Allocation of Excitation Signals for Generic Identifiability of Dynamic Networks

Xiaodong Cheng Shengling Shi Paul M.J. Van den Hof

Control Systems Group, Department of Electrical Engineering, TU Eindhoven

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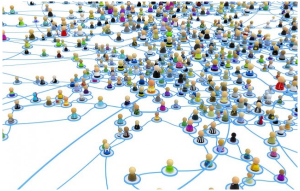
Outline

Introduction – dynamic networks

- Appear in a wide range of applications

Manufacturing

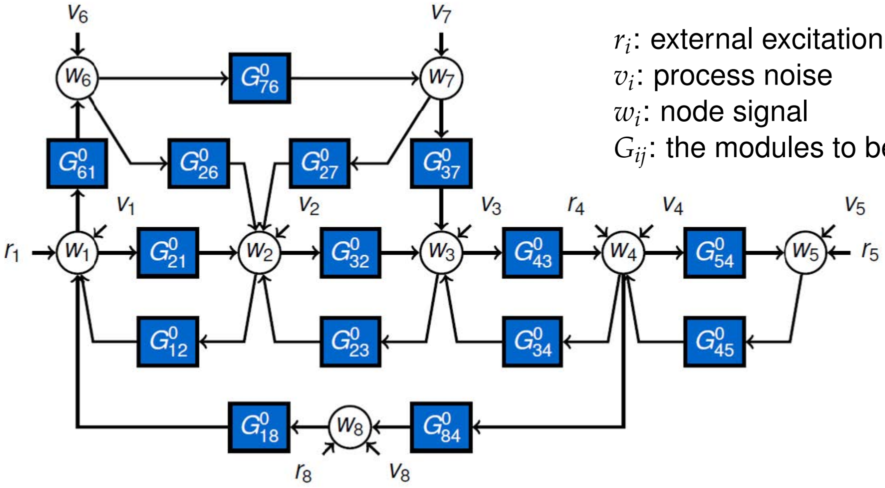
Power Grid



Social Network

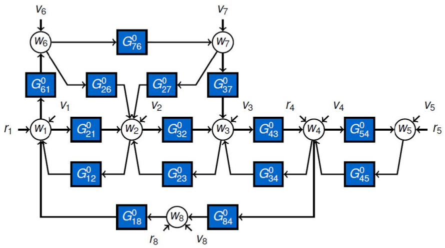
- How to address data-driven modeling problems in a network setting

Dynamic network setup



r_i: external excitation signals
v_i: process noise
w_i: node signal
G_{ij}: the modules to be identified

Dynamic network setup



Assume:

- G_{ij} are proper transfer functions
- the network is well-posed and stable

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & \cdots & G_{1L} \\ G_{21} & 0 & \ddots & G_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ G_{L1} & G_{L2} & \cdots & G_{LL} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + R \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_L \end{bmatrix}$$

Network identifiability

$$w = Gw + Rr + v, \text{ with } v = He,$$

- Transfer function: $T = (I - G)^{-1}[R \;\; H]$ typically can be identified from data

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- A network model is denoted by $\sigma = (G, R, H)$;
- A *network model set* is defined by $\Sigma = \{\sigma(\theta) = (G(\theta), R(\theta), H(\theta)), \theta \in \Theta\}$

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Definition (Network identifiability[1])

Σ *is identifiable if* $T(\theta_1) = T(\theta_0) \Rightarrow \sigma(\theta_1) = \sigma(\theta_0)$ *holds for all* $\theta_0, \theta_1 \in \Theta$*.*

Generic identifiability holds if this is true for almost all models in Σ (except models in the measure zero set).

[1] H. H. M. Weerts, P. M. J. Van den Hof, and A. G. Dankers, "Identifiability of linear dynamic networks," *Automatica* 2018.

Problem

Consider the network $\mathcal{G}$, and all the node signals w_j are measured.

Problem (Generic identifiability condition)

Under what conditions(i.e., the number and locations of external excitation signals), Σ is generically identifiable?

Problem (Allocation of excitation signals)

Where to put external excitation signals in $\mathcal{G}$ to achieve generic identifiability of Σ.

*For simplicity, we will not consider noises, and all the modules are to be identified.

Generic identifiability - revisit

Lemma (Path-based condition[2,3])

Let $\mathcal{R} \subseteq \mathcal{V}$ be the set of vertices that are excited. Σ is generically identifiable if and only if the maximum number of vertex disjoint paths from $\mathcal{R}$ to $\mathcal{N}_i^-$ is equal to $|\mathcal{N}_i^-|$ for all $i \in \mathcal{V}$.

Relevant notations:

- $\mathcal{N}_i^-$: the set of the in-neighbors of vertex i;
- $|\mathcal{N}_i^-|$ the cardinality of the set $\mathcal{N}_i^-$.

[2] J. M. Hendrickx, M. Gevers, and A. S. Bazanella, "Identifiability of dynamical networks with partial node measurements," *IEEE TAC*, 2018.

[3] H. H. M. Weerts, P. M. J. Van den Hof, and A. G. Dankers, "Single module identifiability in linear dynamic networks," *CDC2018*.

Generic identifiability - revisit

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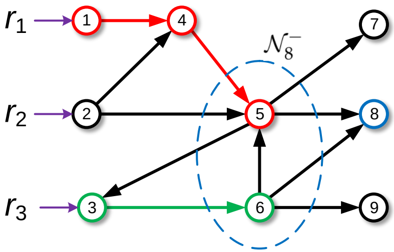

Generic identifiability - revisit

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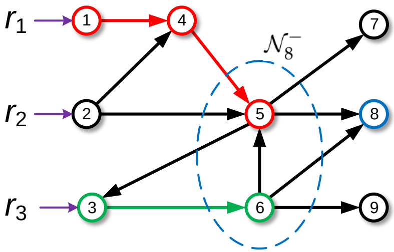
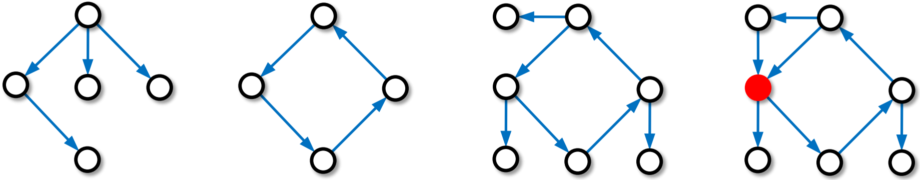


Result provides an analysis tool, but is less suited for the signal allocation problem

Pseudo-trees

Definition (Directed pseudo-trees)

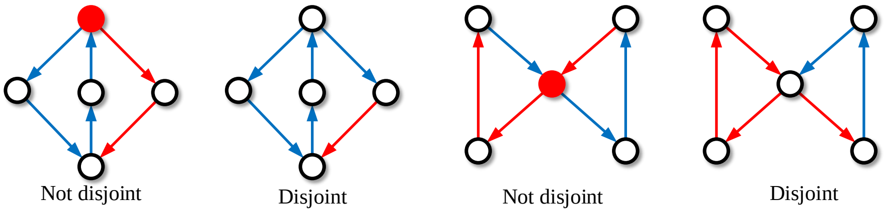
A connected directed graph $\mathcal{T}$ is called a (directed) pseudo-tree if $|\mathcal{N}_i^-| \leq 1$, for all $i \in V(\mathcal{T})$.



Disjoint pseudo-trees

Two pseudo-trees are (edge)-disjoint if

- Any two pseudo-trees should not share common edges
- Each node and its out-going edges are in the same pseudo-tree



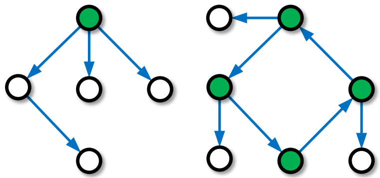
Any directed network can be decomposed into a set of disjoint pseudo-trees

Generic identifiability - new condition

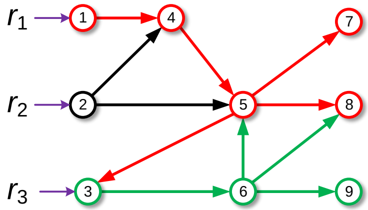
Theorem (Pseudo-tree covering condition[4])

A network is generically identifiable if

- *it can be decomposed into K disjoint pseduotrees,*
- *and there are K independent excitation signals allocated at a root of each pseudotree.*



Left: Tree with root in green
Right: Cycle with outgoing trees; Any node in cycle is root



[4] X. Cheng, S. Shi, and P. M. J. Van den Hof, "Allocation of excitation signals for generic identifiability of dynamic networks," *CDC2019*

Excitation allocation

Excitation allocation problem

Given a directed network, what is the minimal number of excitation signals for the network to be generic identifiable?

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Given a directed network, what is the minimal number of excitation signals for the network to be generic identifiable?

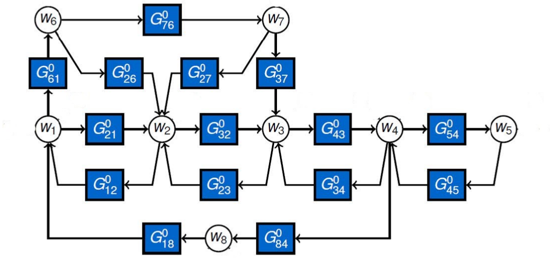
Now, we solve

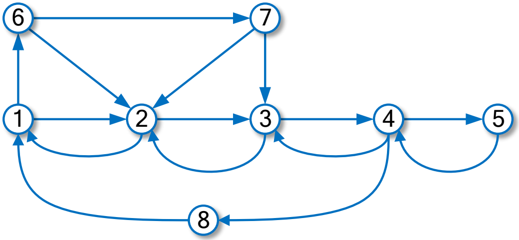
Graph covering problem

Given a directed network, find the minimal number of disjoint pseudo-trees such that all the edges are covered.

Excitation allocation

For simple network, it is easy:





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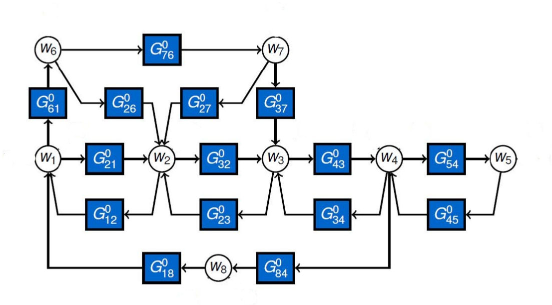
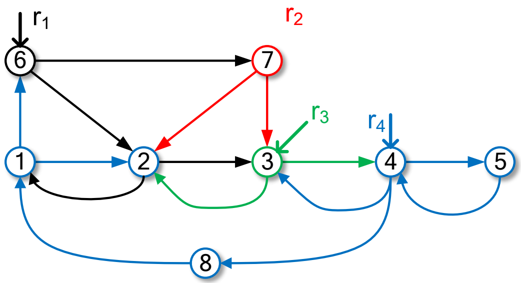
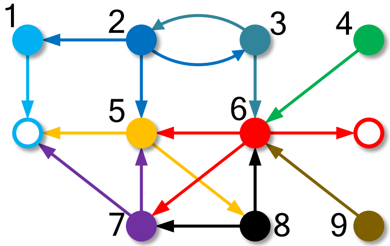




Pseudo-tree merging

- A *minimal* pseudo-tree is a graph which only contains one root and all the out-neighbors of this root.
- Any directed network can be covered by a set of disjoint minimal pseudo-trees.



We start with a covering of $\mathcal{G}$ with minimal pseudo-trees.

Mergeability

Definition (Mergeability)

Consider two disjoint pseudo-trees $\mathcal{T}_1$ and $\mathcal{T}_2$. We say $\mathcal{T}_1$ is mergeable to $\mathcal{T}_2$, if

1. *the union of $\mathcal{T}_1$ and $\mathcal{T}_2$ is also a pseudo-tree;*
2. *there is a directed path from the roots of $\mathcal{T}_2$ to each node in $\mathcal{T}_1$.*

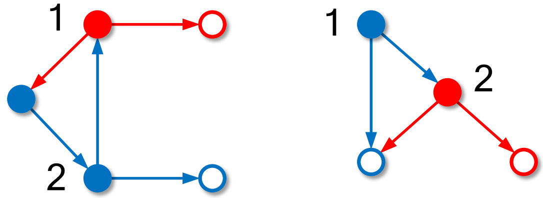


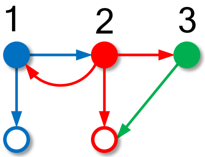
Figure: Left: $\mathcal{T}_1$ is mergeable to $\mathcal{T}_2$, and $\mathcal{T}_2$ is mergeable to $\mathcal{T}_1$
Right: $\mathcal{T}_1$ is NOT mergeable to $\mathcal{T}_2$, and $\mathcal{T}_2$ is mergeable to $\mathcal{T}_1$

Mergeability

Definition (Mergeability matrix)

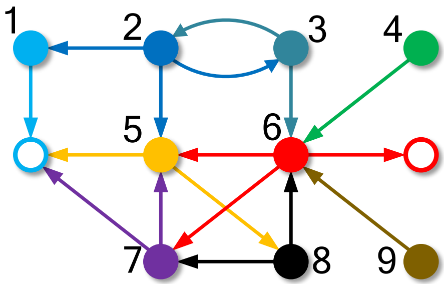
Denote $\mathbb{M} := \{1, \varnothing, 0\}$. *The mergeability matrix* $\mathcal{M}$ *is defined with the* (i,j)*-th entry as*

$$\mathcal{M}(i,j) = \begin{cases} 1 & \text{if } \mathcal{T}_i \text{ is mergeable to } \mathcal{T}_j; \\ \varnothing & \text{if } V(\mathcal{T}_i) \cap V(\mathcal{T}_j) = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$



$$\begin{array}{cc} & \begin{array}{ccc} \mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 \end{array} \\ \mathcal{M} = & \begin{pmatrix} 0 & 1 & \varnothing \\ 1 & 0 & 0 \\ \varnothing & 0 & 0 \end{pmatrix} \begin{array}{c} \mathcal{T}_1 \\ \mathcal{T}_2 \\ \mathcal{T}_3 \end{array} \end{array}$$

Pseudo-tree merging - step 1



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\end{array}
$$

- Find a row of $\mathcal{M}$ with only a single “1” entry, while the others are either “0” or “$\varnothing$”.
- If there exist multiple rows containing a single “1” entry, then we choose the one with more “$\varnothing$” entries.

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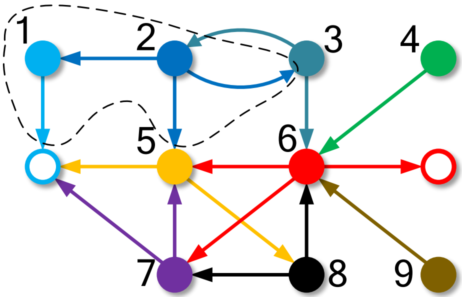

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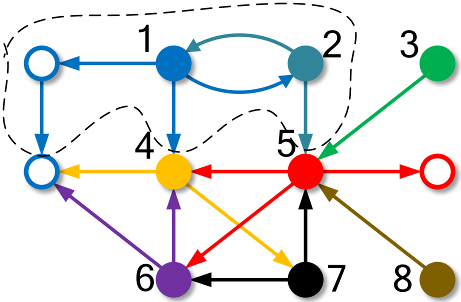

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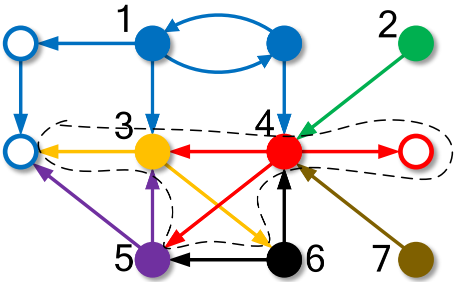

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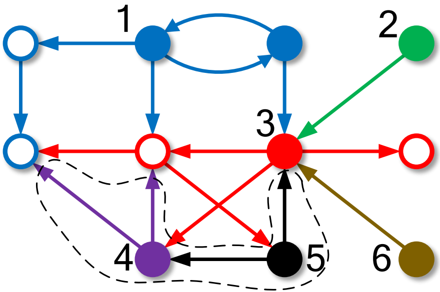

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\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \varnothing & 0 & 0 \\
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0 & \varnothing & 0 & 0 & \boxed{1} & \varnothing \\
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\end{pmatrix} &
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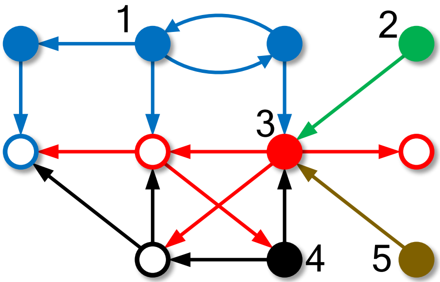
Pseudo-tree merging - step 1



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\begin{array}{cc}
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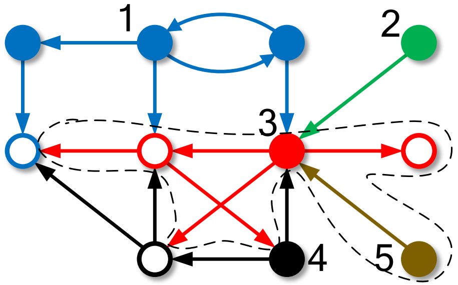
$$

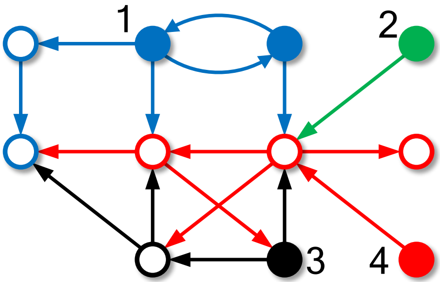
Pseudo-tree merging - step 2



$$\begin{array}{cc} \begin{array}{ccccc} \mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 & \mathcal{T}_4 & \mathcal{T}_5 \end{array} & \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} \mathcal{T}_1 \\ \mathcal{T}_2 \\ \mathcal{T}_3 \\ \mathcal{T}_4 \\ \mathcal{T}_5 \end{array} \end{array}$$

- Find a row with at least one “1” entry, and then merge the two pseudo-trees corresponding to an arbitrary “1” entry of this row.
- If there exists multiple rows containing “1” entries, select a row with more “$\varnothing$” entries

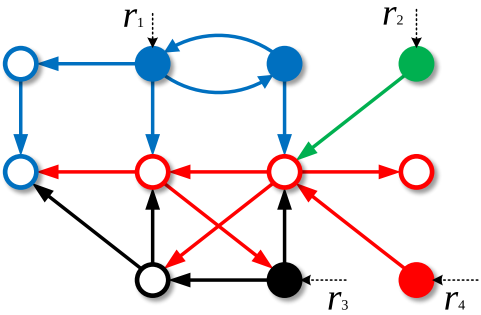
Pseudo-tree merging - step 2



$$\begin{array}{ccccc} \mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 & \mathcal{T}_4 & \\ \multicolumn{4}{c}{\multirow{4}{*}{$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$}} & \mathcal{T}_1 \\ &&&& \mathcal{T}_2 \\ &&&& \mathcal{T}_3 \\ &&&& \mathcal{T}_4 \end{array}$$

- There are no mergeable pairs of pseudo trees $\Rightarrow$ end of the algorithm
- Allocate the external excitation signals to the roots

Pseudo-tree merging - step 2



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Conclusions

- A graphical condition to characterize the generic identifiability of dynamic network using disjoint pseudo-trees
- An greedy pseudo-tree merging algorithm to solve excitation allocation problem

Extensions

- correlated noises, non-parameterized modules [5]
- better algorithm for the decomposition of a graph into pseudo-trees
- a graph-based condition for the partial measurement case

Thank you for your attention!
Questions?

[5] X. Cheng, S. Shi, and P. M. J. Van den Hof, "Allocation of excitation signals for generic identifiability of linear dynamic networks," *Submitted*