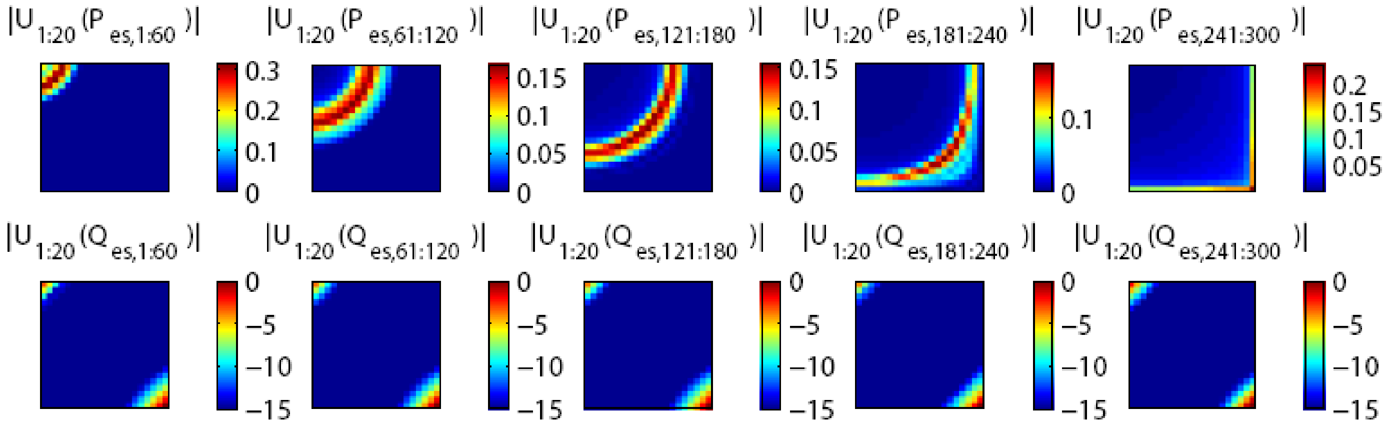
Controllability and Observability in Two-phase Porous Media Flow

Jorn Van Doren[1], Jan-Dirk Jansen[1], Paul Van Den Hof[1,2]

Delft University of Technology

Eindhoven University of Technology



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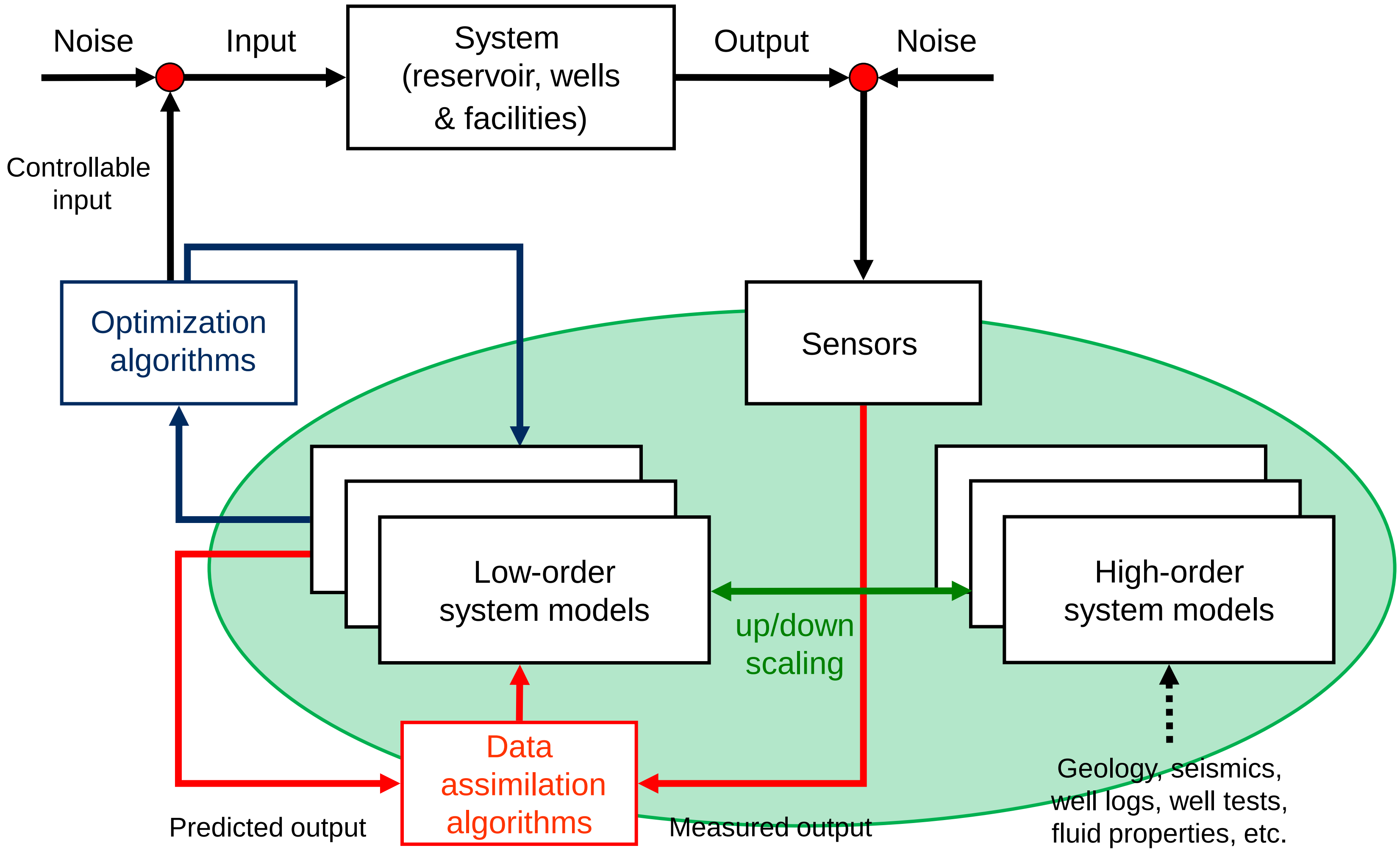
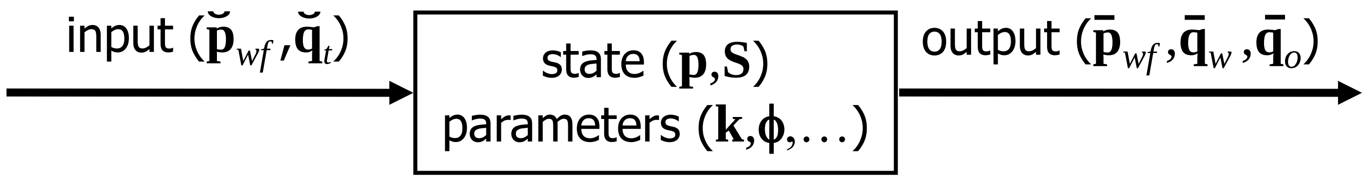
Closed-loop reservoir management



System-theoretical aspects

- Controllability of a dynamic system is the ability to influence the states through manipulation of the inputs.
- Observability of a dynamic system is the ability to determine the states through observation of the outputs.
- Identifiability of a dynamic system is the ability to determine the parameters from the input-output behavior.

System model



- Well-defined theory for linear systems. More difficult for nonlinear ones.

Single-phase flow, linear dynamics

$$\mathbf{u} = \begin{bmatrix} \breve{\mathbf{p}}_w^T & \breve{\mathbf{q}}_w^T \end{bmatrix}^T \qquad \mathbf{y} = \begin{bmatrix} \bar{\mathbf{p}}_w^T & \bar{\mathbf{q}}_w^T \end{bmatrix}^T$$

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k \end{aligned} \qquad \mathbf{x} = \mathbf{p}$$

$$\dot{\mathbf{p}} = \left[\mathbf{V}^{-1}(-\mathbf{T} - \mathbf{J}_p)\right]\mathbf{p} + \left[\mathbf{V}^{-1}(\mathbf{I}_q + \mathbf{J}_p)\mathbf{L}_{qu}\right]\mathbf{u}$$

$$\mathbf{y} = \left[\mathbf{L}_{yp}(\mathbf{I}_q + \mathbf{J}_p)\right]\mathbf{p} + \left[\mathbf{L}_{yp}(\mathbf{J}_p + \mathbf{J}_q)\mathbf{L}_{qu}\right]\mathbf{u}$$

Dynamics completely captured by $\Sigma = \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$

Observability & controllability

Which states are easiest to reach (i.e. are most controllable)?

$$\mathbf{W}_{\mathcal{C}} = \sum_{k=0}^{\infty} \mathbf{A}^k \mathbf{B}\mathbf{B}^T (\mathbf{A}^T)^k$$

Controllability Gramian

Which states give strongest output (i.e. are most observable)?

$$\mathbf{W}_{\mathcal{O}} = \sum_{k=0}^{\infty} (\mathbf{A}^T)^k \mathbf{C}^T \mathbf{C} \mathbf{A}^k$$

Observability Gramian

$$\left.\begin{aligned} \mathbf{A}\mathbf{W}_{\mathcal{C}}\mathbf{A}^{\mathrm{T}} - \mathbf{W}_{\mathcal{C}} + \mathbf{B}\mathbf{B}^{\mathrm{T}} &= \mathbf{0} \\ \mathbf{A}^{\mathrm{T}}\mathbf{W}_{\mathcal{O}}\mathbf{A} - \mathbf{W}_{\mathcal{O}} + \mathbf{C}^{\mathrm{T}}\mathbf{C} &= \mathbf{0} \end{aligned}\right\}$$ Lyapunov equations

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Observability Gramian

'Energy' interpretation:

$$E_{con}(\mathbf{x}) = \mathbf{x}^T \mathbf{W}_{\mathcal{C}}^{-1} \mathbf{x}$$

$$E_{obs}(\mathbf{x}) = \mathbf{x}^T \mathbf{W}_{\mathcal{O}} \mathbf{x}$$

Balanced input and output

- In a balanced realization the controllability and observability Gramians are equal and diagonal. We define a similarity transformation $\mathbf{T}$ such that:

$$\bar{\mathbf{x}} = \mathbf{T}\mathbf{x}$$

$$\dot{\bar{\mathbf{x}}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\bar{\mathbf{x}} + \mathbf{T}\mathbf{B}\mathbf{u}, \ \cdots$$

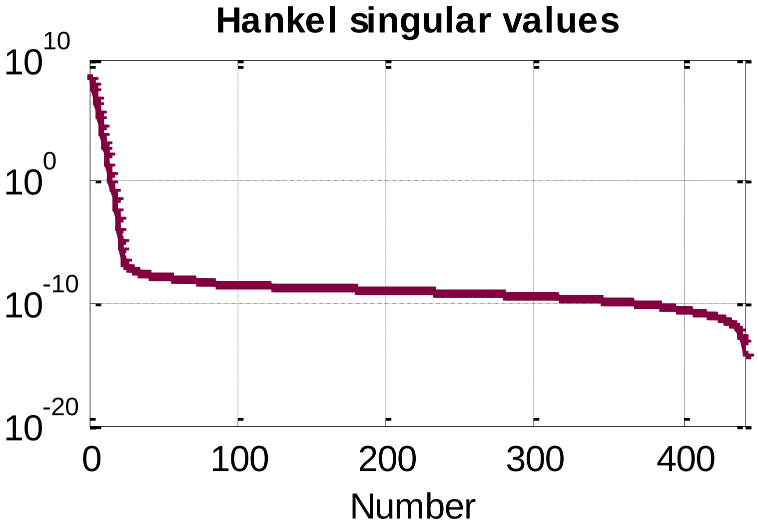
$$\bar{\mathbf{W}}_{\mathcal{C}} = \bar{\mathbf{W}}_{\mathcal{O}} = diag(\boldsymbol{\sigma}_h) = \begin{bmatrix} \sigma_{h,1} & 0 & \cdots & 0 \\ 0 & \sigma_{h,1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{h,n} \end{bmatrix}$$

- Balancing equalizes the input-to-state and state-to-output energies, such that the states that are difficult to reach are precisely those that are difficult to observe.

Hankel singular values (HSV)

- Mathematically

$$\sigma_H = \sqrt{\lambda_i(\mathbf{W}_{\mathcal{C}}\mathbf{W}_{\mathcal{O}})}$$

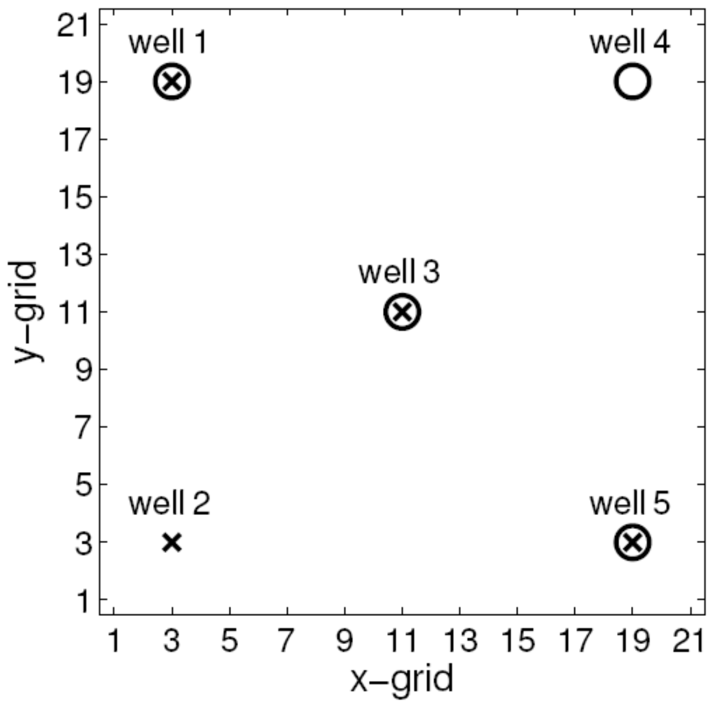


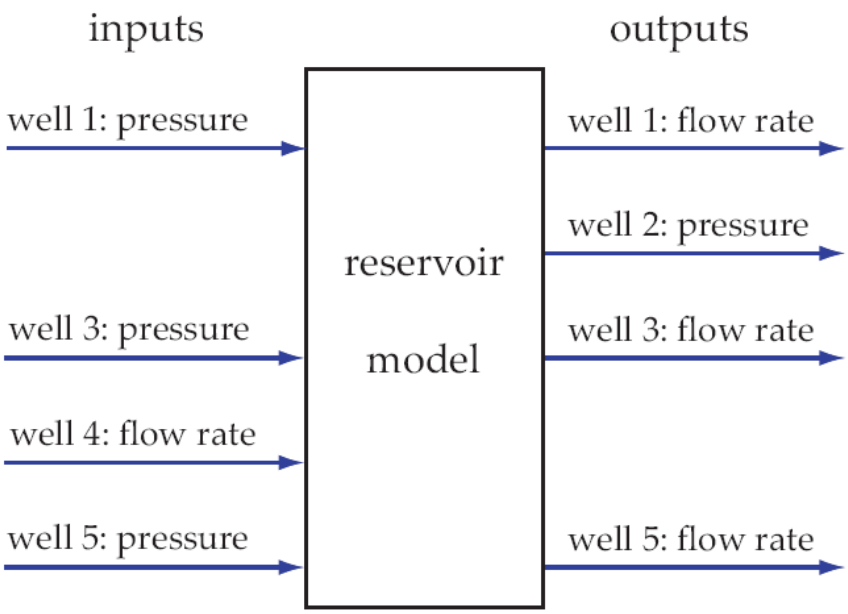
- Physically
  - They give a measure of the energy of individual states.
  - They measure the combined controllability and observability of a system.
  - They give a measure of contribution of each state to the input/output behavior.

- Invariant to internal system description (unlike $\mathbf{W}_{\mathcal{C}}$, $\mathbf{W}_{\mathcal{O}}$)

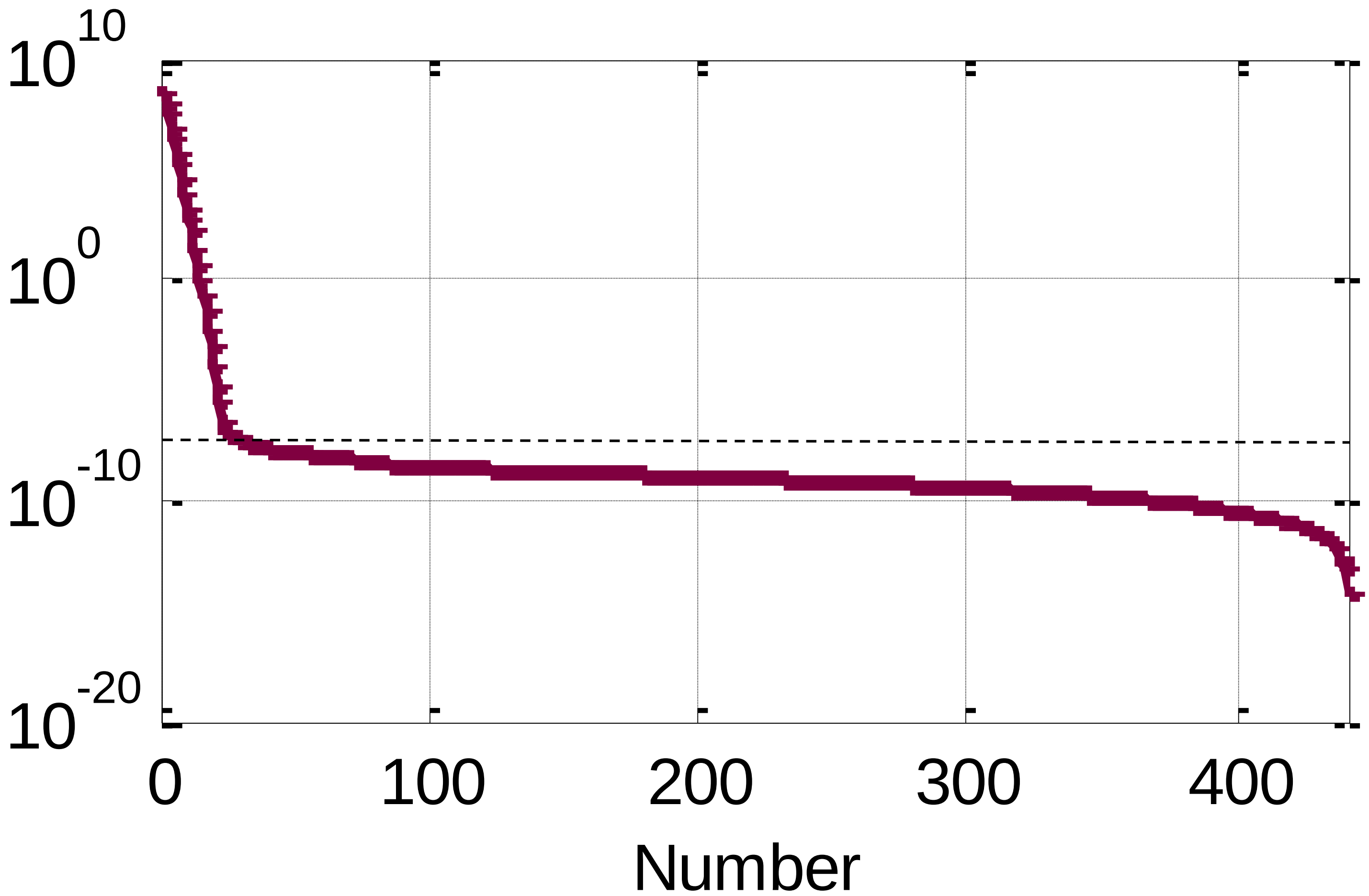
Example: 5-well toy model

o – flow meter
x – pressure gauge



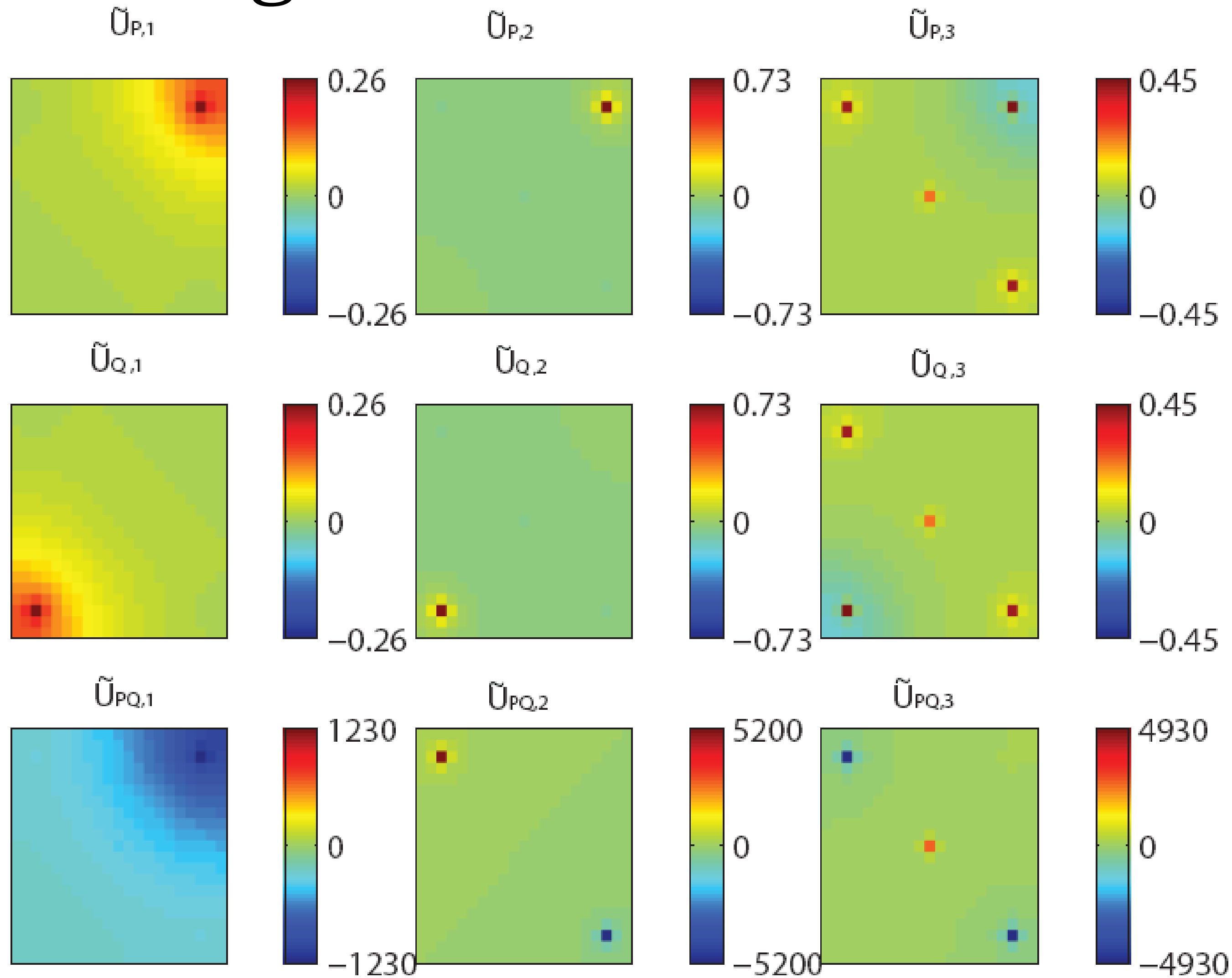


Example: Hankel singular values

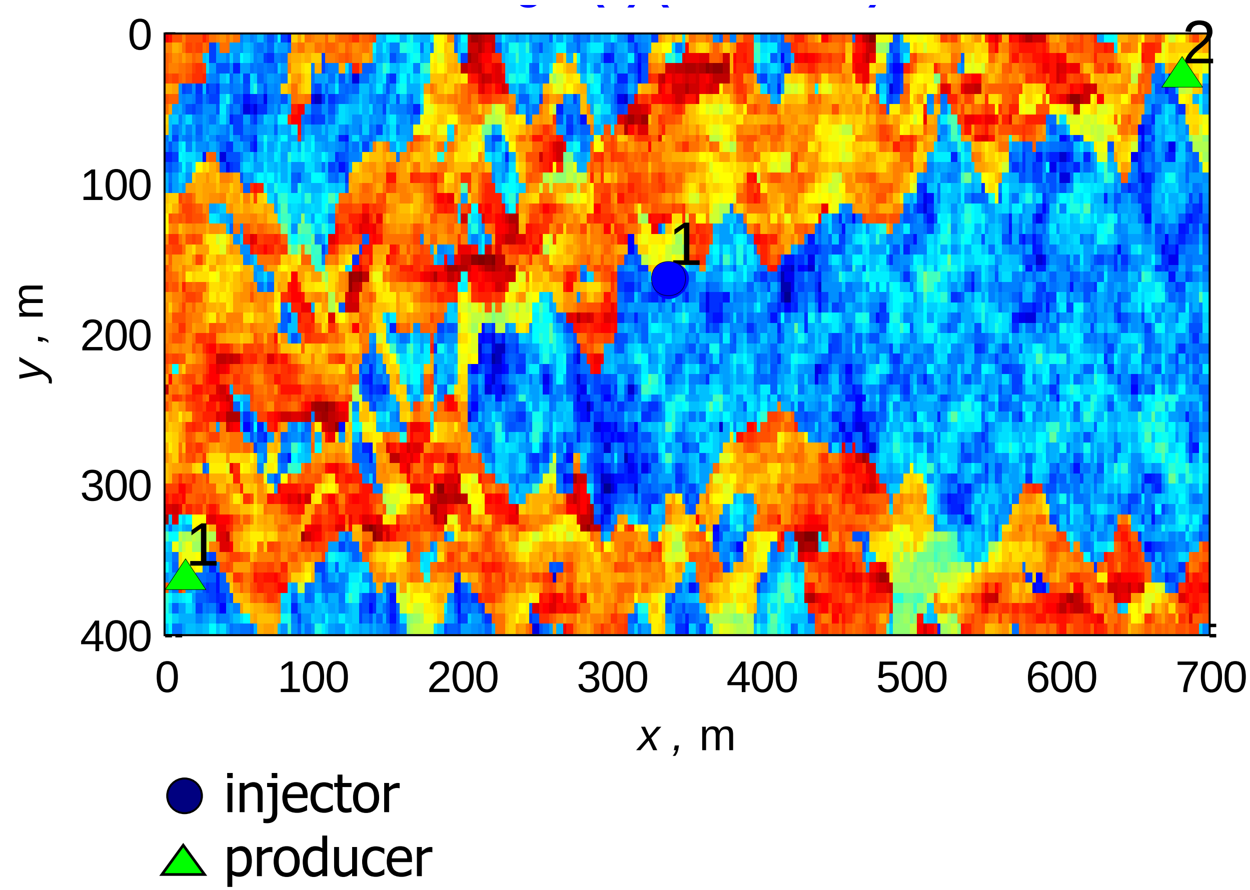


First three singular vectors

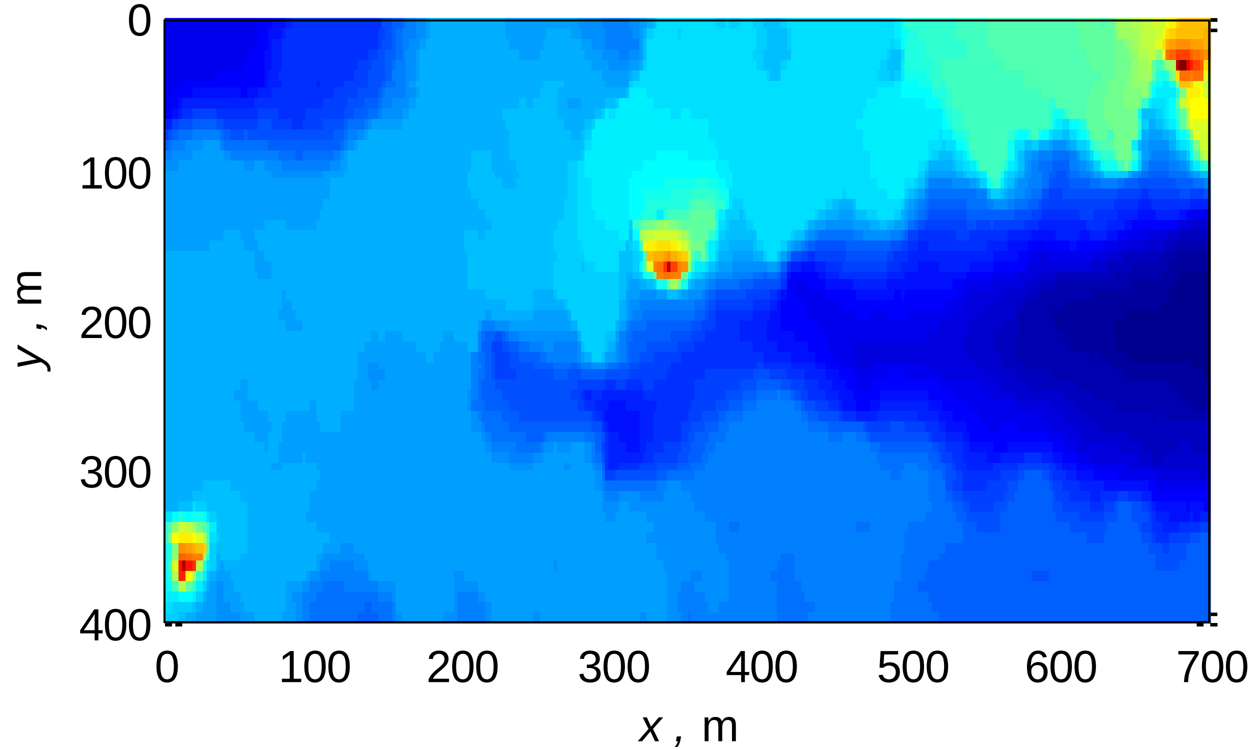
- Controll.
- Observ.
- Balanced



Top layer SPE 10

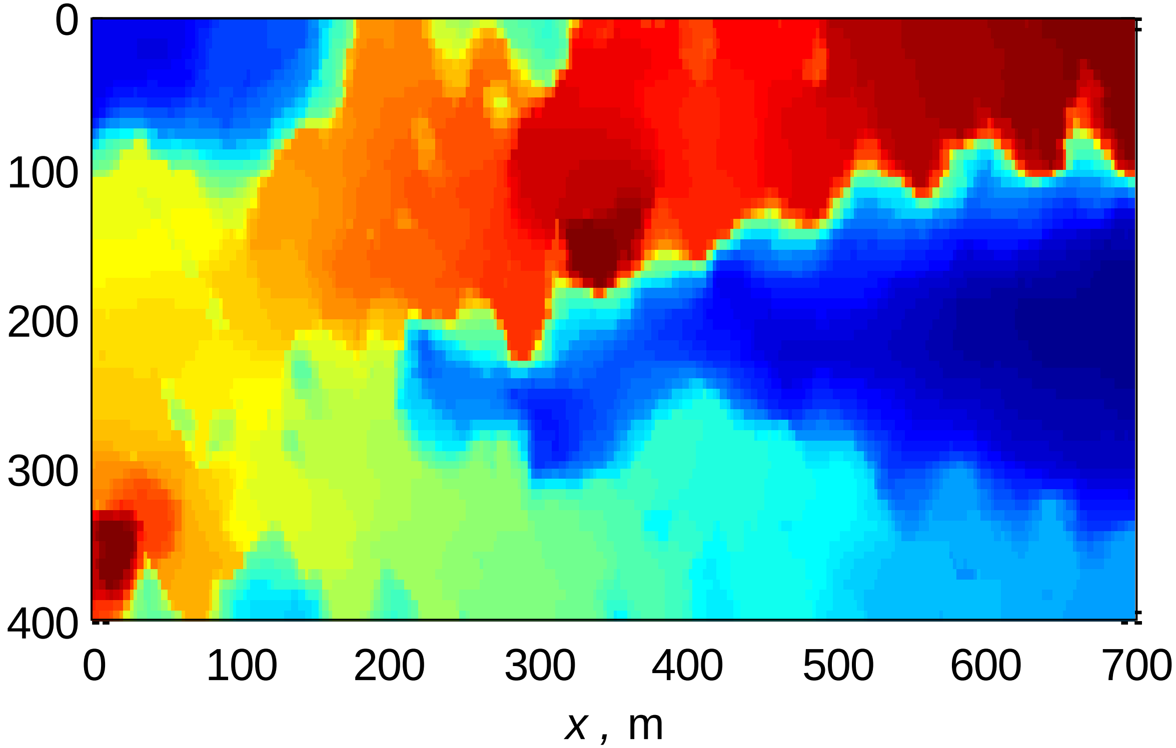


Combined observability/controllability



Singular value-weighted first 52 log 10 Hankel singular vectors

Combined observability/controllability



Singular value-weighted first 52 'grid block importance maps'

Two-phase flow, nonlinear dynamics

$$\mathbf{u} = \begin{bmatrix} \breve{\mathbf{p}}_w^T & \breve{\mathbf{q}}_w^T \end{bmatrix}^T \qquad \mathbf{y} = \begin{bmatrix} \bar{\mathbf{p}}_w^T & \bar{\mathbf{q}}_w^T \end{bmatrix}^T$$

$$\mathbf{x}_k = \mathbf{f}_k\left(\mathbf{u}_k, \mathbf{x}_{k-1}\right)$$
$$\mathbf{y}_k = \mathbf{h}_k\left(\mathbf{u}_k, \mathbf{x}_k\right)$$

$$\mathbf{x}_{k+1} = \bar{\mathbf{A}}_k \mathbf{x}_k + \bar{\mathbf{B}}_k \mathbf{u}_k$$
$$\mathbf{y}_k = \bar{\mathbf{C}}_k \mathbf{x}_k + \bar{\mathbf{D}}_k \mathbf{u}_k$$

$$\mathbf{x} = \begin{bmatrix} \mathbf{p}^T & \mathbf{s}^T \end{bmatrix}^T$$

$$\bar{\mathbf{A}}_k = \frac{\partial \mathbf{f}_k}{\partial \mathbf{x}_k}, \quad \bar{\mathbf{B}}_k = \frac{\partial \mathbf{f}_k}{\partial \mathbf{u}_k}$$

$$\bar{\mathbf{C}}_k = \frac{\partial \mathbf{g}_k}{\partial \mathbf{x}_k}, \quad \bar{\mathbf{D}}_k = \frac{\partial \mathbf{g}_k}{\partial \mathbf{u}_k}$$

Local dynamics (linearized along trajectory) captured by $\left\{\bar{\mathbf{A}}_{1:K}, \bar{\mathbf{B}}_{1:K}, \bar{\mathbf{C}}_{1:K}, \bar{\mathbf{D}}_{1:K}\right\}$

Empirical controllability Gramian

$$\mathbf{W}_c = \sum_{k=0}^{\infty} \mathbf{A}^k \mathbf{B}\mathbf{B}^T (\mathbf{A}^T)^k$$ Ordinary controllability Gramian

$$\mathbf{x}_1 = \mathbf{A}\mathbf{x}_0 + \mathbf{B}\mathbf{u}_1$$

$$\mathbf{x}_2 = \mathbf{A}^2\mathbf{x}_0 + \mathbf{A}\mathbf{B}\mathbf{u}_1 + \mathbf{B}\mathbf{u}_2$$

$$\mathbf{x}_3 = \mathbf{A}^3\mathbf{x}_0 + \mathbf{A}^2\mathbf{B}\mathbf{u}_1 + \mathbf{A}\mathbf{B}\mathbf{u}_2 + \mathbf{B}\mathbf{u}_3$$

etc.

For unit impulse input: $\mathbf{x}_0 = \mathbf{0}$, $\mathbf{u}_1 = \mathbf{1}$, and $\mathbf{u}_2 = \ldots = \mathbf{0}$

$$\mathbf{x}'_k = \mathbf{A}^{k-1}\mathbf{B}$$

$$\mathbf{W}_c = \sum_{k=1}^{\infty} \mathbf{x}'_k \left(\mathbf{x}'_k\right)^T$$

Empirical controllability Gramian (2)

$$\mathbf{W}_{\mathcal{C}} = \sum_{k=1}^{\infty} \mathbf{x}'_k \left(\mathbf{x}'_k\right)^T$$

Ordinary controllability Gramian

$$\bar{\mathbf{W}}_{\mathcal{C}} = \begin{bmatrix} \mathbf{x}_1\mathbf{x}_1 & \mathbf{x}_1\mathbf{x}_2 & \cdots & \mathbf{x}_1\mathbf{x}_K \\ \mathbf{x}_2\mathbf{x}_1 & \mathbf{x}_2\mathbf{x}_2 & \cdots & \mathbf{x}_2\mathbf{x}_K \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_K\mathbf{x}_0 & \mathbf{x}_K\mathbf{x}_2 & \cdots & \mathbf{x}_K\mathbf{x}_K \end{bmatrix}$$

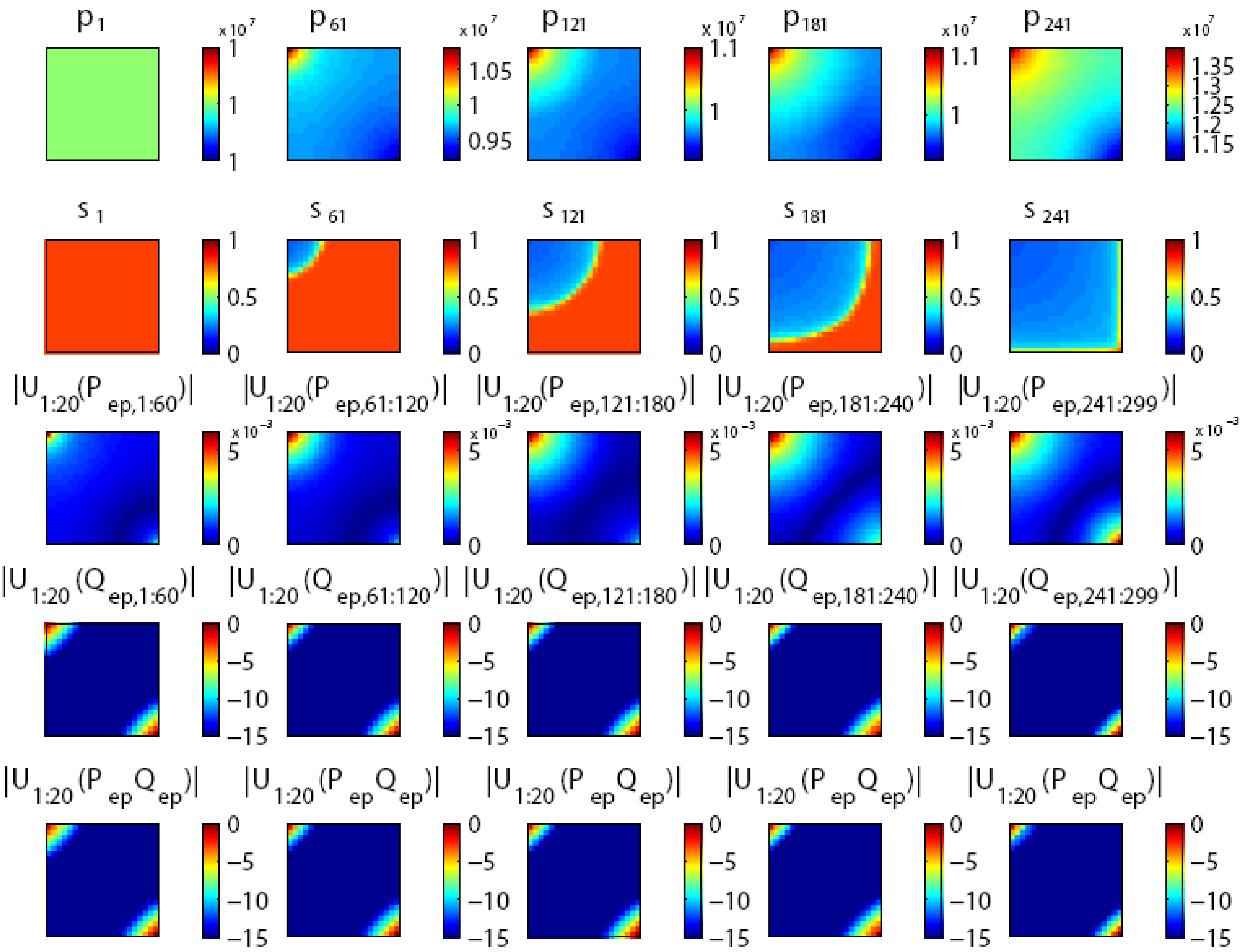
$$= \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_K \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_K \end{bmatrix}}_{\text{snapshot matrix}} = \mathbf{X}\mathbf{X}^T$$

When repeated for many inputs: Empirical Gramian (Lall et al., 2002)

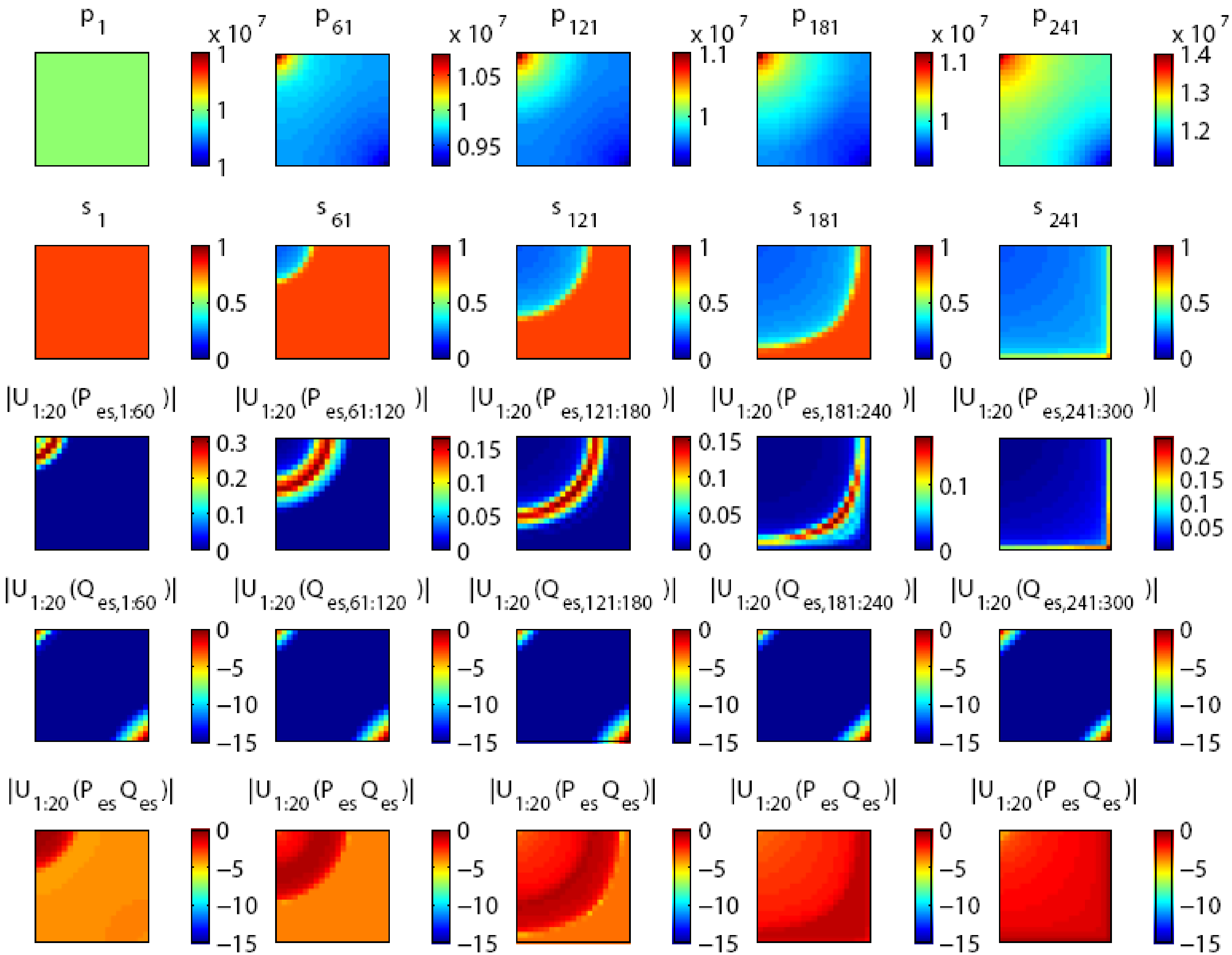
Empirical Gramians, balancing & POD

- Empirical Gramian is formally only defined for unit impulse input (Lall et al., 2002).
- Can be made more robust by repeating for different inputs or parameters ('empirical covariance matrices'; Hahn et al., 2003)
- Recall: Covariance matrix $\mathbf{C} = \sum_{k=1}^{K} \left(\mathbf{x}_k - \bar{\mathbf{x}}\right)\left(\mathbf{x}_k - \bar{\mathbf{x}}\right)^T \Big/ \left(K-1\right)$
- Remains only valid locally, i.e. around given state trajectory
- Similar expression for empirical observability Gramian. Alternatively defined in terms of snapshots of adjoint simulation
- Empirical balanced realization can be obtained by combining forward and adjoint snapshots (Willcox, 2002; Antoulas, 2005)
- SVD of $\mathbf{X}$ or eigenvalue decomposition of $\mathbf{X}\mathbf{X}^T$ (or $\mathbf{X}^T\mathbf{X}$) gives basis vectors for reduced-order modeling: Proper Orthogonal Decomposition (POD)

Controllab./observab. pressures



Controllab./observab. saturations



Identifiability

- Structural identifiability: can we determine parameters from ideal (persistently exciting) inputs and perfect measurements?
- Theoretical maximum number of identifiable parameters:

$$N_{\max} = \left(N_u N_y\right) n_{eff} + N_u N_y$$

- Practical identifiability: can we determine parameters from a given input-output sequence?
- Closely related to (dimensionless) sensitivity matrix (Al Reynolds et al.):

$$\mathbf{S} = \frac{\partial \mathbf{h}}{\partial \boldsymbol{\theta}} \quad \text{or} \quad \tilde{\mathbf{S}} = \mathbf{P}_d^{-\frac{1}{2}} \mathbf{S} \mathbf{P}_m^{\frac{1}{2}}$$

- If you think observability is poor, try identifiability!
- For more; see dissertation Jorn van Doren (2010)

References

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