Identification of linear dynamic networks with reduced-rank noise

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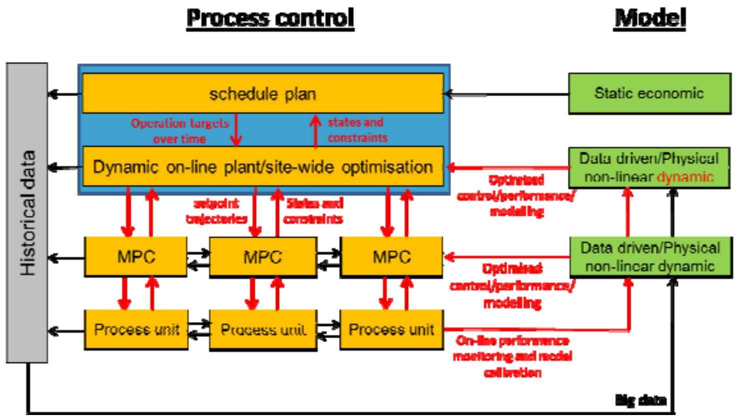
European Research Council

TU/e Technische Universiteit Eindhoven University of Technology

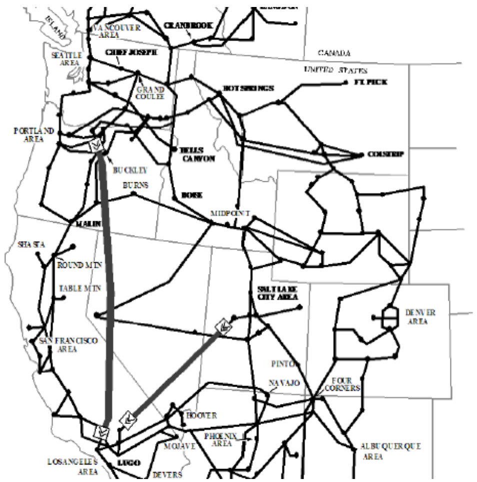
Where innovation starts

Introduction – dynamic networks

Decentralized process control

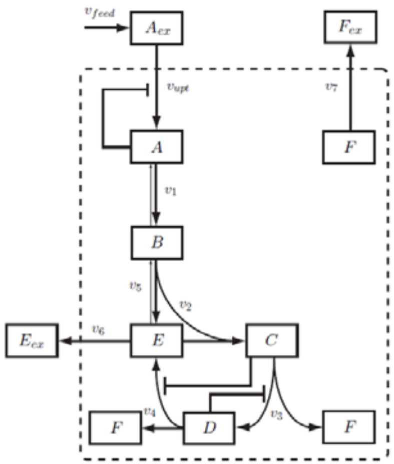


Power grid



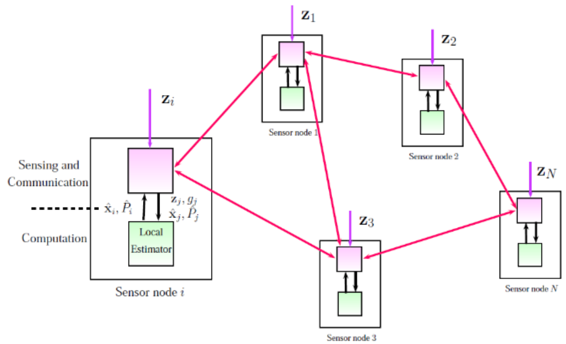
Pierre et al. (2012)

Metabolic network



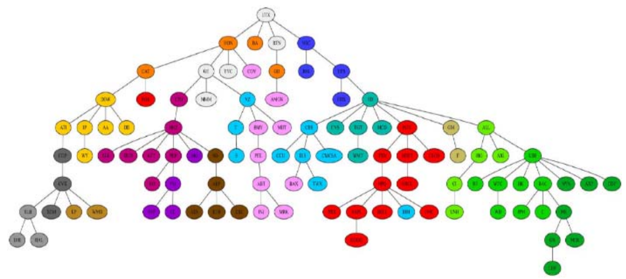
Hillen (2012)

Distributed control (robotic networks)



Simonetto (2012)

Stock market



Materassi et al. (2010)

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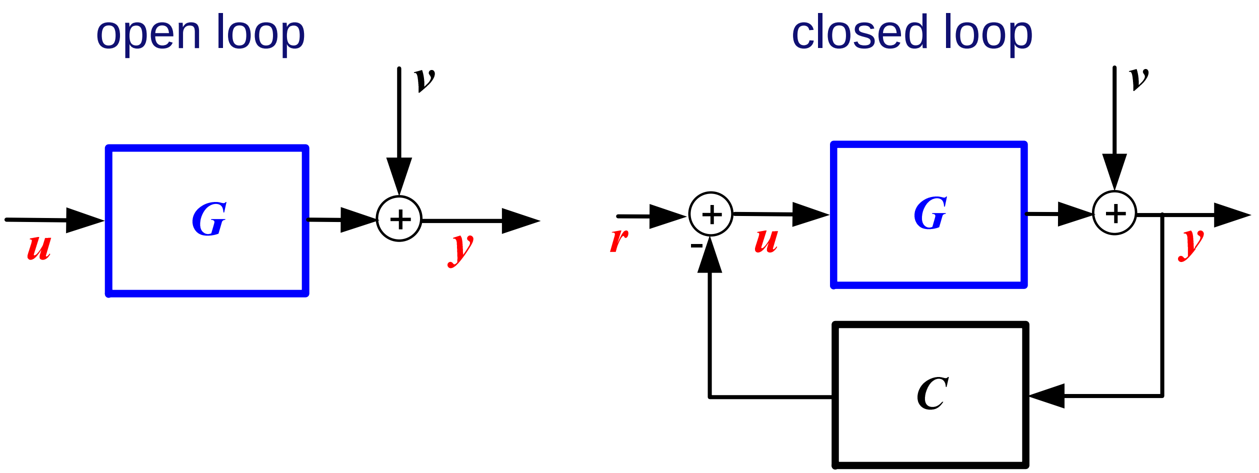
Introduction – identification

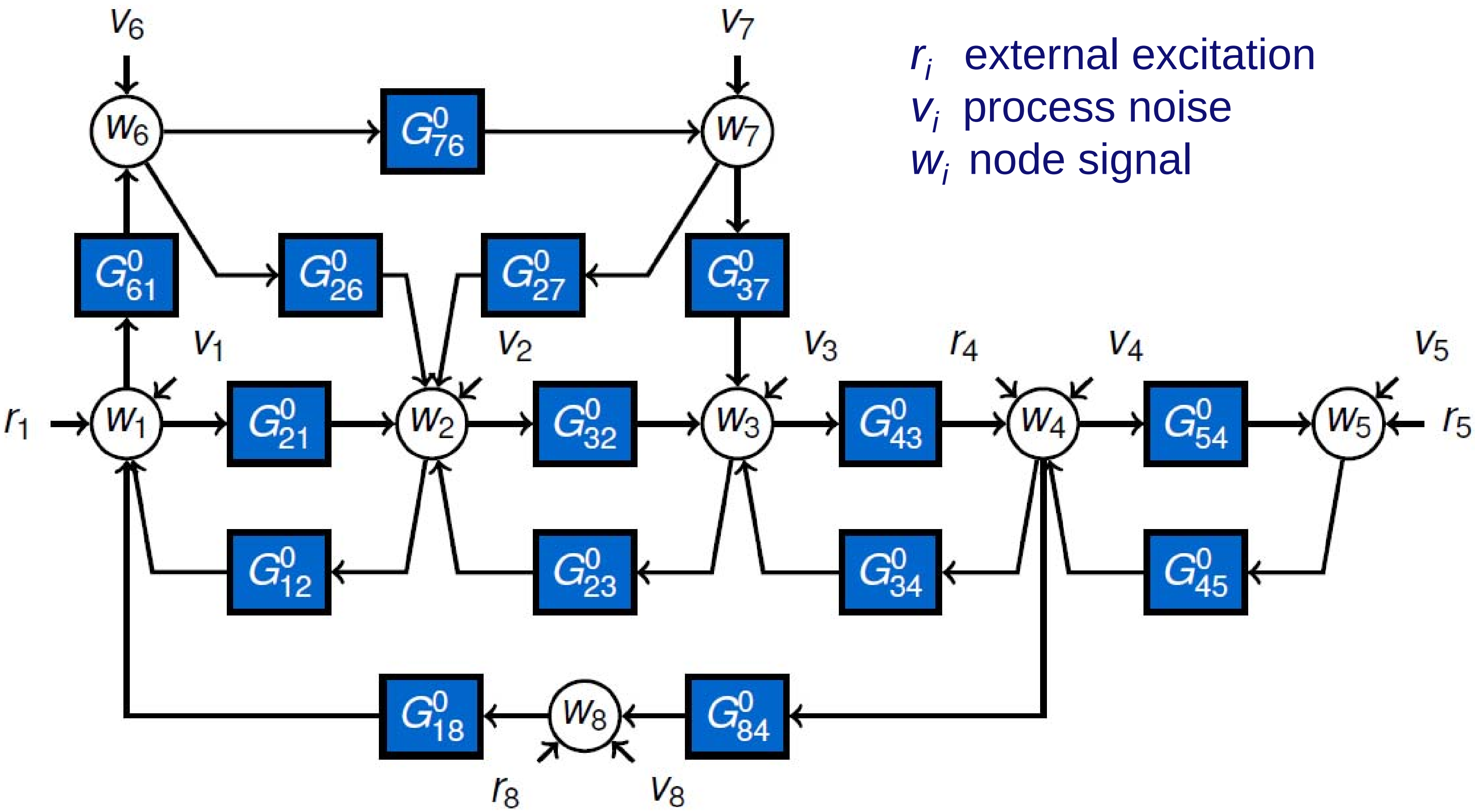
The classical (multivariable) identification problems: [Ljung (1999)]



Identify a plant model $\hat{G}$ on the basis of measured signals *u, y* *(*and possibly *r)*

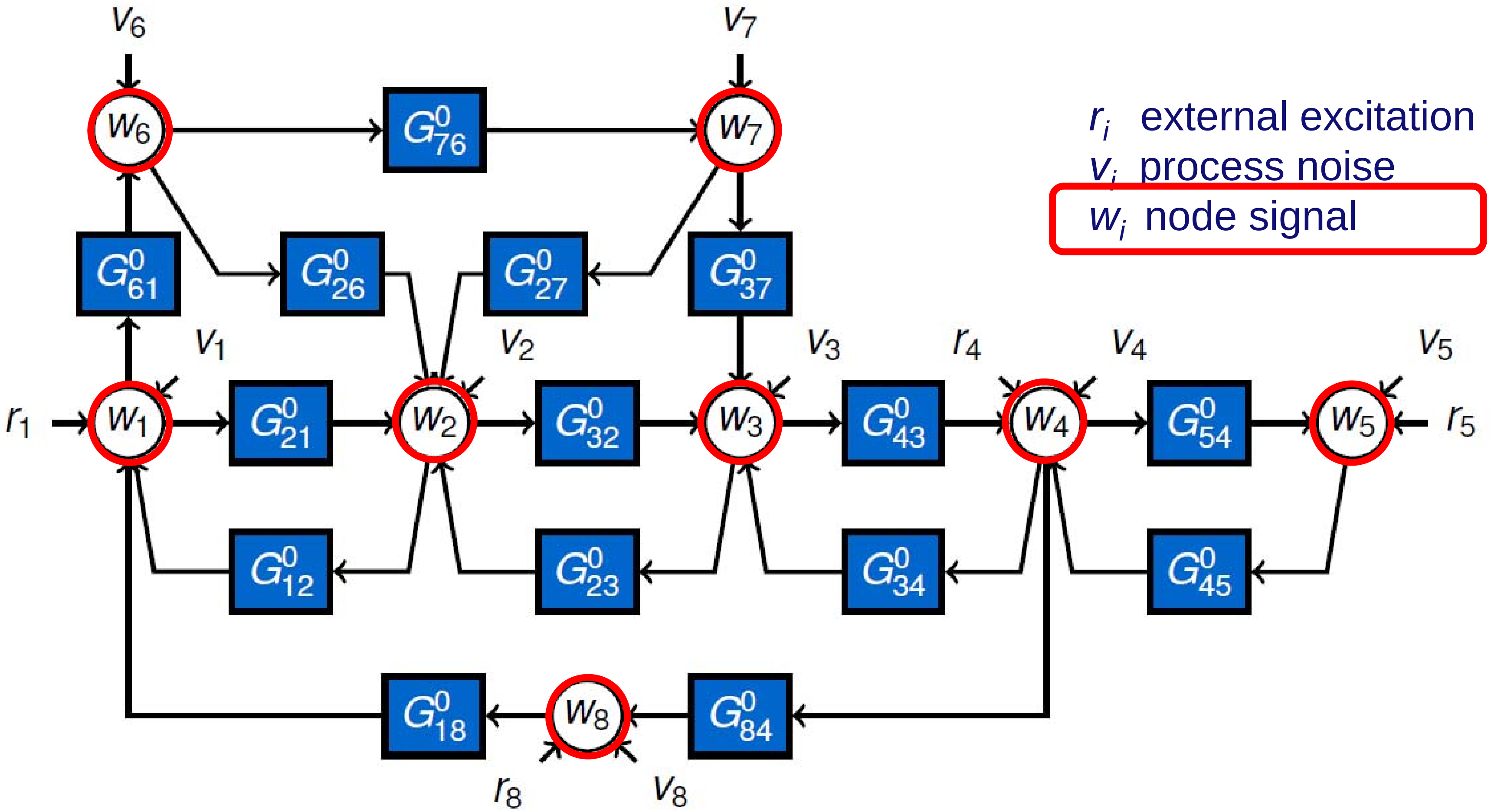
We have to move from a fixed and known configuration to deal with and exploit ***structure*** in the problem.

Dynamic network: what is it?



r_i external excitation
v_i process noise
w_i node signal

Dynamic network: what is it?



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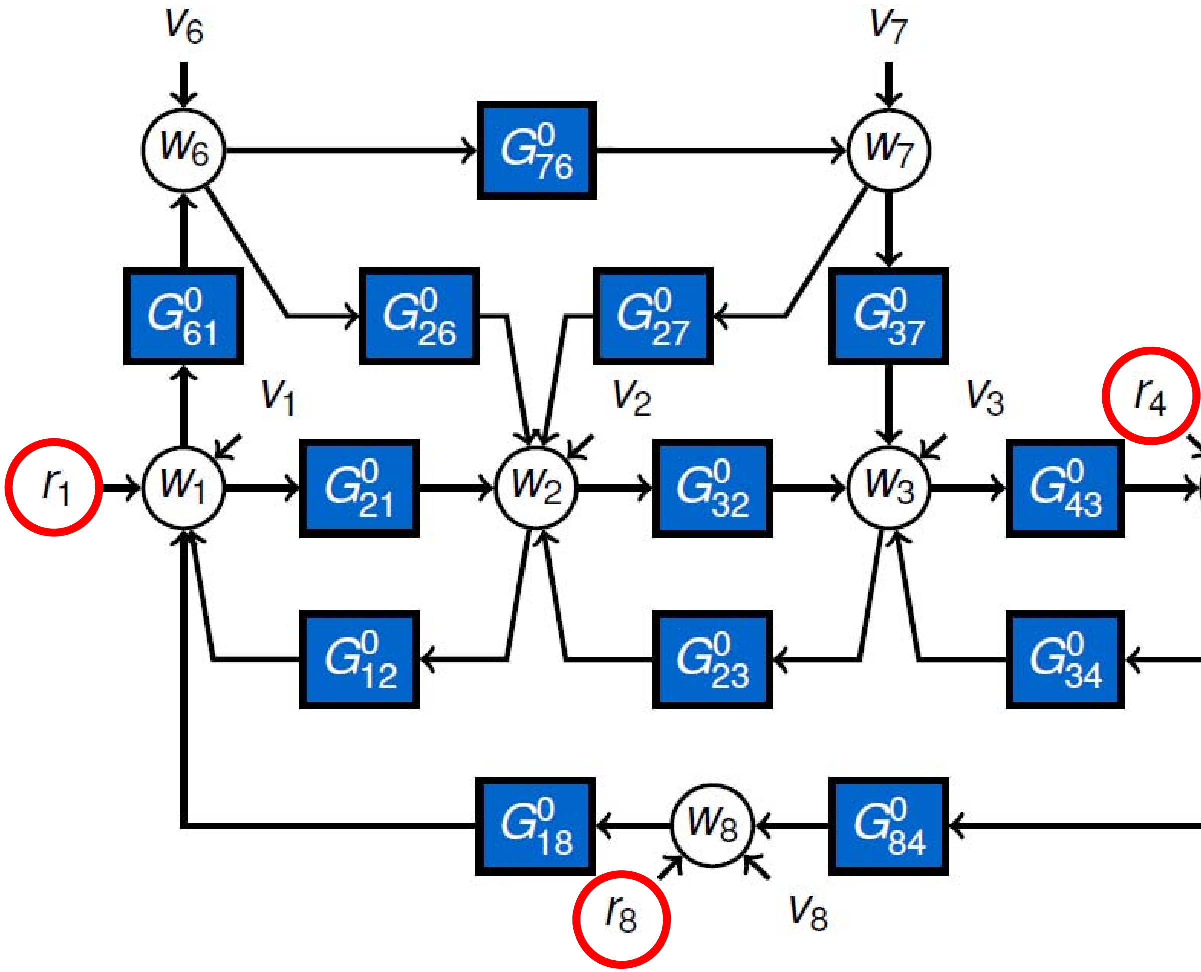

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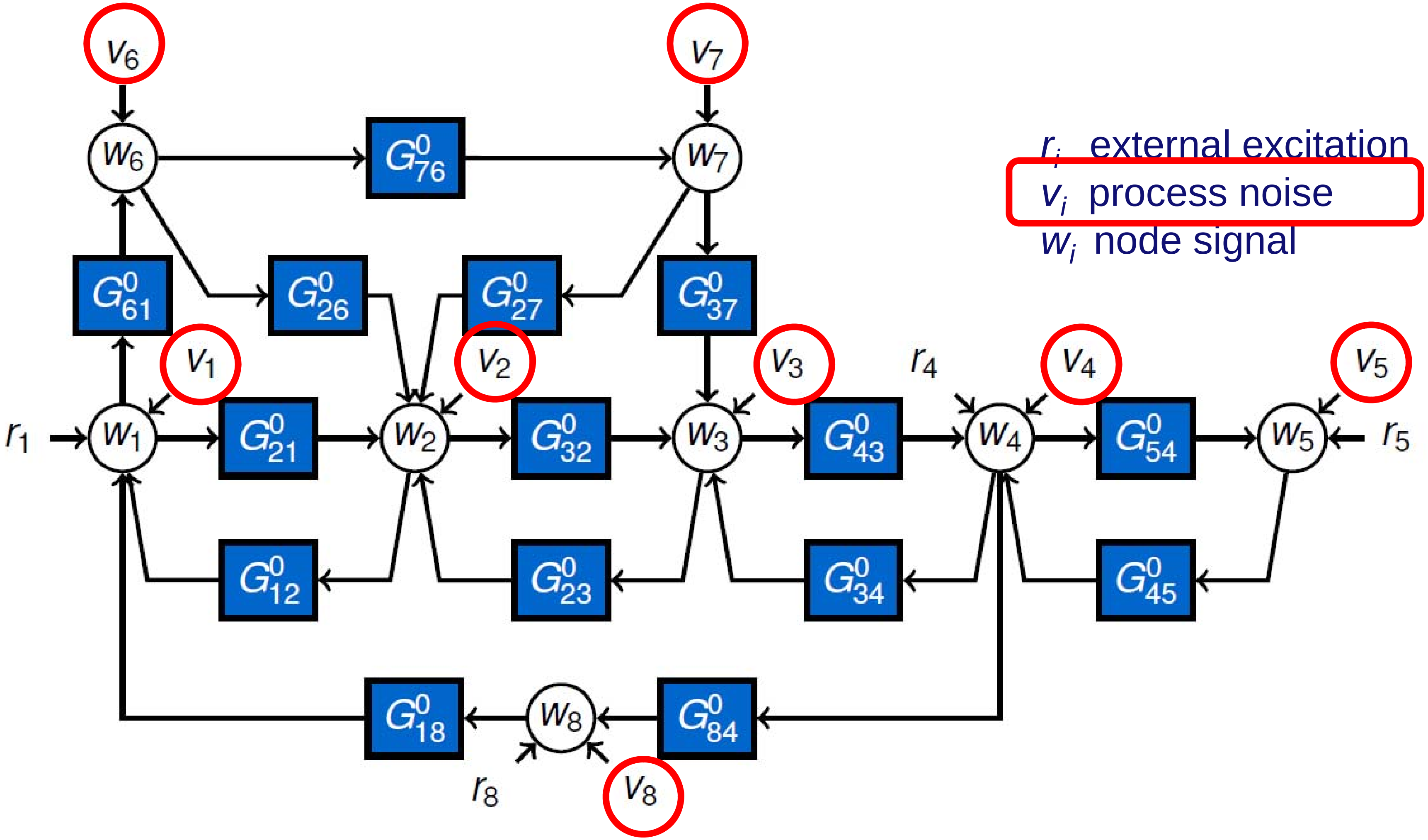

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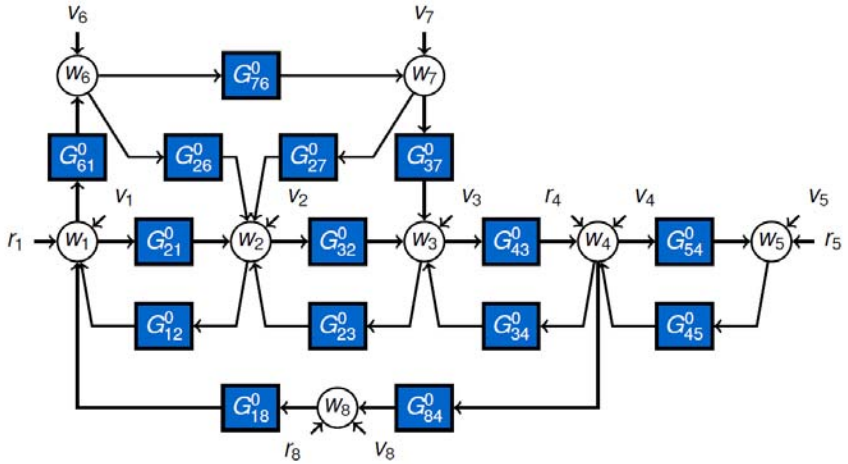
Introduction



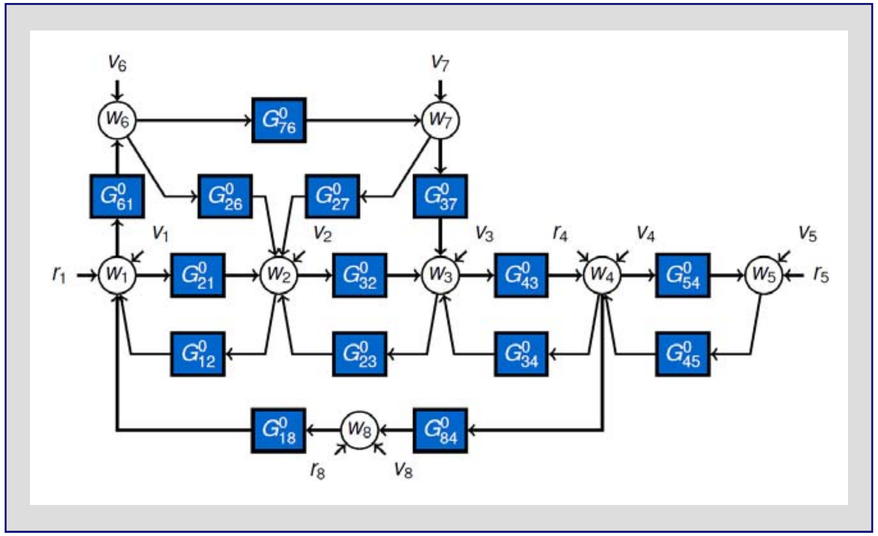
r_i external excitation
v_i process noise
w_i node signal

$$v(t) = \begin{bmatrix} v_1(t) \\ \vdots \\ v_L(t) \end{bmatrix}$$

What are assumptions on process noises when identifying (parts of) a network?

- Independent white noise processes
- Vector stochastic process with full rank spectrum, $rank\ \Phi_v(\omega) = L\ \ a.e.$ leading to a square noise model: $v(t) = H(q)e(t)$
- If $dim(e) < L$ then we have “singular” or “reduced-rank” noise

Network Setup



Assumptions:

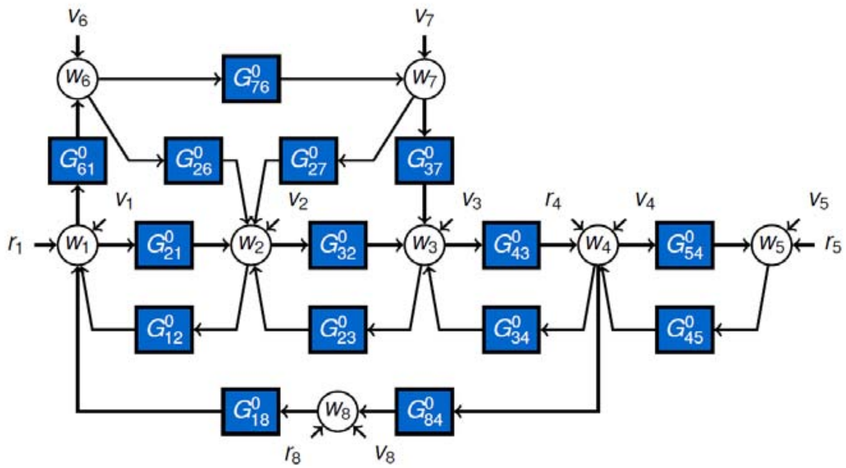
- Total of *L* nodes
- Network is well-posed and stable
- Modules may be unstable
- Node signals and excitation signals can be measured

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & G^0_{12} & \cdots & G^0_{1L} \\ G^0_{21} & 0 & \ddots & G^0_{2L} \\ \vdots & \ddots & \ddots & \vdots \\ G^0_{L1} & G^0_{L2} & \cdots & 0 \end{bmatrix}}_{G^0} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + R^0(q) \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{bmatrix} + \underbrace{H^0(q) \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_p \end{bmatrix}}_{v}$$

$$w = G^0 w + R^0 r + H^0 e$$

- P.M.J. Van den Hof, A.G. Dankers, P.S.H. Heuberger and X. Bombois. *Automatica, 49*, 2994-3006, 2013.

Introduction



r_i external excitation
v_i process noise
w_i node signal

$$\begin{bmatrix} v_1(t) \\ \vdots \\ v_L(t) \end{bmatrix} = H^0(q) \begin{bmatrix} e_1(t) \\ \vdots \\ e_p(t) \end{bmatrix}$$

Main question:
How to identify (parts of) a dynamic network,
when the process noise is of reduced rank *(p < L)*?

Contents

- Modelling a reduced-rank stochastic process
- Multi-output identification in a dynamic network
  the joint-direct method with weighted LS
- Constrained LS and maximum likelihood estimation
- Variance-free estimation, minimum variance and the CRLB
- Simulation example

Modelling reduced rank noise

Assumption

The node signals w_j are ordered in such a way that the first *p* noise components $v_j, \;\; j = 1, \cdots p$ constitute a full rank process.

Modelling reduced rank noise

A reduced-rank stochastic process $\boldsymbol{v}$ with dimension *L* and rank *p* can equivalently be described in two ways:

a) $\boldsymbol{v(t) = \check{H}^0(q)\check{e}(t)}$

With $\boldsymbol{\check{H}^0} \in \mathbb{R}^{\boldsymbol{L\times L}}\boldsymbol{(z)},\ \boldsymbol{\check{e}(t)} \in \mathbb{R}^{\boldsymbol{L}}$ a white noise process,
$\boldsymbol{\check{H}^0}$ stable, stably invertible, and monic, and
$\boldsymbol{cov(\check{e}) = \check{\Lambda}^0}$ having rank *p*

b) $\boldsymbol{v(t) = H^0(q)e(t)}$

With $\boldsymbol{H^0} \in \mathbb{R}^{\boldsymbol{L\times p}}\boldsymbol{(z)},\ \boldsymbol{e(t)} \in \mathbb{R}^{\boldsymbol{p}}$ a white noise process,
$\boldsymbol{H^0} = \begin{bmatrix} \boldsymbol{H_a^0} \\ \boldsymbol{H_b^0} \end{bmatrix}$ with $\boldsymbol{H_a^0}$ square, stable, stably invertible, and monic,
$\boldsymbol{cov(e) = \Lambda^0}$ having full rank *p*

Modelling reduced rank noise

Relations between descriptions:

$$v(t) = \check{H}^0(q)\check{e}(t) = \begin{bmatrix} H_a^0(q) & 0 \\ H_b^0(q) - \Gamma^0 & I \end{bmatrix} \begin{bmatrix} e \\ \Gamma^0 e \end{bmatrix}$$

with $\Gamma^0 = \lim_{z \to \infty} H_b^0(z)$

while $\check{\Lambda}^0 = \begin{bmatrix} I \\ \Gamma^0 \end{bmatrix} \Lambda^0 \begin{bmatrix} I \\ \Gamma^0 \end{bmatrix}^T$ and $\begin{bmatrix} \Gamma^0 & -I \end{bmatrix} \check{e}(t) = 0$

Both noise models $\check{H}^0$ and $H^0 = \begin{bmatrix} H_a^0 \\ H_b^0 \end{bmatrix}$ will be used.

Joint-direct identification method

We follow a prediction error approach, by predicting **all** node variables:

$$\hat{w}(t|t-1) := \mathbb{E}\left\{w(t) \mid w^{t-1},\ r^{t}\right\}$$

Then: $$\hat{w}(t|t-1) = W_w^0(q)w(t) + W_r^0(q)r(t)$$

with:
$$\begin{aligned} W_w^0(q) &= I - (\check{H}^0(q))^{-1}(I - G^0(q)), \\ W_r^0(q) &= (\check{H}^0(q))^{-1}R^0(q). \end{aligned}$$

being the unique predictor filters.

Joint-direct identification method

The **network** is defined by: $(G^0, R^0, H^0, \Lambda^0)$

a network model is denoted by: $M = (G, R, H, \Lambda)$

and a **network model set** by:

$$\mathcal{M} = \{M(\theta) = (G(\theta), R(\theta), H(\theta), \Lambda(\theta)), \theta \in \Theta\}$$

Then the parametrized predictor:

$$\hat{w}(t|t-1) = W_w(q,\theta)w(t) + W_r(q,\theta)r(t)$$

leads to the prediction error: $\varepsilon(t,\theta) = w(t) - \hat{w}(t|t-1;\theta)$

Weighted LS criterion:

$$\hat{\theta}_N^{WLS} = \arg\min_{\theta\in\Theta} \frac{1}{N}\sum_{t=1}^{N} \varepsilon^T(t,\theta)\ Q\ \varepsilon(t,\theta) \qquad Q > 0$$

Joint-direct identification method

Weighted LS criterion:

$$\hat{\theta}_N^{WLS} = \arg\min_{\theta\in\Theta} \frac{1}{N}\sum_{t=1}^{N} \varepsilon^T(t,\theta)\; Q\; \varepsilon(t,\theta) \qquad\qquad \boldsymbol{Q > 0}$$

Properties:

- Consistent estimate under regularity conditions,
- Provided model set large enough, appropriate excitation, global network identifiability,
- But for minimum variance an optimal $\boldsymbol{Q}$ has to be chosen

Typical choice, leading to minimum variance estimatorm for $\boldsymbol{Q} \in \mathbb{R}^{\boldsymbol{L\times L}}$

$$\boldsymbol{Q = [cov(\breve{e})]^{-1} = (\breve{\Lambda}^0)^{-1}}$$

but in our situation $\boldsymbol{\breve{\Lambda}^0}$ is singular

Constrained LS and Maximum Likelihood

The WLS estimator does not take account of the dependencies in the innovation:

$$\begin{bmatrix}\boldsymbol{\Gamma}^0 & -\boldsymbol{I}\end{bmatrix} \check{e}(t) = 0$$

or differently formulated:

$$\begin{bmatrix}\boldsymbol{\Gamma}^0 & -\boldsymbol{I}\end{bmatrix} \begin{bmatrix}\varepsilon_a(t,\theta_0) \\ \varepsilon_b(t,\theta_0)\end{bmatrix} = 0$$

This can be imposed, by restricting the parametrized model to satisfy:

$$\underbrace{\Gamma(\theta)\varepsilon_a(t,\theta) - \varepsilon_b(t,\theta)}_{:= Z(t,\theta)} = 0$$

We denote: $:= Z(t,\theta)$

Constrained LS and Maximum Likelihood

Constrained LS criterion:

$$\hat{\theta}_N^{CLS} = \arg\min_{\theta \in \Theta} \frac{1}{N} \sum_{t=1}^{N} \varepsilon_a^T(t,\theta)\, Q_a\, \varepsilon_a(t,\theta) \qquad Q_a > 0$$

$$\text{subject to } \frac{1}{N} \sum_{t=1}^{N} Z^T(t,\theta) Z(t,\theta) = 0$$

Properties:

- Consistent estimate under similar conditions as WLS
- The choice
  $$Q_a = (\Lambda^0)^{-1}$$
  leads to minimum variance, and ML properties in case of Gaussian noise.
- For indendently parametrized $\Lambda(\theta)$, the cost function turns into a determinant function

Constrained LS and Maximum Likelihood

Implementation:

In practice, constraints could be unfeasible, e.g. in case $\mathcal{S} \notin \mathcal{M}$

Constraint relaxation:

$$\hat{\theta}_N^{rel} = \arg\min_{\theta} \frac{1}{N}\sum_{t=1}^{N}\Big(\varepsilon_a^T(t,\theta)Q_a\varepsilon_a(t,\theta) + \lambda Z^T(t,\theta)Z(t,\theta)\Big), \quad \lambda \in \mathbb{R}$$

with tuning parameter $\lambda \in \mathbb{R}$

For $\lambda > 0$ the consistency result remains true.
For $\lambda \to \infty$ constraint satisfaction

The criterion is equivalent to WLS with

$$Q(\theta) = \begin{bmatrix} Q_a + \lambda\Gamma^T(\theta)\Gamma(\theta) & -\lambda\Gamma^T(\theta) \\ -\lambda\Gamma(\theta) & \lambda I \end{bmatrix}$$

Minimum variance and CRLB

Asymptotic criterion:

$$\theta^\star = \arg\min_{\theta\in\Theta} \ \bar{\mathbb{E}}\ \varepsilon^T(t,\theta)\ Q_a\ \varepsilon(t,\theta)$$ subject to

$$\bar{\mathbb{E}} Z(t,\theta) Z^T(t,\theta) = 0$$

When linearizing $Z(t,\theta)$ in the neighbourhood of the optimum:

$$Z(t,\theta) \approx Z(t,\theta^*) + A(t)(\theta-\theta^*)$$

the constrained parameter space can be characterized by

$$\theta = S\rho + C \qquad \rho \in \mathbb{R}^{n_\rho}$$ of reduced dimension

with S, C determined by:

$$\begin{cases} \Pi S &= 0 \\ C &= -\Pi^\dagger \Pi \theta^* \end{cases}$$

and S full rank, where $\bar{\mathbb{E}} A^T(t) A(t) = \Pi^T \Pi$

$\Pi^\dagger$ right inverse

P. Stoica and B.C. Ng, *IEEE Signal Processing Letters, 5* (7), 177-179, 1998.