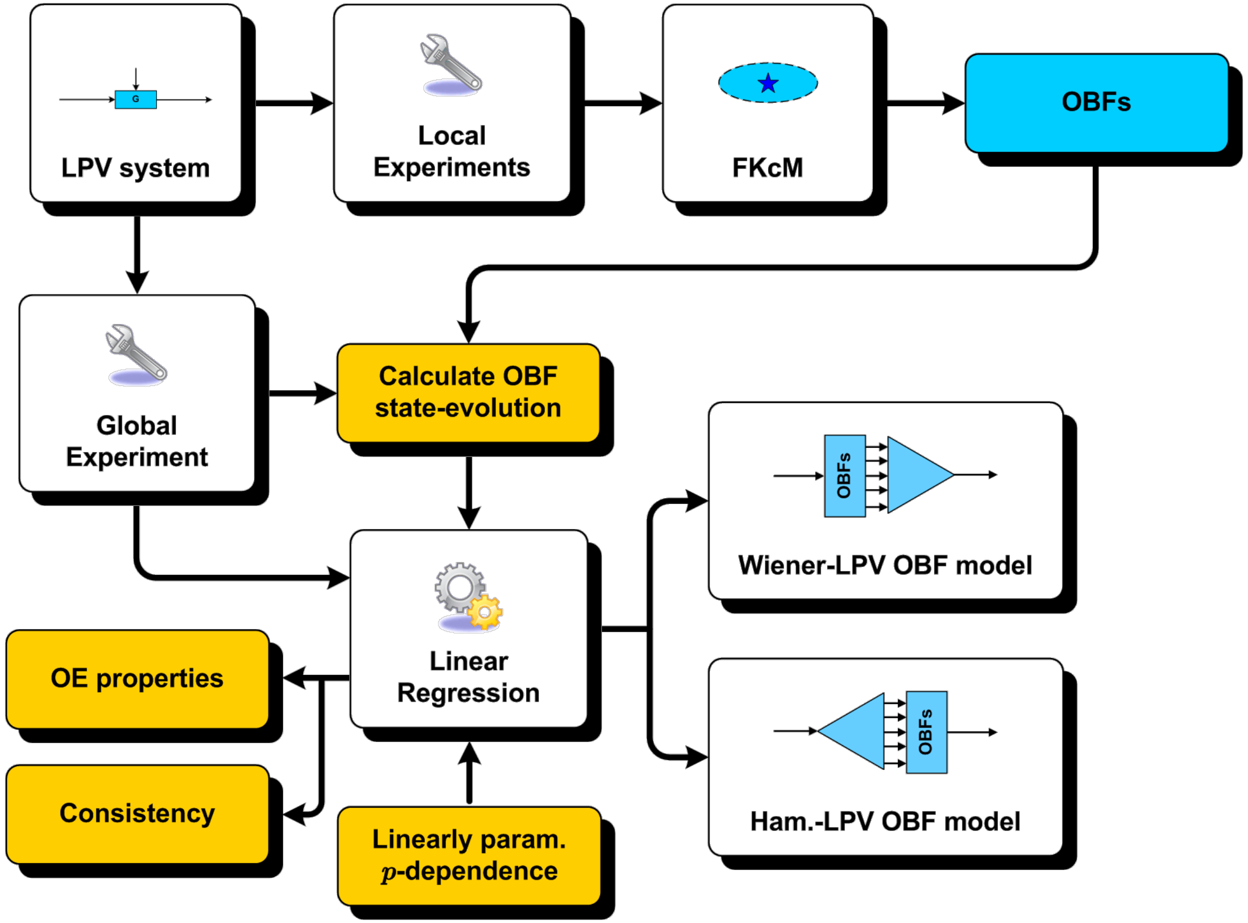
Tóth et. al., CDC, 2007

Tóth, Springer, 2010

Bachnas, JPC, 2014

Identification with LPV OBF models

- **The global approach**



Extensions:

- Oder selection (Lasso)
- Kernelized form (RKHS)
- Bayesian form (GP)

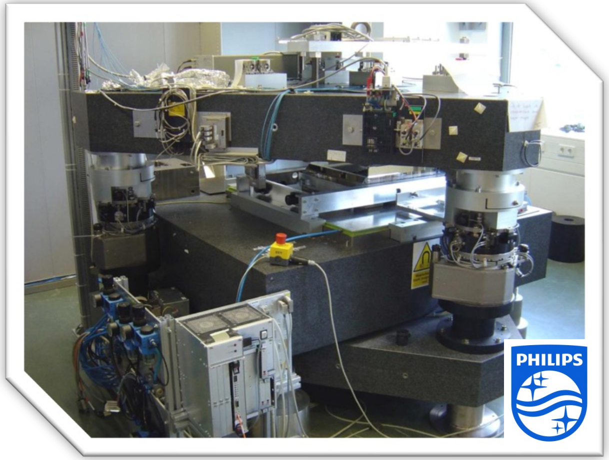
(Darwish, PhD thesis, 2017)

Tóth et. al., Automatica, 2009

Tóth, Springer, 2010

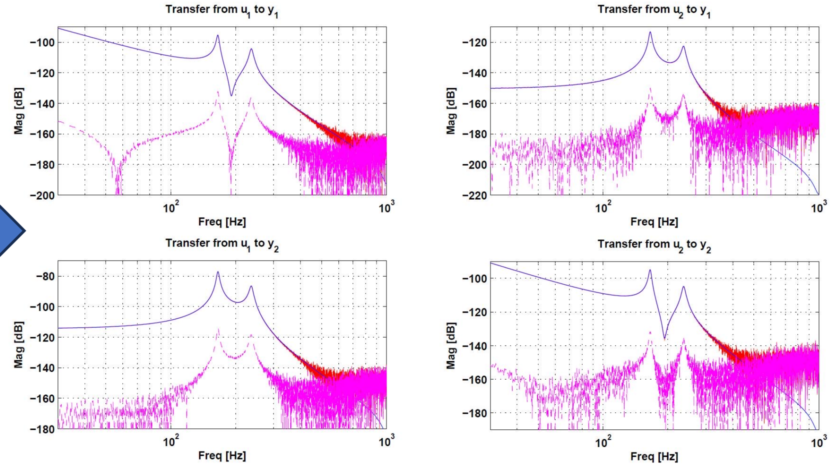
The LPV OBF approach

- **Identification of the PAS5500/300D wafer stage**

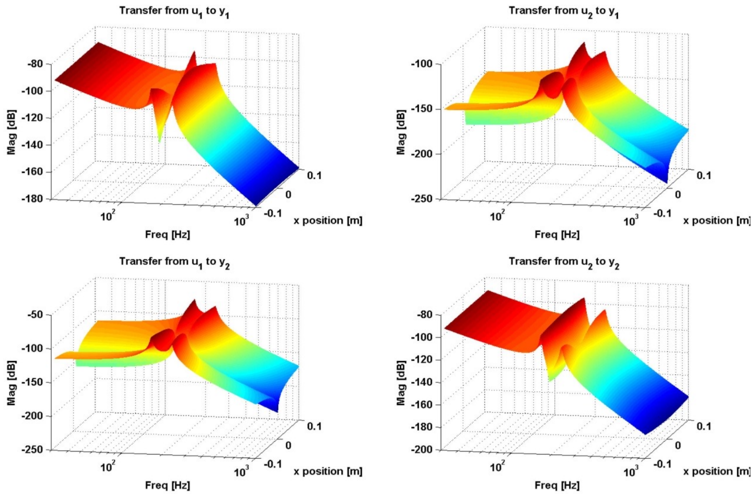


Data accusation:

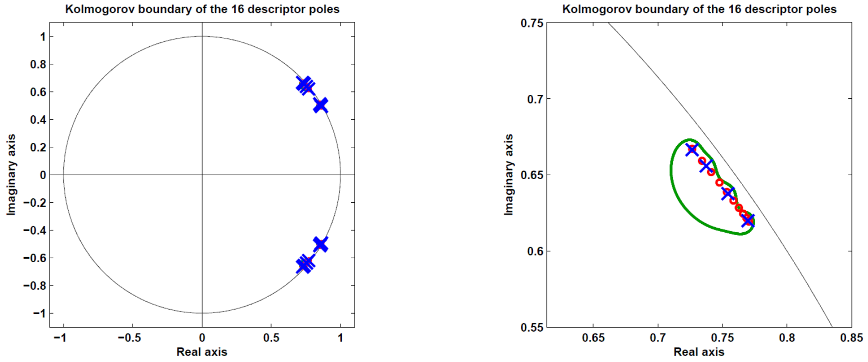
- 21 local (x,y) positions
- 25 periods of multisines
- 16K excited frequencies (Tóth et. al., ACC, 2011)



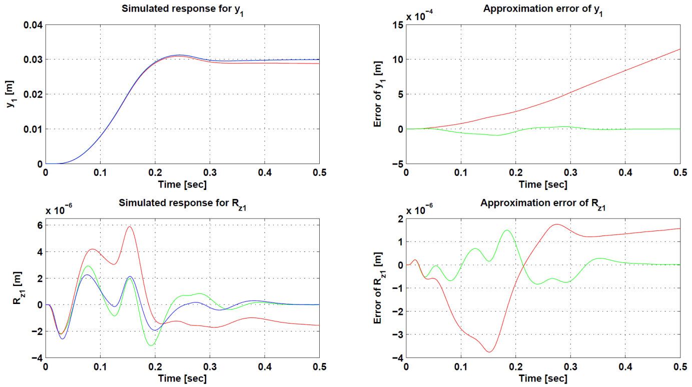
Local FRF estimation



Local OBF approach with polynomial interpolation



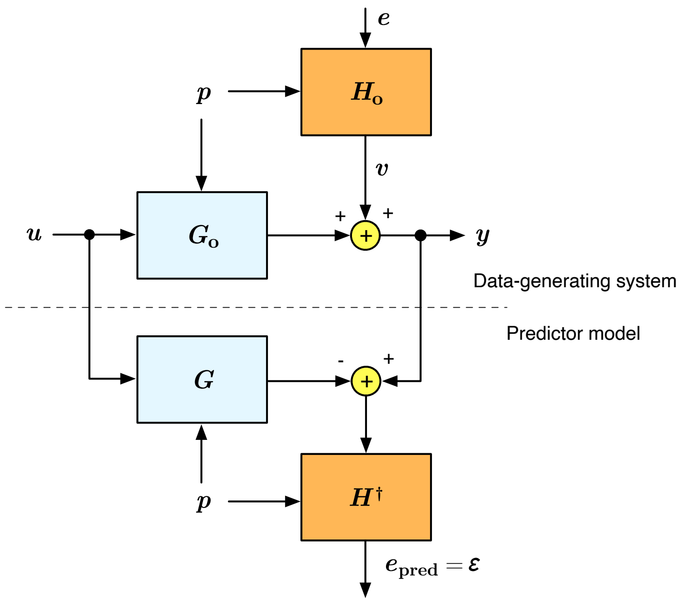
Optimization of 16 OBF poles

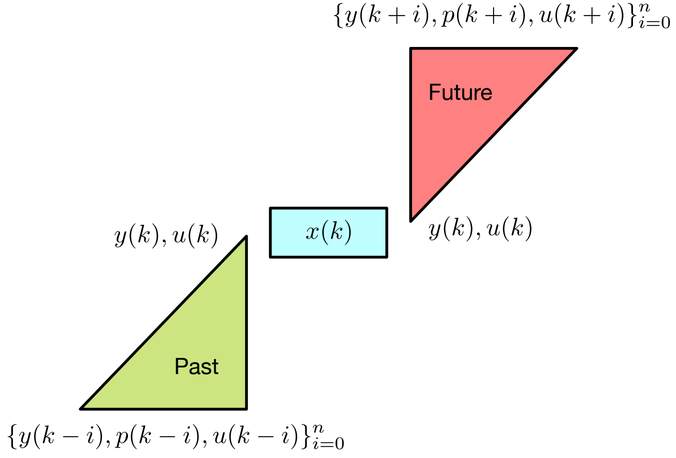


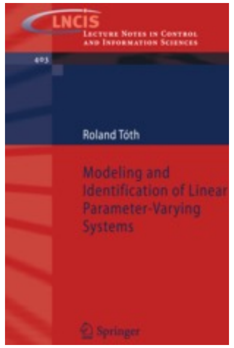
Open and closed-loop model response w.r.t. measured response

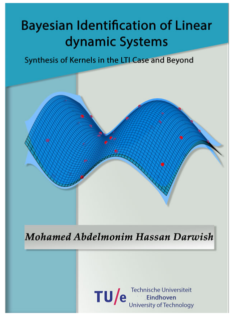
Progress of LPV identification

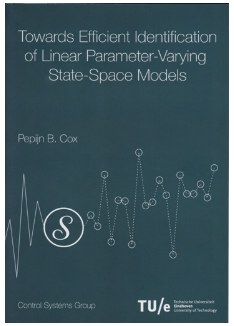
- Wide-range of methods
  - Unified Prediction Error Minimization framework
    - ARX, ARMAX, OE, BJ, SS model structures
    - Maximum Likelihood estimators and statistical efficiency
    - Instrumental variable methods
    - Automatic model structure discovery (LASSO, NNG, Compressive sensing)
  - Unified Subspace Identification framework
    - Hankel-based realization theory
    - Stochastic realization via CCA
    - SS-ARX, PBSID, N4SID, etc.
  - ML-based approaches
    - Kernelized OBF methods (Bayesian OBF, IRR)
    - RKHS-based estimators (LS-SVM, IV-SVM, GP)
    - DNN estimators (SUBNET)











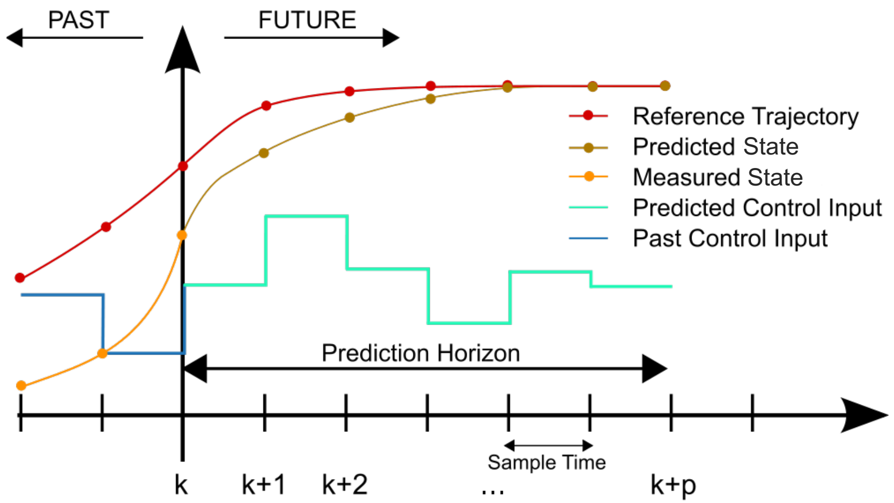
LPV CORE — Toolbox for efficient control and identification based on LPV models

Drebble erc

Content

OBF models in control

- **Model predictive control (LTV)**
  - Exploiting OBF models as predictors (Patwardhan, 2005)
  - Linear-in-the parameter property
  - LTV / adaptive formulations
    (Bachnas et. al., 2015, Ettefagh et. al., 2018)



$$\min_{u_0^{N-1}} \sum_{i=0}^{N-1} \ell_{i|k}(x_{i|k}, \Delta u_{i|k}) + V_\mathrm{f}(x_{N|k})$$

OBF model

$$\begin{pmatrix} x_{1|k} \\ x_{2|k} \\ \vdots \\ x_{N|k} \end{pmatrix} = \begin{pmatrix} A \\ A^2 \\ \vdots \\ A^N \end{pmatrix} x(k) + \begin{pmatrix} B & 0 & \cdots & 0 \\ BA & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ BA^{N-1} & BA^{N-2} & \cdots & B \end{pmatrix} \begin{pmatrix} u_{0|k} \\ u_{1|k} \\ \vdots \\ u_{N-1|k} \end{pmatrix}$$

$$\Phi_{n_\mathrm{g}} = \left[\begin{array}{c|c} A & B \\ \hline I & 0 \end{array}\right]$$

Recursively adapted

$$y_{i|k} = \theta_k^\top x_{i|k}$$

Constraints

$$u_\mathrm{min} \leq u_{i|k} \leq u_\mathrm{max} \qquad x_{N|k} \in \mathcal{X}_\mathrm{f}$$
$$y_\mathrm{min} \leq y_{i|k} \leq y_\mathrm{max}$$

Cost

$$\ell_{i|k}(x, \Delta u)) = (x - x_{\mathrm{ref}|k})^\top \theta_k^\top Q_k \theta_k (x - x_{\mathrm{ref}|k}) + \Delta u^\top R \Delta u$$
$$\Delta u_{i|k} = u_{i|k} - u_{i-1|k}$$

Properties:
- Stability guarantees
- Fast adaptation

OBF models in control

- **Model predictive control (LPV)**
  - Exploiting OBF models as predictors
  - LPV formulation
    (Bachnas et. al., 2015, Bachnas, 2023)

Main change:

Known LPV-OBF model

$$y_{i|k} = C(p_{i|k})x_{i|k}$$

Properties:

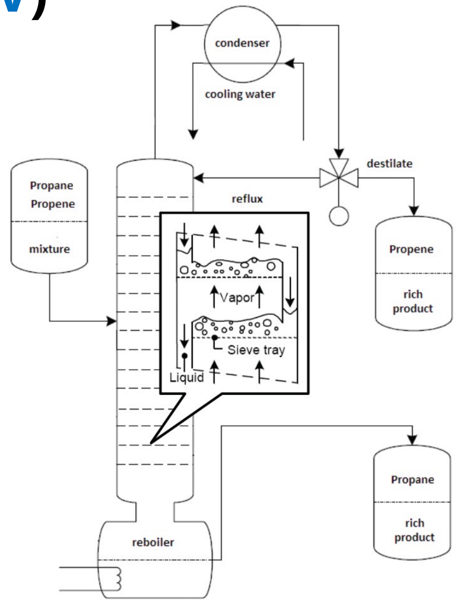
- Stability guarantees
- Scheduling trajectory: Iterated or predicted

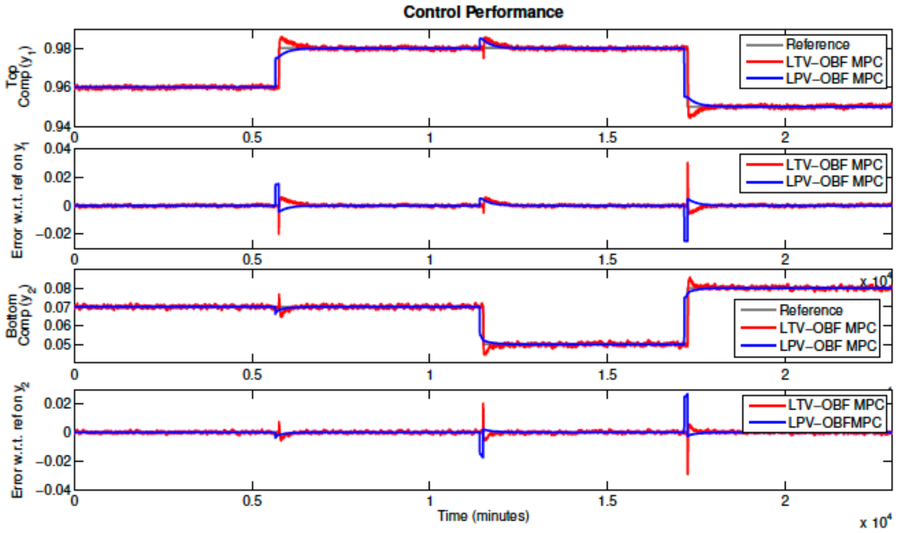
- **Data-driven control (LPV)**

High-purity distillation column

IPCOS

eurostars™





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Properties:

- Stability guarantees
- Scheduling trajectory: Iterated or predicted

- **Data-driven control (LPV)**
  - Direct local-type of frequency-domain controller synthesis
  - Co-prime OBF parameterization of LPV / robust controllers
    (Karimi et. al. 2017, Blomers et. al., 2022, Blomers, 2023)

- **More …**

Content

- Introduction
- Beyond LTI representations with OBFs
- Identification of LPV-OBF models
- A glimpse of control with OBF models
- Conclusions

Conclusions

- **Surrogate modelling of NL/TV systems**
  - OBF models provide a flexible series expansion representation (LPV and LTV forms)
  - Many attractive representation and parameterization properties
  - Main problem of pole selection has been solved
- **Identification with surrogate OBF models**
  - Simple but powerful TD and FD methods with consistency guarantees
  - Local and global LPV approaches
  - Series-expansion representations were crucial for the development of PEM
- **Control with surrogate OBF models**
  - Exploiting simplicity of OBF predictive models for adaption and parameter-variation
  - Efficiency of parameterizing controllers in a co-prime form

Two decades of history

2004 Nassau
(my first CDC)

2024 Budapest

Thanks for the 20 years of collaboration,
guidance and friendship!