Likelihood Based Uncertainty Bounding in Prediction Error Identification using ARX models

A Simulation Study

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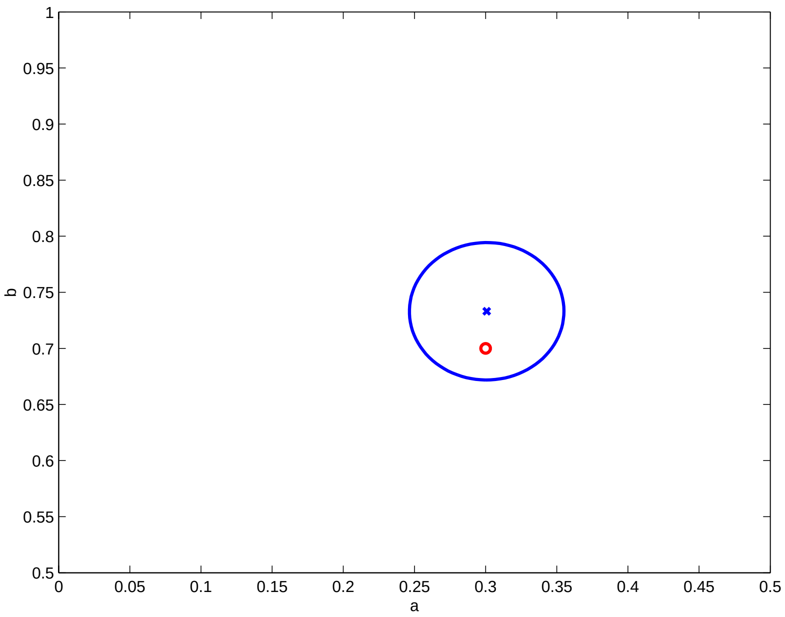
Confidence regions

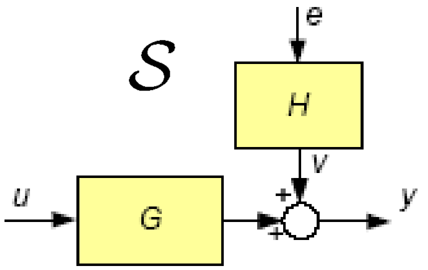
A 95% confidence region is a region in parameter space that attempts to cover the "true" parameter with probability 0.95.



Outline

- Data generating system and predictor model
- Statistical inference in PE identification
- Confidence regions and hypothesis tests
- Test statistics
- ARX modelling
- Simulation results
- Conclusions

Data generating system and predictor model



$$y(t) = G_0(q)u(t) + H_0(q)e(t)$$

- input (deterministic): $u^N = \{u(t)\}_{t=1,\ldots,N}$
- output (stochastic): $y^N = \{y(t)\}_{t=1,\ldots,N}$
- $e(t)$ Gaussian white noise

Predictor model set ($\mathcal{S} \in \mathcal{M}$):

$$\mathcal{M} \;:=\; \{(G(q,\theta), H(q,\theta))|\theta \in \Theta \subset \mathbb{R}^n\}$$

$$\exists\theta_0 \quad \ni \quad G(q,\theta_0) = G_0(q) \text{ and } H(q,\theta_0) = H_0(q)$$

Statistical inference

One-step ahead predictor:

$$\hat{y}(t|t-1;\theta) = H^{-1}(q,\theta)G(q,\theta)u(t) + [1 - H^{-1}(q,\theta)]y(t)$$

Prediction errors:

$$\begin{aligned} \epsilon(t,\theta) &= y(t) - \hat{y}(t|t-1;\theta) \\ \epsilon(t,\theta_0) &= e(t) \sim \mathcal{N}(0,\sigma^2) \end{aligned}$$

Joint probability density function of y^N:

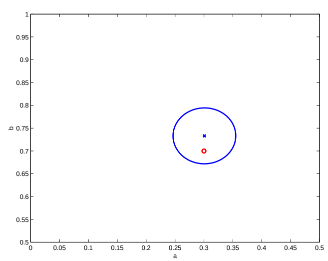
$$f_y(y^N;\theta_0) = \prod_{t=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2} (\underbrace{y(t) - \hat{y}(t|t-1;\theta_0)}_{\epsilon(t,\theta_0)})^2 \right]$$

Statistical inference

$$S(\theta) \;=\; \frac{\partial \log f_y(y^N;\theta)}{\partial\theta} = \frac{-N}{2\sigma^2}\frac{\partial V_N(\theta)}{\partial\theta}$$

$$F(\theta) \;=\; -\mathbb{E}\left[\frac{\partial^2 \log f_y(y^N;\theta)}{\partial\theta^2}\right] = \frac{N}{2\sigma^2}\mathbb{E}\left[\frac{\partial^2 V_N(\theta)}{\partial\theta^2}\right]$$

$$\hat{\theta}_N = \arg\max_{\theta} f_y(\theta; y^N) = \arg\min_{\theta} \underbrace{\frac{1}{N}\sum_{t=1}^{N}\epsilon(t,\theta)^2}_{V_N(\theta)}$$



Confidence region for θ_0?

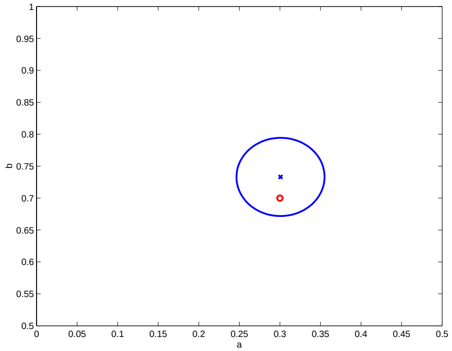
Construction of confidence regions

$$H_0 : \theta_0 = \theta$$
$$H_1 : \theta_0 \neq \theta$$

To test H_0 against H_1 at significance level α, choose a test statistic $T(y^N, \theta)$ with a known distribution under H_0 and decide H_1 if:

$$T(y^N, \theta) > c(\alpha)$$

with $Pr[T(y^N, \theta) > c(\alpha)] = \alpha$ under H_0.



$100(1-\alpha)\%$ confidence region for θ_0:

$$\left\{\theta | T(y^N, \theta) \leq c(\alpha)\right\}$$

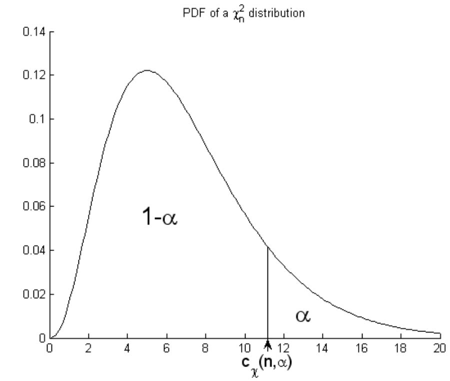
Test statistics

$\hat{\theta}_N \overset{as.}{\sim} \mathcal{N}(\theta_0, F^{-1}(\theta_0))$	$(\hat{\theta}_N - \theta_0)^T F(\theta_0)(\hat{\theta}_N - \theta_0) \overset{as.}{\sim} \chi^2_n$
$S(\theta_0) \overset{as.}{\sim} \mathcal{N}(0, F(\theta_0))$	$S(\theta_0)^T F^{-1}(\theta_0) S(\theta_0) \overset{as.}{\sim} \chi^2_n$
$-2\log \frac{f_y(\theta_0; y^N)}{f_y(\hat{\theta}_N; y^N)} = \frac{N}{\sigma^2}(V_N(\theta_0) - V_N(\hat{\theta}_N)) \overset{as.}{\sim} \chi^2_n$:	

$$\left.\begin{aligned} T_T(y^N, \theta) &= (\hat{\theta}_N - \theta)^T F(\theta)(\hat{\theta}_N - \theta) \\ T_W(y^N, \theta) &= (\hat{\theta}_N - \theta)^T F(\hat{\theta}_N)(\hat{\theta}_N - \theta) \\ T_R(y^N, \theta) &= S(\theta)^T F^{-1}(\theta) S(\theta) \\ T_{LR}(y^N, \theta) &= \frac{N}{\sigma^2}(V_N(\theta) - V_N(\hat{\theta}_N)) \end{aligned}\right\} \overset{as.}{\sim} \chi^2_n \text{ under } H_0$$

$(100 - \alpha)\%$ confidence regions for θ_0:

$$\left\{\theta \,|\, T(y^N, \theta) \leq c_\chi(n, \alpha)\right\}$$



ARX modelling

$$
\begin{aligned}
G(q,\theta) &= \frac{q^{-n_k}(b_0 + b_1 q^{-1} + \cdots + b_{n_b-1} q^{-n_b+1})}{1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}} \\
H(q,\theta) &= \frac{1}{1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a})} \\
\theta^T &= [a_1 \cdots a_{n_a} b_0 \cdots b_{n_b-1}] \\
\varphi^T(t) &= [-y(t-1) \cdots - y(t-n_a) u(t-n_k) \cdots u(t-n_k-n_b+1)] \\
\Phi &= \begin{pmatrix} \varphi^T(1) \\ \vdots \\ \varphi^T(N) \end{pmatrix} \\
\mathbf{y}^T &= [y(1) \cdots y(N)] \\
& N(V_N(\theta) - V_N(\hat{\theta}_N)) = (\theta - \hat{\theta}_N)^T \Phi \Phi^T (\theta - \hat{\theta}_N) \\
S(\theta) &= \frac{1}{\sigma^2} \Phi^T (\mathbf{y} - \Phi\theta); \qquad F(\theta_0) = \frac{1}{\sigma^2} \underbrace{\mathbb{E}\left[\Phi^T \Phi\right]}_{R(\theta_0)}
\end{aligned}
$$

Confidence regions for ARX parameters

$100(1-\alpha)\%$ confidence regions for θ_0:
$$\left\{\theta | (\theta - \widehat{\theta}_N)^T X (\theta - \widehat{\theta}_N) \leq \sigma^2 c_\chi(n,\alpha)\right\}$$

Test statistic	X
T_T	$R(\theta)$
T_W (Wald)	$R(\widehat{\theta}_N)$
T_R (Rao)	$\Phi^T \Phi R^{-1}(\theta) \Phi^T \Phi$
T_{LR} (Likelihood Ratio)	$\Phi^T \Phi$

$$R(\theta_0) = \mathbb{E}\left[\Phi^T \Phi\right].$$

Result from asymptotic theory (θ_0 known)

Asymptotically valid expression for $cov(\widehat{\theta}_N)$:

$$P_{\theta_0} = \frac{\sigma^2}{N}\left(\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\mathbb{E}_{ue}\left[\varphi(t)\varphi^T(t)\right]\right)^{-1}$$

$100(1-\alpha)\%$ confidence region for θ_0

$$\left\{\theta | (\widehat{\theta}_N - \theta)^T P_{\theta_0}^{-1}(\widehat{\theta}_N - \theta) \leq c_\chi(n,\alpha)\right\}$$

Simulation experiment

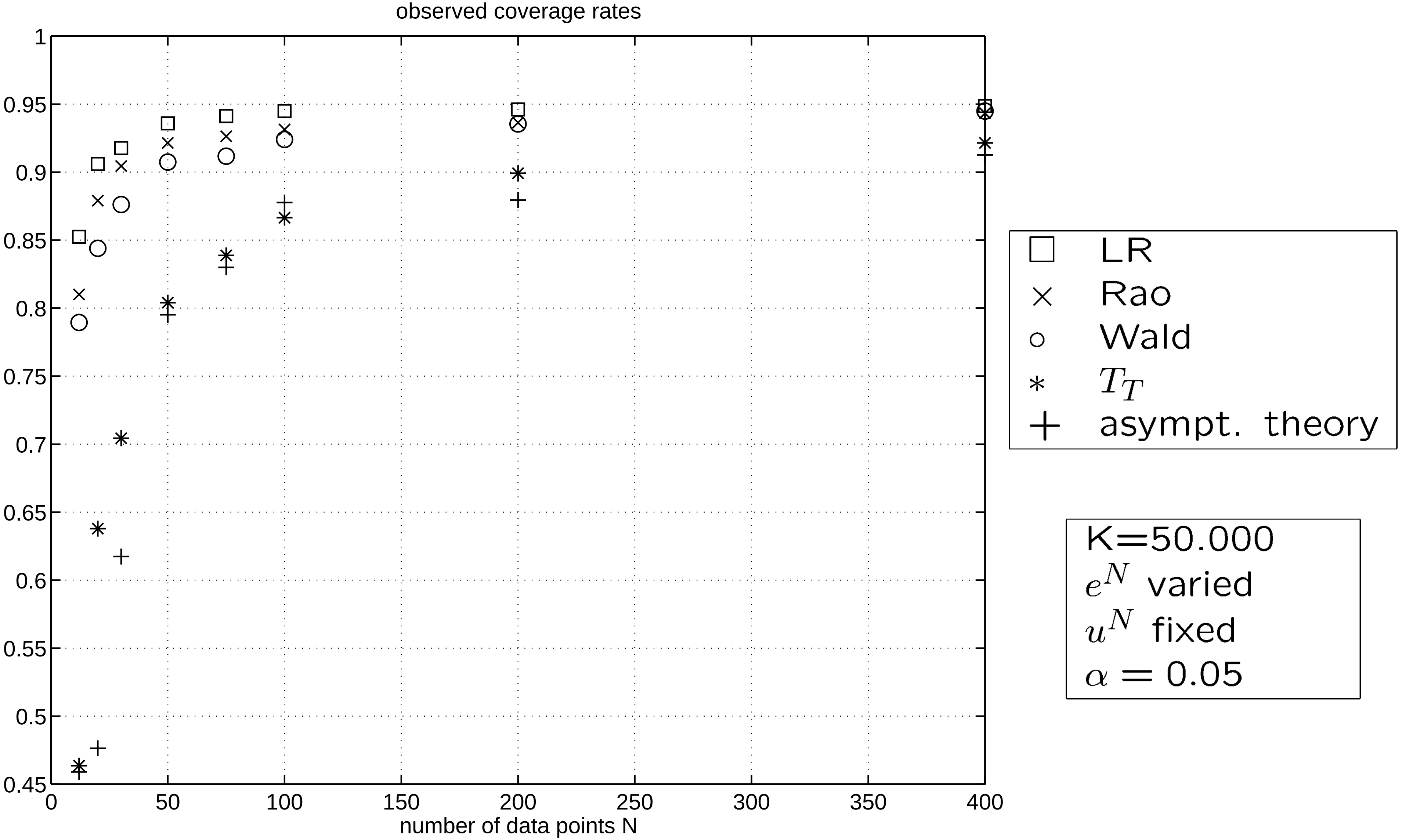
50.000 data sets (y^N, u^N)

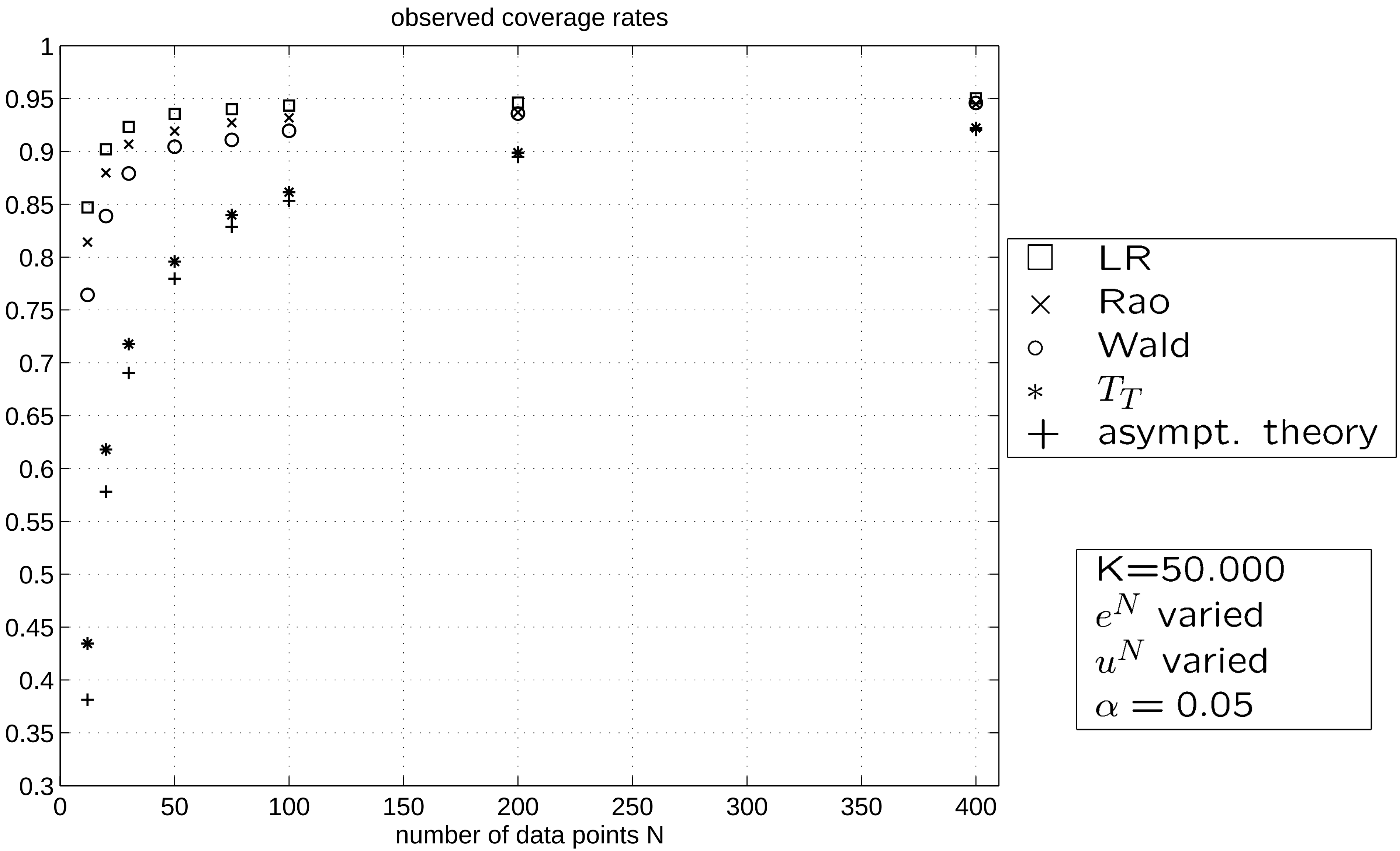
Data generating system $\mathcal{S}$:

$$y(t) - 1.5578y(t-1) + 0.5769y(t-2) = \\ 0.1047u(t-1) + 0.0872u(t-2) + e(t)$$

u^N (known) and e^N (unknown) realizations of Gaussian white noise proces, $\sigma_u^2 = 1, \sigma_e^2 = 0.5$.

For each data set, the model was identified and it was recorded if the constructed confidence regions contained θ_0.


observed coverage rates
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
0.45
0
50
100
150
200
250
300
350
400
number of data points N
LR
Rao
Wald
T_T
asympt. theory
K=50.000
e^N varied
u^N fixed
$\alpha = 0.05$


observed coverage rates
number of data points N
LR
Rao
Wald
T_T
asympt. theory
K=50.000
e^N varied
u^N varied
$\alpha = 0.05$

Conclusions

$100(1-\alpha)\%$ confidence regions for ARX:

$$\left\{\theta | (\theta - \widehat{\theta}_N)^T X (\theta - \widehat{\theta}_N) \leq \sigma^2 c_\chi(n, \alpha)\right\}$$

- Confidence regions that incorporate information on the particular noise realization e^N in X (via Φ or $\widehat{\theta}_N$) are most reliable.

- Confidence region based on LR test statistic ($X = \Phi^T\Phi$) performs best (in simulations).

ARX modelling

$$\varphi(t,\theta_0) = s_u(t,\theta_0) + s_e(t,\theta_0),$$

$$s_u(t,\theta_0) = \frac{\Lambda_G(q^{-1},\theta_0)}{H(q^{-1},\theta_0)}u(t); \qquad s_e(t,\theta_0) = \frac{\Lambda_H(q^{-1},\theta_0)}{H(q^{-1},\theta_0)}e(t),$$

with $\Lambda_G(q^{-1},\theta)$ and $\Lambda_H(q^{-1},\theta)$ the $n \times 1$ gradient vectors of the transfer function $G(q^{-1},\theta)$ and $H(q^{-1},\theta)$ with respect to θ, respectively.

$$R(\theta_0) = \sum_{t=1}^{N} s_u(t,\theta_0)s_u^T(t,\theta_0) + \sum_{t=1}^{N} \mathbb{E}\left[s_e(t,\theta_0)s_e^T(t,\theta_0)\right]$$

Result from asymptotic theory (θ_0 known)

Asymptotically valid expression for $cov(\widehat{\theta}_N)$:

$$P_{\theta_0} = \frac{\sigma^2}{2\pi N}\left(\int_{-\pi}^{\pi}\left(\Gamma_G(e^{i\omega},\theta_0)\Phi_u(\omega) + \Gamma_H(e^{i\omega},\theta_0)\sigma^2\right)d\omega\right)^{-1}$$

$$\Gamma_G(e^{i\omega},\theta_0) = \frac{\Lambda_G(e^{i\omega},\theta_0)\Lambda_G^*(e^{i\omega},\theta_0)}{H(e^{i\omega},\theta_0)H^*(e^{i\omega},\theta_0)},$$

$$\Gamma_H(e^{i\omega},\theta_0) = \frac{\Lambda_H(e^{i\omega},\theta_0)\Lambda_H^*(e^{i\omega},\theta_0)}{H(e^{i\omega},\theta_0)H^*(e^{i\omega},\theta_0)}$$

with $\Phi_u(\omega)$ the power spectrum of $u(t)$.

$$\theta_0 \in \left\{\theta | (\widehat{\theta}_N - \theta)^T P_{\theta_0}^{-1}(\widehat{\theta}_N - \theta) \leq c_\chi(n,\alpha)\right\} \quad \text{w.p. } 1-\alpha$$