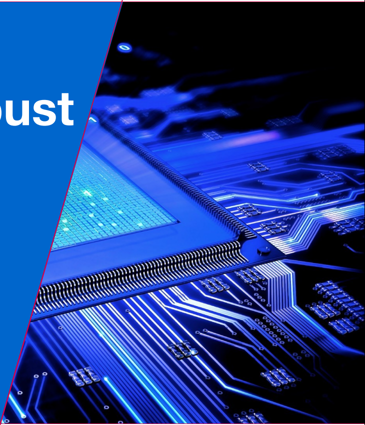
On Moment Based Robust MPC Formulations

M.B. Saltik L. Özkan J. Ludlage
S. Weiland P.M.J. Van den Hof

Control Systems Group,
Dept. of Electrical Engineering



Technische Universiteit
Eindhoven
University of Technology

November 15, 2016

Where innovation starts

Motivation

- Various areas of applications
  1. Chemical processes
     - Distillation columns;
     - Glass furnaces;
     - Ultra-filtration (membrane) systems.
  2. Constrained multibody systems
- Modeling the process
  1. Model based applications - State predictions
  2. Large-scale and nonlinear vs. black-box models
- Uncertainties vs. predictive control

1. How to cope with uncertainty?
   - Accept nominal predictions?
   - Stochastic vs. worst-case reasoning
2. Practical applicability
   - Accept pessimism vs. operational risks?
   - Computational expenses

TU/e Technische Universiteit Eindhoven University of Technology

Outline

Problem Statement

- Design MPC such that:
  1. controller operates in real-time (complexity)
  2. controller is risk-aware (unrealistic predictions)
- Robustness $\neq$ Risk-awareness

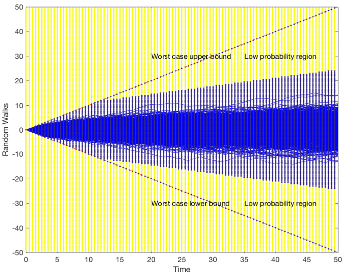


Figure 1. Random walks of state trajectory

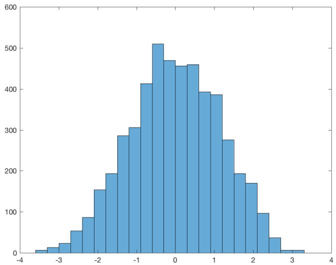


Figure 2. Histogram of state (at $t = 5$)

Problem Statement

- Design MPC such that:
  1. controller operates in real-time (complexity)
  2. controller is risk-aware (unrealistic predictions)
- Robustness $\neq$ Risk-awareness

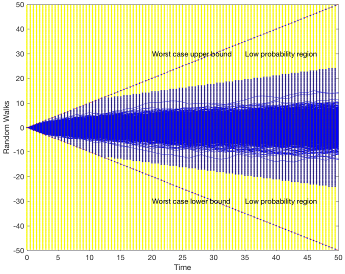


Figure 3. Random walks of state trajectory

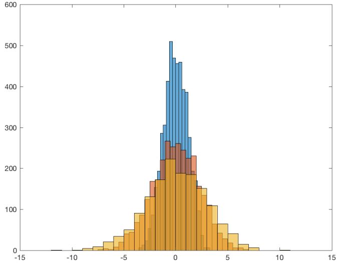


Figure 4. Histogram of state (at $t = 25$)

Technische Universiteit Eindhoven University of Technology

Problem Statement

- Design MPC such that:
  1. controller operates in real-time (complexity)
  2. controller is risk-aware (unrealistic predictions)
- Robustness $\neq$ Risk-awareness

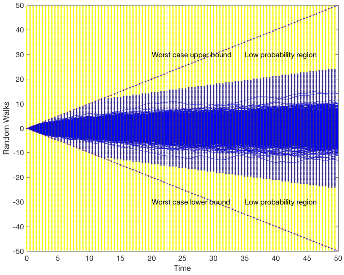


Figure 5. Random walks of state trajectory

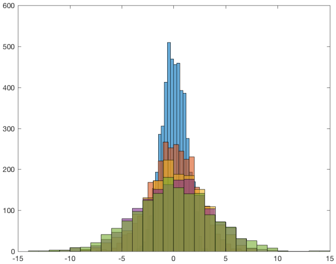
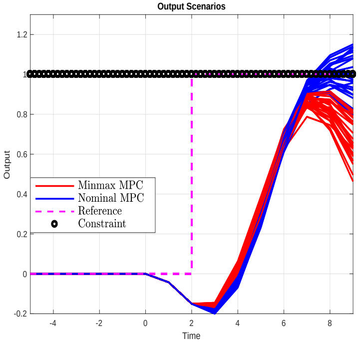
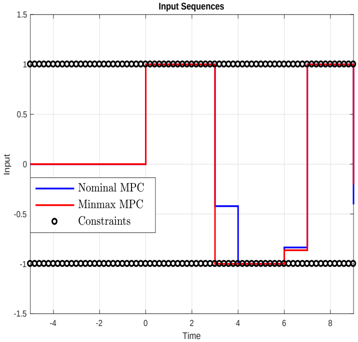


Figure 6. Histogram of state (at $t = 45$)

Risk-aware MPC

- Uncertain predictions and RMPC



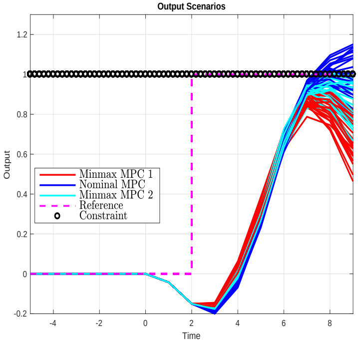


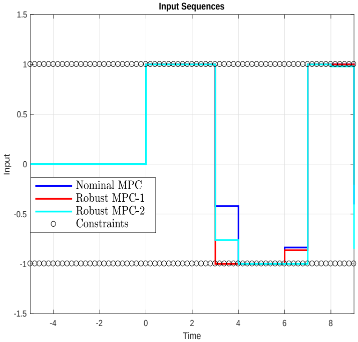
- Sources of uncertainty
  1. External disturbances (perturbations),
  2. Model-plant mismatch
  3. Initial condition mismatch

Technische Universiteit Eindhoven University of Technology

Risk-aware MPC

- Uncertain predictions and RMPC





- Sources of uncertainty
  1. External disturbances (perturbations),
  2. Model-plant mismatch
  3. Initial condition mismatch

Technische Universiteit Eindhoven University of Technology

Risk-aware MPC - Current Approaches

- Uncertain cost and constraint functions

1. Deterministic approach
   - Min-max and derivatives
   - Pessimism - Misery of worst-case thinking
2. Stochastic approach
   - Chance (probabilistic) constraints
   - Implementation - non-convexity

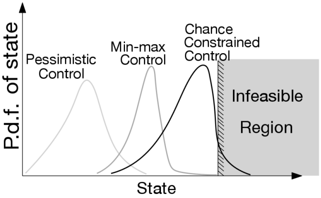


Figure 7. Pessimism in control

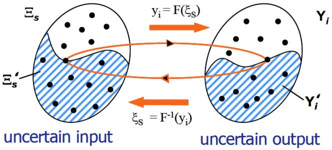


Figure 8. Non-convex constraints[1].

[1] Arellano-Garcia, H. "Chance constrained optimization of process systems under uncertainty." TUB, (2006).

Constructing Moment-based MPC 1

Uncertain MPC problems - Risk based approach
Reshape:

- Predictions
- Constraint violations
- Computational complexity

$$x_{k+1} = A(\delta_k)x_k + B(\gamma_k)u_k + Fw_k;$$
$$y_k = Cx_k,\ x_0 = x^0,$$
$$J = f(x_j, u_j, \delta_j, \gamma_j, w_j),$$
$$\forall j \in \mathbb{Z}_{[0,N_p-1]}$$

Natural candidates for 'reshaping'

1. Min-max
2. Scenarios
3. Chance constraints
4. Moments

$$\mathcal{P}(k): \begin{cases} \min\limits_{u_{[0,N_p-1]|k}} & \mathcal{R}^{cost}(J(x_k)) \\ \text{s.t.} & \mathcal{R}^{const}(c_i(x_{j|k})) \leq 0, \\ & \text{Prediction model}, \\ & \forall j \in \mathbb{Z}_{[0,N_p-1]}, \\ & \forall i \in \mathbb{Z}_{[1,N_c^j]}, \end{cases}$$

Constructing Moment-based MPC 2

Moment-based MPC

- New cost function: Moments of uncertain cost function

$$\min_{u_{[0,N_p-1]|k}} J(x,u,\zeta) \rightarrow \min_{u_{[0,N_p-1]|k}} \mathbb{E}\{J\} + \lambda_v \mathbb{D}\{J\} + \lambda_s \mathbb{S}\{J\} \ldots$$

- New constraints: Moments of uncertain constraint function

$$c_i(x,u,\zeta) \leq 0 \rightarrow \mathbb{E}\{c_i(x,u,\zeta)\} + \lambda_{v,i}\mathbb{D}\{c_i(x,u,\zeta)\} \leq 0$$

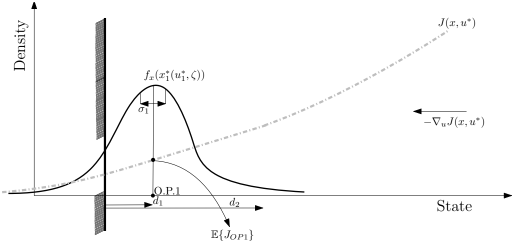


Figure 9. Operating point selection.

Constructing Moment-based MPC 2

Moment-based MPC

- New cost function: Moments of uncertain cost function

$$\min_{u_{[0,N_p-1]|k}} J(x,u,\zeta) \rightarrow \min_{u_{[0,N_p-1]|k}} \mathbb{E}\{J\} + \lambda_v \mathbb{D}\{J\} + \lambda_s \mathbb{S}\{J\} \ldots$$

- New constraints: Moments of uncertain constraint function

$$c_i(x,u,\zeta) \leq 0 \rightarrow \mathbb{E}\{c_i(x,u,\zeta)\} + \lambda_{v,i} \mathbb{D}\{c_i(x,u,\zeta)\} \leq 0$$

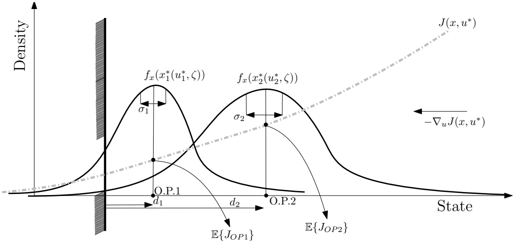


Figure 10. Operating points for two different cases.

Moment based MPC

- Nominal and Mean case

$$\bar{x}_{k+1} = A\bar{x}_k + Bu_k; \qquad x_{k+1} = Ax_k + Bu_k + Fw_k;$$

$$\bar{y}_k = C\bar{x}_k,\ \bar{x}_{0|k} = x_k. \qquad y_k = Cx_k,\ x_{0|k} = x_k.$$

$$J^N = \sum_{j=0}^{N_p-1} \|\bar{x}_{j|k}\|_Q^2 + \|u_{j|k}\|_R^2, \qquad J = \sum_{j=0}^{N_p-1} \|x_{j|k}\|_Q^2 + \|u_{j|k}\|_R^2,$$

$$J^N = \xi^\top H_M^{N_p} \xi, \qquad J^M = \mathbb{E}\{J(x_k)\},$$

$$J^M = \underbrace{\xi_k^\top H_M^{N_p} \xi_k}_{\text{Nominal cost}} + \underbrace{f^M}_{\text{Constant}},$$

$$\xi_k^\top = \begin{bmatrix} x_{0|k}^\top & u_{[0,N_p-1]|k}^\top \end{bmatrix} \qquad f^M = \mathrm{Tr}(T_F^\top \bar{Q} T_F \Sigma_w)$$

- Nominal MPC = Mean MPC
  1. Same control action - same minimizer

Moment based MPC - MV and MVS Case 1

- Mean-Variance and MVS case

$$J_w^{MV} = \mathbb{E}\{J\} + \lambda_v \mathbb{D}\{J\},$$
$$J_w^{MV} = \sum_{j=0}^{N_p-1} \|\bar{x}_{j|k}\|^2_{Q^{MV}_{j,w}} + \|u_{j|k}\|^2_R$$
$$Q^{MV}_{j,w} = Q + 4\lambda_v Q_v^w(j),$$
$$Q_v^w(j) = Q\Upsilon^w(j)Q,$$

$$J_w^{MVS} = J_w^{MV} + \lambda_s \mathbb{S}\{J\},$$
$$J_w^{MVS} = \sum_{j=0}^{N_p-1} \|\bar{x}_{j|k}\|^2_{Q^{MVS}_{j,w}} + \|u_{j|k}\|^2_R$$
$$Q^{MVS}_{j,w} = Q^{MV}_{j,w} + 24\lambda_s Q_s^w(j),$$
$$Q_s^w(j) = Q\Upsilon^w(j)Q\Upsilon^w(j)Q,$$

$$\Upsilon^w(j) = \sum_{i=0}^{j-1} A^{j-1-i} F \Sigma_w F^\top A^{j-1-i^\top}$$

- Stage-wise varying Q matrix $\rightarrow$ $Q^{MV}_{j+1,w} \succ Q^{MV}_{j,w}$

Technische Universiteit Eindhoven University of Technology

Moment based MPC - MV and MVS Case 2

- Complexity reduction
  1. Stage-varying Q matrix
  2. Nominal MPC complexity
- Stability
  1. No need for terminal cost
  2. Decrease in variance
- Disturbance rejection with larger Q matrix
  1. Modern control: high gain controller
  2. Classical control: higher bandwidth
- Generalization to different cases
  1. Even p.d.f. (MV case)
  2. Reference tracking - (nominally) time varying systems
  3. Initial condition mismatch $\rightarrow$ disturbance at one time instant.

Moment based MPC - Model Mismatch

- Mean case - Different nominal MPC problem

$$x_{k+1} = A(\delta_k)x_k + B(\gamma_k)u_k; \quad J^M(x_k) = \mathbb{E}\left\{\xi_k^\top \tilde{H}^{N_p}(\delta,\gamma)\xi_k\right\},$$
$$y_k = Cx_k, \; x_{0|k} = x_k. \qquad = \xi_k^\top \mathbb{E}\left\{\tilde{H}^{N_p}(\delta,\gamma)\right\}\xi_k,$$

1. Both Q and R varies over prediction stages
2. Explicitly available stage-wise backwards accumulation

$$Q^M_{j-1,\delta} \succ Q^M_{j,\delta}, \qquad R^M_{j-1,\delta} \succ R^M_{j,\delta}$$

3. Depending on the bounds of δ and γ:
   - Higher gain controller - Increase bandwidth - Disturbance rejection
   - Lower gain controller - Decrease bandwidth - Low freq. robustness

- Problem: Exponentially increasing number of terms in $Q^M_{j,\delta}$ or $R^M_{j,\delta}$

Technische Universiteit Eindhoven University of Technology

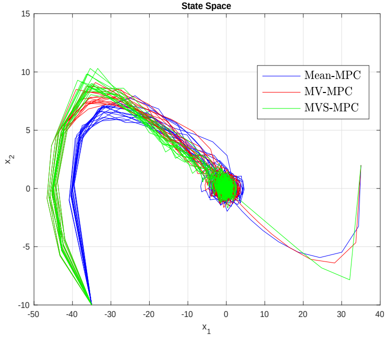
Simulation Examples - Simple LTI System

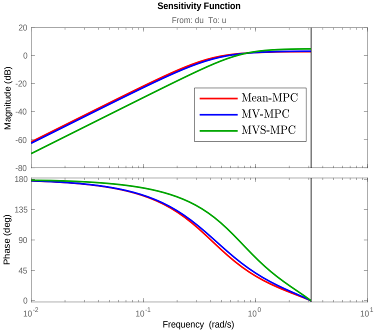
$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} u_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_k$$

$$y_k = Cx_k, \; x_0 = \left\{ \begin{bmatrix} 35 \\ 2 \end{bmatrix}, \begin{bmatrix} -35 \\ -10 \end{bmatrix} \right\}$$

$$\begin{aligned} J &= Quad(x, u, Q, R, N_p) \\ J^M &= \mathbb{E}\{J\} \\ J^{MV} &= \mathbb{E}\{J\} + \lambda_v \mathbb{D}\{J\} \\ J^{MVS} &= J^{MV} + \lambda_s \mathbb{S}\{J\} \end{aligned}$$

$$\lambda_v = 10, \quad \lambda_s = 1000, \quad N_p = 10$$





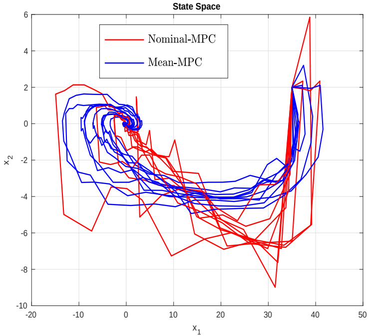
Technische Universiteit Eindhoven University of Technology

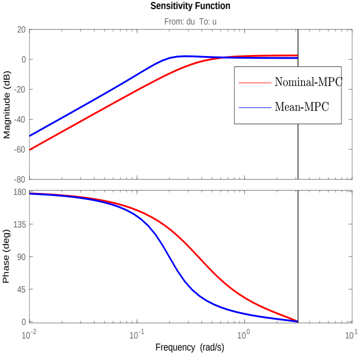
Simulation Examples - Model mismatch

- Double integrator system: $A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B_0 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

$$A = A_0 + A_1\delta_k, \quad B = B_0 + B_1\gamma_k, \qquad A_1 = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$\delta_k \sim \mathcal{N}(0,1), \quad \gamma \sim \mathcal{N}(0,1) \qquad J^M = \mathbb{E}\{J\}, \quad N_p = 5$$





Conclusions

- Robust operation $\neq$ Worst-case approach
- Moment-based MPC
  1. Complexity - Nominal MPC problem if unconstrained
  2. Closed-loop performance - similar with classical control
- Incorporation of risk & deviation metrics into MPC formulation
  1. Economic MPC
  2. Convex approximations of chance constraints
  3. Different risk & deviation metrics - CVaR, Semi variance etc.

The End

Questions or requests about this presentation?

⟹ Send an email to m.b.saltik@tue.nl