Validity of the standard cross-correlation test for model structure validation

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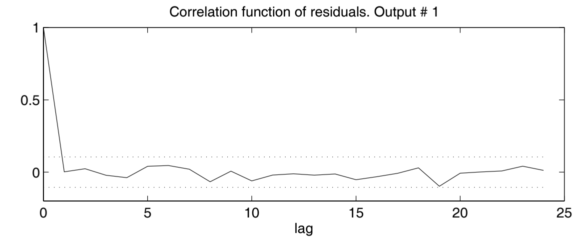
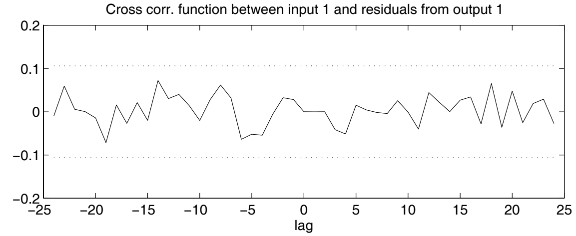
Validation tests in Matlab toolbox

$\hat{R}_{\varepsilon}(\tau)$



$\hat{R}_{\varepsilon u}(\tau)$



Contents

Model uncertainty bounding

Crucial in any application of identified models.

Uncertainty due to:

noise - finite data - limited complexity models

$\longrightarrow$ variance and bias

Approaches (in prediction error identification)

- Assume no bias ($G_0 \in \mathcal{G}$) and use analytical expressions for variance error
- Bound the bias (undermodelling) and exploit variance expressions (Goodwin et al, Hakvoort, De Vries)

First approach requires verification of the assumption
$\longrightarrow$ model validation

Question:

When a model passes the model validation test, is it justified to use analytical model uncertainty expressions (based on the *assumption* $G_0 \in \mathcal{G}$) to bound the model uncertainty?

If **YES:** model validation test is crucial for constructing uncertainty bounds
If **NO:** how to verify/justify the assumption $G_0 \in \mathcal{G}$?

Technical set-up

Data-generating system: $y(t) = G_0(u(t)) + v(t)$
with $v(t) = H_0(q)e(t)$ and $v(t)$ uncorrelated with $u(t)$.

Model class: $\mathcal{M}(\theta) : [G(q,\theta), H(q,\theta)]$

PE estimate: $\hat{\theta}_N = \arg\min_{\theta} \frac{1}{N}\sum_{t=1}^{N} \varepsilon^2(t,\theta)$

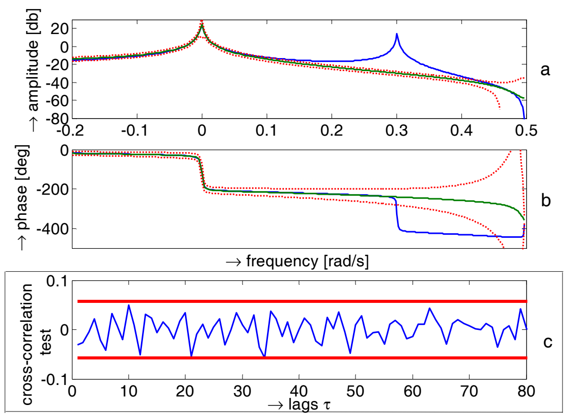
with

$$\varepsilon(t,\theta) = H^{-1}(q,\theta)(y(t) - G(q,\theta)u(t))$$

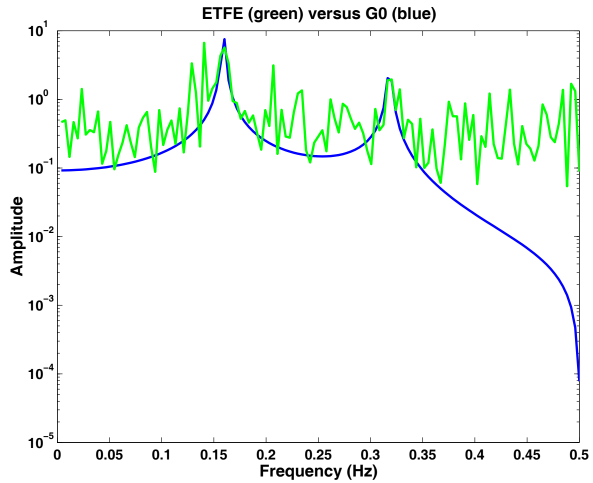
Standard **validation test** for $G(q,\hat{\theta}_N)$:
cross-correlation test $R_{\varepsilon u}(\tau) = 0$

Example with the standard test

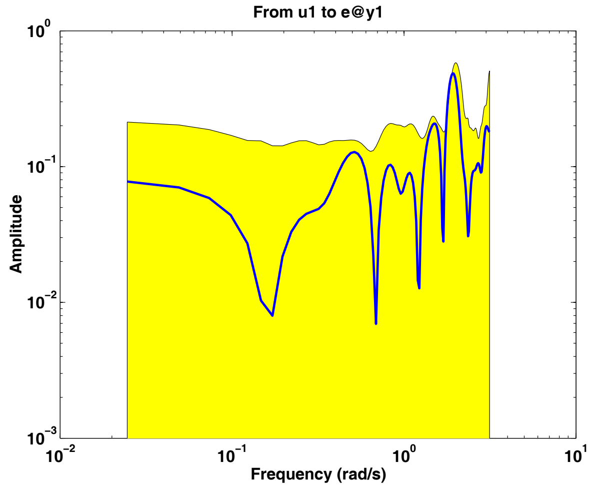
4th order system (blue); 2nd order OE model (green); u white



Wouldn't the second peak be clear from the ETFE?



Wouldn't the correlation in the residual appear in the f-domain?



Observation

Serious undermodelling is not always detected by the model validation test.
This leads to unreliable uncertainty bounds.

Analysis

$$\begin{aligned}\varepsilon(t,\hat\theta_N) &= H^{-1}(q,\hat\theta_N)[G_0(u(t)) - G(q,\hat\theta_N)u(t) + v(t)]\\ &= \underbrace{H^{-1}(q,\hat\theta_N)(G_0(u(t)) - G(q,\theta^*)u(t))}_{bias\ \beta_b} + \\ &\quad + \underbrace{H^{-1}(q,\hat\theta_N)(G(q,\theta^*)u(t) - G(q,\hat\theta_N)u(t))}_{variance\ \beta_v} + \\ &\quad + \underbrace{H^{-1}(q,\hat\theta_N)v(t)}_{noise\ \varepsilon_v}\end{aligned}$$

$$\varepsilon(t,\hat\theta_N) = \beta_b(t,\hat\theta_N,G_0,\theta^*) + \beta_v(t,\hat\theta_N,\theta^*) + \varepsilon_v(t,\hat\theta_N)$$

Philosophy behind a model structure validation test

$$\varepsilon(t,\hat\theta_N) = \beta_b(t,\hat\theta_N,G_0,\theta^*) + \beta_v(t,\hat\theta_N,\theta^*) + \varepsilon_v(t,\hat\theta_N)$$

Test the hypothesis

$$\Upsilon_0 : G_0 = G(q,\theta^*) \in \mathcal{M}(\theta), \quad i.e.\ \beta_b = 0.$$

Standard test: assume $\hat\theta_N = \theta^*$ ($\beta_v = 0$), and check whether

$$\hat R_{\varepsilon u}(\tau) = \hat R_{\varepsilon_v u}(\tau) \in \mathcal{N}(0, P/N)$$

with $P = \sum_{\tau=-\infty}^{\infty} R_u(\tau) R_{\varepsilon_v}(\tau)$.

Philosophy behind a model structure validation test

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If P is estimated by $\hat{P} = \sum_{\tau=-\infty}^{\infty} R_u(\tau)\hat{R}_{\varepsilon}(\tau)$, it will be overestimated if $\beta_b \neq 0$.

Observations

- Standard test uses statistical bounds that are dependent on the assumption $\theta^* = \theta_0$.
- The test is only valid under the same assumption that it intends to validate.

Actions for improving test

$$\varepsilon(t,\hat{\theta}_N) = \beta_b(t,\hat{\theta}_N,G_0,\theta^*) + \beta_v(t,\hat{\theta}_N,\theta^*) + \varepsilon_v(t,\hat{\theta}_N)$$

- **(1)** Signal ε_v should be isolated from data to estimate the appropriate bound P
- **(2)** Test should be performed vector-valued (results not independent of τ)
- Hypothesis test should verify whether

$$\varepsilon(\hat{\theta}_N) = \beta_v(\hat{\theta}_N,\theta^*) + \varepsilon_v$$

(taking account of β_v)

(1) Isolating noise model from undermodelling

- Apply periodic input data and average out undermodelling term;
- Identify a noise model from data with $u = 0$.
- Identify high order (auxiliary) plant model to be used as carrier (no bias), and estimate a time series model for modelling v and $R_{\varepsilon_v}(\tau)$
  (cf. bootstrap method of Tjärnström and Ljung, 2002)

(2) Vector-valued test

If $\hat{\theta}_N = \theta^*$, consider

$$\hat{\mathrm{R}}^T_{\varepsilon u} = [\hat{R}_{\varepsilon u}(0) \cdots \hat{R}_{\varepsilon u}(n_\tau - 1)]$$

Then

$$\hat{\mathrm{R}}_{\varepsilon u} = \underbrace{\frac{1}{N}\begin{bmatrix} u(1) & u(2) & \cdots & \cdots & u(N) \\ & \ddots & \cdots & \cdots & \vdots \\ & & u(1) & \cdots & u(N-n_\tau+1) \end{bmatrix}}_{P_u} \underbrace{\begin{bmatrix} \varepsilon(1) \\ \vdots \\ \varepsilon(N) \end{bmatrix}}_{\epsilon}$$

so that

$$cov(\hat{\mathrm{R}}_{\varepsilon u}) = P_u \Lambda_\varepsilon P_u^T$$

with

$$\Lambda_\varepsilon = \mathbb{E}(\epsilon \epsilon^T)$$

Based on

$$\sqrt{N}\hat{\mathrm{R}}_{\varepsilon u} \rightarrow \mathcal{N}(0, P_u \Lambda_\varepsilon P_u^T) \qquad N \rightarrow \infty$$

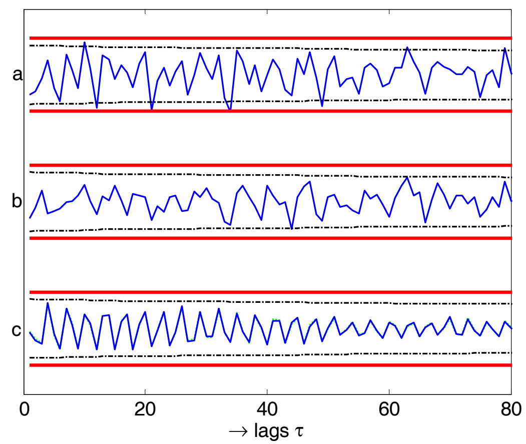
it follows that $G(q, \hat{\theta}_N)$ is not invalidated if

$$\hat{\mathrm{R}}^T_{\varepsilon u}[P_u \Lambda_\varepsilon P_u^T]^{-1} \hat{\mathrm{R}}_{\varepsilon u} \leq c_\chi(\alpha, n_\tau)/N$$

Computation of Λ_ε requires a noise model.

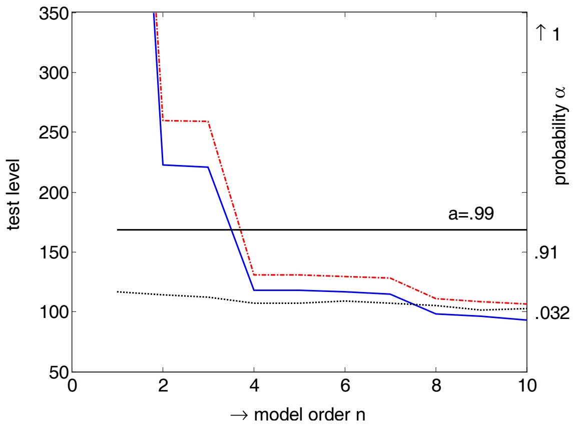
More example

- Same 4th order system as before; white output noise
- $N = 256$
- Input signal is white noise
- Auxiliary high order model: OE of order $N/5$;
- Automated time series model on output residuals (Broersen)
- Only improvement step is: noise modelling and adapting bound on $\hat{R}_{\varepsilon u}(\tau)$.



red = classic; dashed = new bound.
(a): $\hat{R}_{\varepsilon u}(\tau)$; (b): $\hat{R}_{\varepsilon_v u}(\tau)$; (c): $\hat{R}_{\beta_b u}(\tau)$

Vector-valued test



Test value $\hat{R}_{\varepsilon u}^T P^{-1} \hat{R}_{\varepsilon u}$ for true noise covariance (blue), noise covariance estimate (red), sample covariance of ε (dash-dotted)

Summary

- Standard validation test embodies circular reasoning (test is valid under the assumption it intends to validate)
- Validation test requires accurate noise modelling
  Separate validation of $G(\hat{\theta}_N)$ and $H(\hat{\theta}_N)$ is limited
- Vector-valued test is more powerful
- Either excitation-free noise modelling, or
- For general (nonperiodic) input this requires bias-free plant modelling
  (cf. model error model, Ljung)