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Non-Euclidean geometry in systems theory and identification

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HUN-REN Institute for Computer Science and Control (SZTAKI)

Symposium: Four decades of data-driven modeling in systems and control – achievements and prospects

AUTONOMOUS SYSTEMS National Laboratory

HUN-REN Hungarian Research Network

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Introduction

BOLYAI AND THE NON-EUCLIDEAN GEOMETRY

V. Postulate: "For any line and an external point in the plane they define, exactly one parallel line can be drawn."

It reveals the sophisticated thinking of greek mathematicians that they felt this postulate as different, more complicated, from the others. During the 17-18 centuries several people attempted to prove the fifth postulate. Only around 1830 did it become clear from the work of Bolyai and Lobachevsky, that this axiom cannot be derived from the others, it is independent of them.

The essence of the revolutionary innovation: assuming that Euclidean geometry is correct, it is possible to define a non-contradictory geometric theory in which the fifth postulate does not hold. Around 1850 radical changes took place in the philosophy of mathematics: they realized that mathematics can describe multiple realities.

Non-contradiction of the new geometry was proved by giving a model, i.e., a set of mathematical objects, with their relationship to each other, in such a way, that the axioms of hyperbolic geometry, are fulfilled.

GEOMETRY: A GLOBAL APPROACH

KLEIN VIEW: (MOVEMENTS)GROUPS + INVARIANTS (PROPERTIES) = GEOMETRY

To visualize hyperbolic geometry, we have to resort to a model. Beltrami was the first to provide a model for hyperbolic geometry (the so-called pseudosphere). Poincaré's models are even simpler.
In the Poincaré model the hyperbolic plane is the unit disk, and points are Euclidean points. Lines are portions of circles intersecting the disk and meeting the boundary at right angles. The angles for the model are the same as Euclidean angles. A model with the property that angles are faithfully represented is called a conformal model. A hyperbolic circle is drawn as a Euclidean circle, but its center becomes lopsided toward the outer edge of the unit disk.

Erlanger Program: geometry should be defined as the study of transforms (symmetries) and of the objects that transforms leave unchanged, or invariant.

1. The set of symmetries of an object has a very nice algebraic structure: they form a group.
2. Klein's approach allows us to relate different (models of the) geometries.

THE HILBERT AXIOMS OF EUCLIDEAN GEOMETRY

Basic Concepts $\Rightarrow$ points, straight lines, planes

Basic relations $\Rightarrow$
betweenness, containment, congruence

Basic groups of axioms $\Rightarrow$

- Incidence axioms
- Betweenness axioms
- Parallelism axiom
- Congruence axioms
- Continuity axioms

HYPERBOLIC GEOMETRY: THE POINCARÉ DISC MODEL

One model of the hyperbolic geometry is the

Poincaré disc model,

named after the French mathematician Henry Poincaré (1854–1912) as proposed by Eugenio Beltrami.

Disc: an open unit disc of the complex plane

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

a hyperbolic plane

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

$\mathbb{D}$ boundary, "torus"

Points: elements of $\mathbb{D}$ — complex numbers of $\mathbb{D}$

Lines: Circles in $\mathbb{D}$ intersecting $\mathbb{T}$ perpendicularly.

HYPERBOLIC GEOMETRY: THE POINCARÉ DISC MODEL

Analytic discussion: the Blaschke function

$$B_{\mathbf{a}}(z) := \varepsilon \frac{z-a}{1-\overline{a}z} \quad (\mathbf{a} = (a, \varepsilon) \in \mathbb{B},\ \mathbb{B} = \mathbb{D} \times \mathbb{T})$$

Properties:

- $B_{\mathbf{a}} : \mathbb{D} \to \mathbb{D},\ \ \mathbb{T} \to \mathbb{T}$ is a bijection.
- $B_{\mathbf{a}}$ $(\mathbf{a} \in \mathbb{B})$ forms a group with respect to the function composition.
- Lines: the image of $(-1, 1)$ generated by $B_{\mathbf{a}}$.

HYPERBOLIC GEOMETRY: THE POINCARÉ DISC MODEL

Illustration:

$\forall w_1, w_2 \in \mathbb{D}\ \exists\, \mathbf{a} \in \mathbb{B}$,
such that

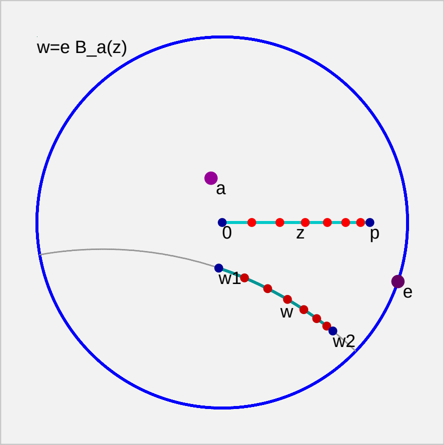
$$B_{\mathbf{a}}(0) = w_1,$$

and

$$B_{\mathbf{a}}(p) = w_2,$$

where

$$p = |B_{(w_1,1)}(w_2)|.$$



$\overline{w_1 w_2}$ hyperbolic section: the image of the interval $[0, p]$ generated by $B_{\mathbf{a}}$

HYPERBOLIC GEOMETRY: THE POINCARÉ DISC MODEL

Congruence: using the definition of hyperbolic lines, the ordering on the hyperbolic plane is equivalent to the ordering on the Euclidean line. The **congruence axioms** are satisfied.

The distance between two points: pseudo-hyperbolic metrics

$$\rho_0(w_1, w_2) = \frac{|w_1 - w_2|}{|1 - \overline{w_1} w_2|} = |B_{(w_1,1)}(w_2)|$$

hyperbolic metrics

$$\rho_0(w_1, w_2) = \text{ath}(\rho_0(w_1, w_2))$$

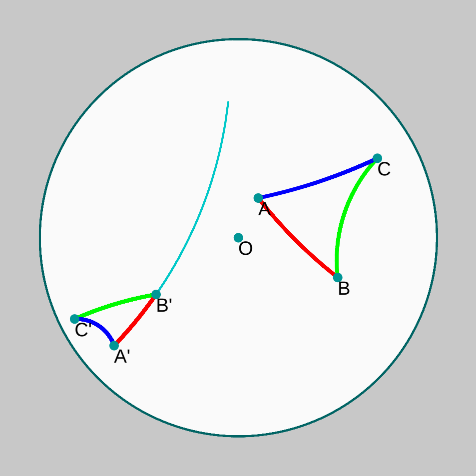
$B_{\mathbf{a}}$ is isometry with respect to the hyperbolic metrics:

$$\rho_0(B_{\mathbf{a}}(w_1), B_{\mathbf{a}}(w_2)) = \rho_0(w_1, w_2)$$

Interpretation of angles: Euclidean angles of the tangent lines.

HYPERBOLIC GEOMETRY: THE POINCARÉ DISC MODEL

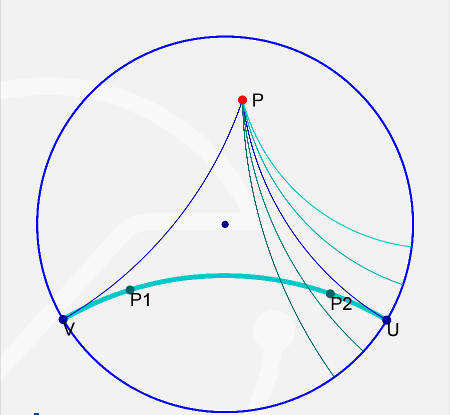
$B_{\mathbf{a}}$ ($\mathbf{a} \in \mathbb{B}$) can be identified as the group of congruency transforms of the hyperbolic plane.



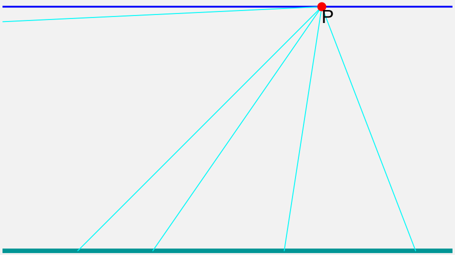
HYPERBOLIC GEOMETRY: THE POINCARÉ DISC MODEL

Parallelism axioms: are not satisfied. It is possible to draw infinitely many hyperbolic lines through the point P not on the line $\overline{P_1 P_2}$ such that none of them intersects $\overline{P_1 P_2}$.

hyperbolic



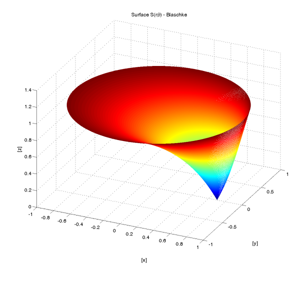
Euclidean

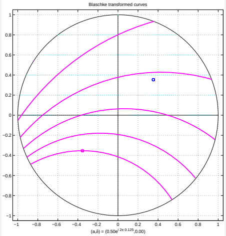


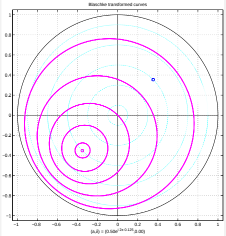
THE BLASCHKE GROUP

THE BLASCHKE FUNCTION

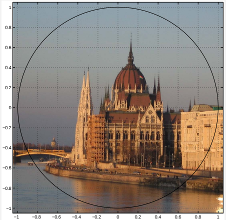
The Blaschke function represents a **hyperbolic map** in the unit disc.

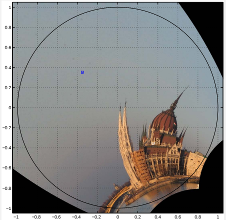






$$B_a(z) := e^{j\delta}\,\frac{z-a}{1-\overline{a}z}$$





- δ – realizes rotation
- a – determines the zero of the function, affects both shape and rotation

$$|B_a(e^{jt})| = 1$$

its absolute value is equal to 1 on the unit circle

THE BLASCHKE GROUP

$$B_{\mathfrak{b}}(z) := \epsilon \frac{z-b}{1-\overline{b}z}$$

$$z \in \mathbb{C}, \mathfrak{b} = (b, \epsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}$$
$$\mathbb{D} := \{z \in \mathbb{C} : |z| > 1\}$$
$$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$$

The Blaschke function forms a **group** with respect to **function composition**:

$$(B_{\mathfrak{b}_1} \circ B_{\mathfrak{b}_2})(z) := B_{\mathfrak{b}_1}(B_{\mathfrak{b}_2}(z)).$$

The collection of parameters $\mathfrak{b} = (b, \epsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}$ forms also a group – $(\mathbb{B}, \circ)$ – that is isomorphic with the group $((B_{\mathfrak{b}}, \mathfrak{b} \in \mathbb{B}), \circ)$, i.e.

$$B_{\mathfrak{b}_1} \circ B_{\mathfrak{b}_2} = B_{\mathfrak{b}_1 \circ \mathfrak{b}_2}$$

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THE BLASCHKE GROUP

With the notations

$$\mathfrak{b}_j := (b_j, \epsilon_j), j \in \{1, 2\} \quad \mathfrak{b} := (b, \epsilon) \quad \mathfrak{b} := \mathfrak{b}_1 \circ \mathfrak{b}_2$$

the group operation can be expressed as

$$b = \frac{b_1 \overline{\epsilon}_2 + b_2}{1 + b_1 \overline{b}_2 \overline{\epsilon}_2} = B_{(-b_2, \overline{\epsilon}_2)}(b_1 \overline{\epsilon}_2)$$

$$\epsilon = \epsilon_1 \frac{\epsilon_2 + b_1 \overline{b}_2}{1 + \epsilon_2 \overline{b}_1 b_2} = B_{(-b_1 \overline{b}_2, \epsilon_1)}(\epsilon_2)$$

The neutral and inverse element of the group $(\mathbb{B}, \circ)$:

$$\mathfrak{e} := (0, 1) \quad \text{and} \quad \mathfrak{b}^{-1} := (-b\epsilon, \overline{\epsilon})$$

Identifying system poles

REPRESENTATIONS OF LTI SYSTEMS
IR MODELS

A discrete time LTI system G can be considered as a linear causal time-invariant operator T_G mapping an input sequence $u_t \in \mathcal{U} \subset l_2$ to an output sequence $y_t \in \mathcal{Y} \subset l_2$.
Applying $\mathcal{Z}$-transform on the input and output sequences, $\mathcal{Z} : l_2 \mapsto \mathcal{H}_2$, it can be proved that the operator $G(z) : U(z) \mapsto Y(z)$ acts as a multiplication operator and $G(z) \in \mathcal{H}_\infty$:

$$G(z) = \sum_{k=1}^{\infty} g_k z^{-k},$$

where $g_k, k = 1, \ldots,$ is the impulse response sequence.
This system representation depends on the use of the canonical shift operator on l_2 that corresponds to the multiplication by z^{-1} in $\mathcal{H}_2$ and $z^{-k}, k = 1, 2, \ldots$ forms a basis in $\mathcal{H}_2$.

GOBF REPRESENTATIONS OF LTI SYSTEMS

Let $V_k(z)$, $k = 0, 1, \ldots$ be an orthogonal basis in $\mathcal{H}_2$. Then an LTI system can be represented as

$$G(z) = \sum_{k=0}^{\infty} G_k V_k(z).$$

Examples:

Canonical basis :

$$G_b = z^{-1}, \qquad V_k(z) = \frac{1}{z} z^{-k}$$

Laguerre basis :

$$G_b(z) = L_a(z) = (1 - az)/(z - a), \quad |a| < 1,$$

$$V_k(z) = L_k^a(z) = \frac{\sqrt{(1 - a^2)}}{z - a} \left(\frac{1 - az}{z - a} \right)^k$$

THE DISCRETE LAGUERRE SYSTEM

For an elementary factor we have

$$F(z) = \frac{1}{1 - \overline{a}\,z} = \sum_{n=0}^{\infty} l_n V_n(z) \quad \Rightarrow \quad l_n = \sqrt{1 - |b|^2}\,\frac{(\overline{a} - \overline{b})^n}{(1 - \overline{a}\,b)^{n+1}}$$

With the term

$$q = \frac{\overline{a} - \overline{b}}{1 - \overline{a}b}$$

geometrical sequences with quotient q are obtained – convergence depends on q.
For $a, b \in \mathbb{D}$

$$\left| \frac{\overline{a} - \overline{b}}{1 - \overline{a}b} \right| < 1,$$

hence the geometrical sequence absolutely converges.

ANALYZING THE CONVERGENCE QUOTIENT

The quotient

$$q = \frac{\overline{a} - \overline{b}}{1 - \overline{a}b}$$

can be interpreted as

$$\overline{q} = \frac{a - b}{1 - \overline{b}a} = B_b(a),$$

where $B_b(z)$ is a Blaschke function of parameter $b \in \mathbb{D}$, i.e.,

$$B_b(z) = \frac{z - b}{1 - \overline{b}z}.$$

The Blaschke function is an *inner function* in the space $\mathcal{H}_2(\mathbb{D})$, i.e., it is a bijection

$$B_b : \mathbb{D} \to \mathbb{D} \qquad B_b : \mathbb{T} \to \mathbb{T}$$

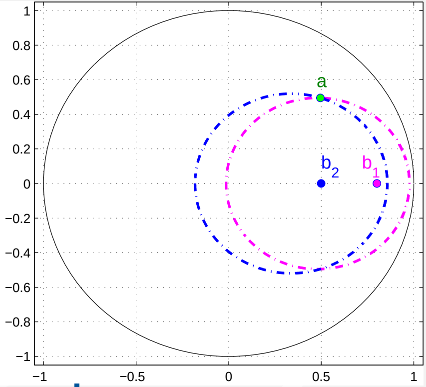
The Blaschke function of this form realizes a **hyperbolic distance measure** between points a and b within the unit circle.

IDENTIFYING A POLE

The hyperbolic distance between two poles a and b ($a, b \in \mathbb{D}$):

$$d = \left| \frac{a - b}{1 - \overline{b}a} \right|$$

Let a be fixed and select alternative b parameters, e.g. b_1 and b_2:



The hyperbolic circles belonging to distances

$$d_1 = \left| \frac{a - b_1}{1 - \overline{b_1}a} \right| \quad d_2 = \left| \frac{a - b_2}{1 - \overline{b_2}a} \right|$$

cross each other on pole a.

This fact can be used to **identify the pole**!

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GENERAL METHOD FOR IDENTIFYING POLES

By selecting a parameter value $|b| < 1$,

1. Estimation of the coefficients ℓ_n of the Laguerre representation based upon specific b.
2. Estimation of the convergence rate q according to the Laguerre coefficients obtained by considering

$$q = \lim_{n \to \infty} \frac{\ell_{n+1}}{\ell_n}$$

3. Apply the hyperbolic transform $a = B_{b^{-1}}(\overline{q})$ to derive pole a.
4. Steps 1 to 3 have to be repeated for new values of parameter b.

In practical cases:

- Finite number of Laguerre coefficients can be estimated.
- The limit can approximately be evaluated .

The $\mathcal{H}_\infty$ Performance Group

GEOMETRY AND CONTROL
LOCAL VS. GLOBAL VIEW

Local view: geometric control theory: based on differential geometry, Lie algebra, algebraic geometry, treats many important system concepts, for example controllability, as geometric properties of the state space or its subspaces. These are the properties that are preserved under change of coordinates, for example, the so-called invariant or controlled invariant subspaces. Linear theory (Wonham,Basile-Marro), nonlinear theory (Isidori). System transforms (diffeomorphism) play a fundamental role to reveal these invariants, e.g., Kalman decomposition.

Global view: an input-output ("coordinate free") framework, centered on a Kleinian approach to the geometry. Transformation groups play fundamental roles, they leave a given property invariant, e.g., stability or $\mathcal{H}_\infty$ norm.

PROBLEM SETTING
INDIRECT VS. DIRECT BLENDING

Solutions of the quadratic performance problems, e.g., a suboptimal $\mathcal{H}_\infty$ design, are parametrized by the elements of the unit ball. However, we cannot define directly an operation on this parameter space in a trivial way that bears a nice algebraic structure.

In this case the group actions that correspond to the addition of stable plants, which will be seen for the Youla parametrization, are the hyperbolic motions of the unit ball, determined by the J-unitary operators.

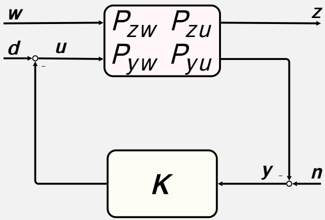
The goal is to provide an explicit parametrization of these operators and the corresponding induced blending on the parameter space. In contrast to these examples for the stability property of the closed loop there exists a direct blending of the stabilizing controllers without the need to introduce an additional parametrization.

Generalized plant

Performance loop, LFT, "gang of nine"

In the generalized plant paradigm two issues are handled:

the loop should be stable: the causal map (z, u, y) to (w, d, n) is invertible and the inverse map $\mathcal{L}(P_g, K) : (w, d, n) \mapsto (z, u, y)$ is stable.



Lower LFT:

$$T_{zw} = \mathfrak{F}_l(P_g, K) = P_{zw} + P_{zu}K(I - P_{yu}K)^{-1}P_{yw}$$

Performance specification: the LFT $\mathfrak{F}_l(P_g, K)$ should satisfy some norm constraints:

optimal $\mathcal{H}_\infty$ design: $\inf_K \|\mathfrak{F}_l(P_g, K)\|_\infty$

suboptimal $\mathcal{H}_\infty$ design: $\|\mathfrak{F}_l(P_g, K)\|_\infty < \gamma$

QUADRATIC PERFORMANCE VS. HYPERBOLIC GEOMETRY
MATRIX BLASCHKE FUNCTION

Any quadratic performance problem is related to the open unit ball through a J-spectral factorization and a Möbius transformation, i.e., the controller is parametrised as:

$$K = \mathfrak{M}_{\Phi}(a), \qquad a \in \mathcal{B} \subset \mathcal{H}_\infty$$

Since J-unitary maps leave the the open unit ball $\mathcal{B}$ invariant, hyperbolic geometry and hyperbolic distance is a natural choice.

$$B_a(x) = (1 - aa^*)^{-1/2}(x - a)(1 - a^*x)^{-1}(1 - a^*a)^{1/2}, \quad a, x \in \mathcal{B}$$

This Blaschke function is a biholomorphic automorphism of $\mathcal{B}$. The hyperbolic distance on $\mathcal{B}$ is $d_{\mathcal{B}}(a, b) = \operatorname{arctanh} \|B_a(b)\|$. On any bounded set (contained in some ρ-ball) the hyperbolic metrics is equivalent to the operator norm.

J-UNITARY MATRICES

For $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ consider the group of J-unitary matrices Φ, i.e., those matrices for which

$$\Phi^* J \Phi = J.$$

These matrices define the movements on the unit (contractive) ball that preserve the hyperbolic distance. There is a correspondence between the contractive ball and the J-unitary matrices:

$$\Phi_a = \begin{pmatrix} N_a & 0 \\ 0 & N_{a^*} \end{pmatrix} \begin{pmatrix} I & -a^* \\ -a & I \end{pmatrix}, \quad a \in \mathcal{B}$$

is J-unitary. We use the notations $D_a = (I - a^* a)^{1/2}$ and $N_a = D_a^{-1}$.
These unitary matrices correspond to the hyperbolic translations. Their Möbius transform defines the multidimensional generalisation of Blaschke products:

$$B_a(z) = \mathfrak{M}_\Phi(z) = N_{a^*}(z - a)(I - a^* z)D_a = -a + D_{a^*} z(I - a^* z)^{-1} D_a$$

Note that

$$B_a(0) = -a, \quad B_a(a) = 0, \; B_{-a}(0) = a \quad B_a \circ B_{-a} = B_{-a} \circ B_a = I.$$

HYPERBOLIC GROUP
Parametrization

While elementary translations form a group in the Euclidean geometry, in the hyperbolic world not. This fundamental difference makes things more complicated: we cannot define a group structure merely on the contractive ball. However, there is a remedy due to the fact that every J-unitary matrix can be expressed as an elementary translation and a block diagonal unitary action.

Every J-unitary matrix can be expressed as

$$\Phi = W_{u,v}\Phi_a, \qquad W_{u,v} = \mathrm{diag}\{u, v\},$$

for a suitable contraction a and unitary matrices u and v.

A less known but an important formula is:

$$\Phi_a W_{u,v} = W_{u,v}\Phi_{v^*au}$$

THE BLASCHKE GROUP

MAIN RESULT: PARAMETER BLENDING FORMULA

$$\Phi_{(u_1,v_1,a_1)}\Phi_{(u_2,v_2,a_2)} = \Phi_{(u,v,a)}$$

defines a group

$$(u, v, a) = (u_1, v_1, a_1) \circ (u_2, v_2, a_2) =$$
$$(u_1 u_2 E_{-a_2}(v_2^* a_1 u_2), v_1 v_2 E_{-a_2^*}(u_2^* a_1^* v_2), B_{-a_2}(a_1)).$$

where $E_a(z)$ is the unitary operator for which

$$B_a(z) = -B_z(a)E_a(z), \quad E_a(z) = Q_a(z)N_{B_a(z)}$$

with

$$Q_a(z) = D_z(I - a^* z)^{-1} D_a.$$

We use the notations $D_a = (I - a^* a)^{1/2}$ and $N_a = D_a^{-1}$.

Feedback stability: a geometric view

FEEDBACK STABILITY

$$d = u + Ky$$
$$n = Pu + y$$

(P, K) form a stable pair if the elements of the matrix

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix}^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix} =$$

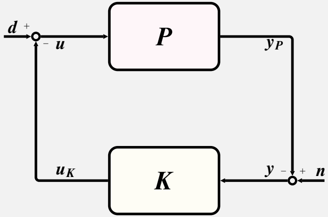
$$= \begin{pmatrix} (I - KP)^{-1} & -K(I - PK)^{-1} \\ -P(I - KP)^{-1} & (I - PK)^{-1} \end{pmatrix}$$

are stable.

$\Longrightarrow$ stable factorization $P = -GE^{-1} = -H^{-1}G\ (K = -FH^{-1} = -E^{-1}F)$

The feedback loop is **stable** if the map $w \to z$ **is bounded and causal.**

$$w = \begin{pmatrix} d \\ n \end{pmatrix},\ z = \begin{pmatrix} u \\ y \end{pmatrix}$$



Projective geometry is a natural framework of the problem formulation.

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INTERNAL STABILITY

PROJECTIVE VIEW

graph subspace: $\mathcal{P} = \operatorname{Im} \begin{pmatrix} I_u \\ P \end{pmatrix} u$

inverse graph subspace: $\mathcal{K}^{-1} = \operatorname{Im} \begin{pmatrix} K \\ I_y \end{pmatrix} y$

Homogeneous coordinates: $P = NM^{-1}$ $\quad K = UV^{-1}$

$$\mathcal{P} \sim \left[\begin{pmatrix} M \\ N \end{pmatrix} \right] = \left\{ \begin{pmatrix} M \\ N \end{pmatrix} T \mid T \text{ invertible} \right\}$$

$$\mathcal{K}^{-1} \sim \left[\begin{pmatrix} U \\ V \end{pmatrix} \right] = \left\{ \begin{pmatrix} U \\ V \end{pmatrix} T \mid T \text{ invertible} \right\}$$

Finite points:

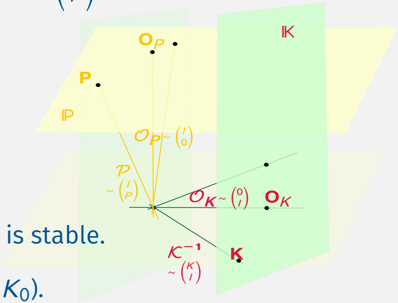
$\underline{\mathbb{P}}_f = \{\underline{P}\}$ ha M invertible

$(\overline{\mathbb{K}}_f = \{\overline{K}\}$ ha V invertible)

Special case: **double coprime factorization**

(P, K) form a stable pair if the inverse $\tilde{\Sigma}_{P,K}$ of $\Sigma_{P,K} = \begin{pmatrix} M & U \\ N & V \end{pmatrix}$ is stable.

Moreover a d. c. factorization $\Sigma\tilde{\Sigma} = I$ determine a stable pair (P, K_0).



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PROJECTIVE GEOMETRY
FINITE POINTS

Projective geometry formalizes the central principles of perspective: parallel lines meet at infinity. Introducing a special hyperplane two subspaces are parallel if they have the same intersection with this special hyperplane.

Special case: **double coprime factorization**

stability of the pair (P, K) means that $\Sigma_{P,K} = \begin{pmatrix} M & U \\ N & V \end{pmatrix}$ has a bounded causal inverse – assuming a double coprime factorization.

In homogeneous coordinates defined by the splitting induced by the hyperplane a point $\underline{P} = [\begin{pmatrix} M \\ N \end{pmatrix}]$ is finite if M is invertible.

Equivalence classes of finite points $\underline{\mathbb{P}}_f$ are the plants P represented by $\underline{P} = [\begin{pmatrix} M \\ N \end{pmatrix}]$.

Controllers $K = UV^{-1}$ are described by the inverse relation (inverse graph), i.e., $\overline{K} = [\begin{pmatrix} U \\ V \end{pmatrix}]$.

PROJECTIVE GROUP: MÖBIUS TRANSFORMATION

Projectivity: action of $S \in GL(\mathcal{U} \oplus \mathcal{Y})$ on $\mathbb{P}$

$S = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ and $\underline{P} \in \mathbb{P}$ the map $\underline{P}^S = [P^S]$ where

$$P^S = \begin{pmatrix} P_1^S \\ P_2^S \end{pmatrix} = S \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} AP_1 + CP_2 \\ BP_1 + DP_2 \end{pmatrix}.$$

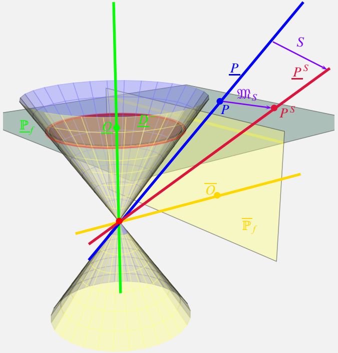
Projectivities of $\mathbb{P}$ form a group under composition.

If $\underline{P}$ is a finite point and $\underline{P}^S$ is also finite, then

$$\begin{pmatrix} I \\ P^S \end{pmatrix} = \begin{pmatrix} I \\ (B + DP)(A + CP)^{-1} \end{pmatrix}.$$

Möbius transformation: restriction of S to the finite points $\mathbb{P}_f$

$$\mathfrak{M}_S(\underline{P}) = (B + DP)(A + CP)^{-1}.$$



Möbius transform inherits the group structure of the linear operators: $\mathfrak{M}_{S_2} \circ \mathfrak{M}_{S_1} = \mathfrak{M}_{S_2 S_1}$

For $P = 0_P$ we have $\mathfrak{M}_{\Sigma_{P,K}}(\underline{0}_P) = (N + V0_P)(M + U0_P)^{-1} = NM^{-1}$.

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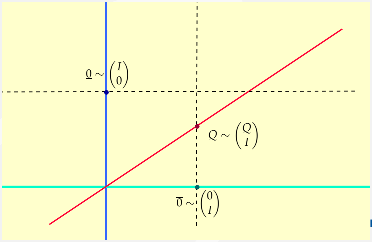
INTERNAL STABILITY: GEOMETRIC PROPERTY

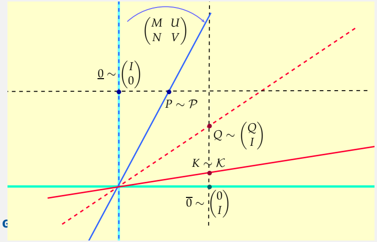
$P = \mathfrak{M}_{\Sigma_{P,K}}(0_P)$ for every $\Sigma_{P,K}$ and the zero plant is stabilized by the entire stable set, and only by that set. The additive group of the stable plants induces the projective subgroup $G_s = \{\begin{pmatrix} I & Q \\ 0 & I \end{pmatrix}\}$. For a fixed stable point (P, K_0), i.e., $\Sigma = \Sigma_{P,K_0}$, the group $G_P = \Sigma G_s \Sigma^{-1}$ keeps P fixed and keeps invariant the stability property of the pairs (P, K). The stabilizing controllers are described by

$$K = \mathfrak{M}_{S_Q}(\overline{K}_0) = \mathfrak{M}_{\Sigma}(Q), \quad S_Q \in G_P,$$

i.e., the well-known Youla parametrization

$$\mathcal{K}_{stab} = \{K \mid K = (U + MQ)(V + NQ)^{-1}, Q \text{ stable}\}.$$





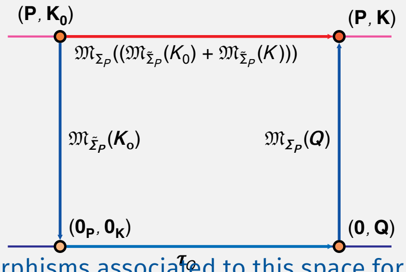
CONTROLLER BLENDING

INDIRECT APPROACH

Indirect blending:

$$0_K = \mathfrak{M}_{\tilde{\Sigma}}(K_0), \; K = \mathfrak{M}_{\Sigma}(Q),$$
$$K = K_1 +_{(P,K_0)} K_2 =$$
$$\mathfrak{M}_{\Sigma}\left(\mathfrak{M}_{\tilde{\Sigma}}(K_1) + \mathfrak{M}_{\tilde{\Sigma}}(K_2)\right)$$

The unit element is K_0 which defines Σ.



We have a parameter space $\mathbb{Q}$, and a group of automorphisms associated to this space formed by simple translations $Q \mapsto \tau_Q$, with

$$\tau_Q = \begin{pmatrix} I & Q \\ 0 & I \end{pmatrix}, \qquad \tau_{Q_1}\tau_{Q_2} = \tau_{Q_1+Q_2}$$

While the group homomorphism between the composition of translations and the addition of parameters is trivial, the mere addition on the Youla parameter level does not lead, in general, to a "simple" operation on the level of controllers.

CONTROLLER BLENDING
DIRECT APPROACH – SEMIGROUP

The observation

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \begin{pmatrix} I & K \\ 0 & I - PK \end{pmatrix},$$

suggests to define the controller blending through

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \begin{pmatrix} I & K_1 \\ 0 & I - PK_1 \end{pmatrix} \begin{pmatrix} I & K_2 \\ 0 & I - PK_2 \end{pmatrix}.$$

It turns out that on controllers the corresponding operation is

$$K = K_1(I - PK_2) + K_2 = K_1 \boxdot_P K_2.$$

The unit of this operation is the zero controller $K = 0_K$ and the corresponding inverse elements are given by $K^{\boxminus_P} = -K(I - PK)^{-1}$.

if 0_K is not a stabilizing controller: it is only a **semigroup.**

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DIRECT APPROACH
GROUP HOMOMORPHISM

By using the notation

$$\begin{pmatrix} I & K \\ P & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \begin{pmatrix} I & K \\ 0 & I - PK \end{pmatrix} = R_P T_K^{(P)}$$

we can express these facts in a more formal term as follows:

$$T_{K_1}^{(P)} T_{K_2}^{(P)} = T_{K_1 \boxdot_P K_2}^{(P)}$$

and

$$K = \mathfrak{M}_{R_P T_K^{(P)} R_P^{-1}}(0_K).$$

In this case the group homomorphism between the composition of translations and the addition of the corresponding parameters is nontrivial.

Controller group
Strictly stable controllers

If there is a stabilizing controller K_0 such that

$$K_0^{\boxminus_P} = -K_0(I - PK_0)^{-1} = \Sigma_{c0}$$

is also a stabilizing controller, then $(\Sigma_P, \boxtimes_P)$ with

$$\begin{aligned} K &= K_1 \boxtimes_P K_2 = K_1 \boxdot_P K_0^{\boxminus_P} \boxdot_P K_2 = \\ &= K_2 + (K_1 - K_0)(I - PK_0)^{-1}(I - PK_2) \end{aligned}$$

is a group with a unit (K_0). The corresponding inverse is given by

$$K^{\boxtimes_P^{-1}} = K_0 - (K - K_0)(I - PK)^{-1}(I - PK_0).$$

This may happen only if the plant is strongly stabilizable: the group of strongly stable controllers.

Conclusion

CONCLUSION

Development of the Bolyai-Lobachevsky geometry, as the first instance of non-euclidean geometries, had a great impact on the evolution of mathematical thinking. Non-Euclidean geometry has turned out to be more than just a logical curiosity, and many of its basic features continue to play important roles in several branches of mathematics and its applications.

We put an emphasize on the Kleinian concept of the geometry and its direct applicability to control problems: through the analogous of the classical geometric constructions not only might get hints for efficient algorithms but also obtain tools for controller manipulations that preserves the property at hand (stability), called controller blending.

Geometry – and also group theory – does not deal with the existence and the actual nature of the objects that are the primitives of the given geometry but rather captures the "rules" they obeys to. It responds to the question "what can be done with these objects" rather than "how to synthesise the object having a given property (e.g., stability)".

CONCLUSION

It is very useful to formulate a control problem in an abstract setting and then translate it into an elementary geometric fact or construction. The basic global geometric structures that are related to feedback stability are closely related to generalised projective geometric ideas. On this projective geometric background one can solve solve the controller blending problem in a general setting.
As a result, an operation is given under which well-posedness is a group while stability is a semigroup. Moreover, an operation was given that makes controllers with strongly stable property a group.

For the blending problem associated to a suboptimal $\mathcal{H}_\infty$ design the relevant geometric structure is related to the hyperbolic geometry while the corresponding group structure is an extension of the so called Blaschke group.

Besides the educative value a merit of the presentation for control engineers might be a unified view on control problems that reveals the main structure of the problem at hand and give a skeleton for the algorithmic development.

Thank You for Your Attention!