Identifiability and data-informativity for single module identification in dynamic networks ⋆

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Abstract: Identifying single modules that are embedded in a linear dynamic network is an identification problem that has been addressed with different techniques. Prediction error identification methods provide sufficient conditions in terms of the selection of an appropriate set of node signals to be measured, an appropriate identification method (e.g. direct or indirect method), and a data-informativity condition, for arriving at consistent and possibly minimum variance module estimates. The question whether at all it is possible to identify a unique module estimate from data, is treated by the concept of identifiability, that is independent of the identification method chosen, and independent of the particular experiment that is done, i.e. independent of the design of the signals. In this presentation we will analyse the relation between these two principal concepts, and show that the data-informativity conditions that are used for a particular identification method are composed of two components: an identifiability-induced component, that is independent of the identification method, and a data-informativity condition that is induced by the particular identification method.

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1. INTRODUCTION

Linear dynamic networks are structured systems that are composed of interconnected linear time-invariant systems. Typically a dynamic network induces a graph, with vertices and edges, that represents the topology of the network. Often a network is represented in a state-space form with states as node signals represented by the vertices in the graph, and the state transitions as links or edges in the graph. However in an identification setting, where not all states of a system are typically measured, it has appeared to be attractive to represent the network in a graph that has (measured) node signals as vertices, and dynamic transfer functions on the links/edges. The basic setting of *Dynamic Structure Functions* that was introduced in Gonçalves and Warnick (2008), was generalized to a stochastic estimation and identification setting in Van den Hof et al. (2013), and has been adopted by several different authors.

In this setting a dynamic network is built up out of L scalar *internal variables* or *nodes* w_j, $j = 1, \ldots, L$, and K *external variables* r_k, $k = 1, \ldots K$. Each internal variable is described as:

$$w_j(t) = \sum_{\substack{l=1 \\ l \neq j}}^{L} G_{jl}(q) w_l(t) + R_j(q) r_j(t) + v_j(t) \qquad (1)$$

where q^{-1} is the delay operator, i.e. $q^{-1} w_j(t) = w_j(t-1)$;

- G_{jl} are proper rational transfer functions, referred to as *modules*.
- r_j are *external variables* that can directly be manipulated by the user and that may or may not be present; if r_j is not present it is replaced by $r_j = 0$.
- v_j is *process noise*, where the vector process $v = [v_1 \cdots v_L]^T$ is modelled as a stationary stochastic process with rational spectral density $\Phi_v(\omega)$, such that there exists a white noise process $e := [e_1 \cdots e_L]^T$, with covariance matrix $\Lambda > 0$ such that $v(t) = H(q)e(t)$, where H is square, stable, monic and minimum-phase.

When combining the L node signals we arrive at the full network expression

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & \cdots & G_{1L} \\ G_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & G_{L-1\ L} \\ G_{L1} & \cdots & G_{L\ L-1} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{bmatrix} + H \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_L \end{bmatrix}$$

which results in the matrix equation:

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$$w = Gw + Rr + He, \tag{2}$$

where by construction the matrix G is hollow, i.e. it has diagonal entries 0.

The single module identification problem to be considered is the problem of identifying one particular module $G_{ji}(q)$ on the basis of measured time-series of a subset of variables in w, and possibly r.

2. IDENTIFICATION METHODS

We can distinguish two main different approaches for addressing the single module identification problem, where the target module is indicated by G_{ji}.

(1) A *direct method*, that is based on selecting a particular set of predictor input signals w_k, $k \in \mathcal{D}$, and a set of predicted output signals w_ℓ, $\ell \in \mathcal{Y}$, with $i \in \mathcal{D}$, $j \in \mathcal{Y}$, and estimating a dynamic model based on a prediction error:

$$\varepsilon(t,\theta) = \bar{H}(q,\theta)^{-1}[w_{\mathcal{Y}}(t) - \bar{G}(q,\theta)w_{\mathcal{D}}(t)], \tag{3}$$

where $\bar{G}(q,\theta)$ and $\bar{H}(q,\theta)$ are parametrized transfer function matrices. The target module is then embedded in the model $\bar{G}(q,\theta)$, and the objective is to estimate the target module consistently and possibly with minimum variance.

(2) An *indirect method*, that is based on selecting a particular set of external excitation signals r_k, $k \in \mathcal{U}$, and a set of predicted outputs w_ℓ, $\ell \in \mathcal{Y}$, and inputs w_k, $k \in \mathcal{D}$ that are used in a predictor model, leading to

$$\varepsilon_{\mathcal{Y}}(t,\theta) = w_{\mathcal{Y}}(t) - \bar{T}_{\mathcal{Y}}(q,\theta)r_{\mathcal{U}}(t) \tag{4}$$

$$\varepsilon_{\mathcal{D}}(t,\rho) = w_{\mathcal{D}}(t) - \bar{T}_{\mathcal{D}}(q,\rho)r_{\mathcal{U}}(t). \tag{5}$$

Since $\bar{T}_{\mathcal{Y}}$ and $\bar{T}_{\mathcal{D}}$ are mappings from external signals (r) to internal signals (w), a processing step is necessary to recover the target module G_{ji} from an estimated $\bar{T}$'s. Consistency of the target module estimate is the typical objective. Different variations of indirect methods exist, including two-stage and instrumental variable (IV) methods.

The direct method provides asymptotically efficient estimates (i.e. consistency and minimum variance for the identification setup) whereas the indirect method and its variations typically provide consistent estimates but - without the use of particularly parametrized noise models - not with minimum variance. This is attributed to the fact that in the indirect method, without dedicatedly parametrized noise models, only the excitation-signal dependent part of the node signals is used for variance reduction, thus not effectively using excitation through the process noise signals.

Note that in both cases it seems natural to only choose a one-dimensional output vector $w_{\mathcal{Y}}$ if our target is the estimation of a single module. However the flexibility in using more than one output can be used in both methods to account for excitation limitations, or to go beyond consistency requirements and achieve minimum variance results.

3. SINGLE MODULE IDENTIFIABILITY

Generic identifiability of a single module G_{ji} in a network model set $\mathcal{M} = \{G(\theta), H(\theta), R(\theta), \Lambda(\theta)\}$ is defined by the property that the transfer function T from present external signals r and e, to measured node signals w generically induce a unique representation of the module G_{ji} in the model set, see e.g. Weerts et al. (2018); Hendrickx et al. (2019); Shi et al. (2019). This property can be verified on the basis of path-based conditions on the graph of the network. It is dependent on the location of external excitation signals (r) and disturbances (e), and on the set of in-neighbors $w_{\mathcal{W}_j}$ of w_j in the network, that are input to parametrized modules G_{jk}. Let $\mathcal{U}$ be the set of external signals that are non-parametrized in the expression for w_j, and let $b_{\mathcal{U}\to\mathcal{W}_j}$ denote the maximum number of vertex disjoint paths from the set $\mathcal{U}$ to $\mathcal{W}_j$. Then G_{ji} is generically identifiable in $\mathcal{M}$ if and only if there exists a disconnecting set $\mathcal{W}_d$ from $\mathcal{U} \cup \{w_i\}$ to $\mathcal{W}_j \backslash \{w_i\}$ such that

$$b_{\mathcal{U}\to\mathcal{W}_d\cup\{w_i\}} = |\mathcal{W}_d| + 1,$$

see Shi et al. (2019).

Note that this identifiability property is not dependent on a particular identification method (direct or indirect). In this way it is a necessary condition for having the possibility of arriving at a consistent estimate of the target module, irrespective of the method chosen.

4. DATA-INFORMATIVITY CONDITIONS FOR CONSISTENT MODULE ESTIMATES - DIAGONAL NOISE SPECTRUM

4.1 The direct method

The first results for consistent estimates of network modules were provided in Van den Hof et al. (2013), where the situation was considered of networks having diagonal noise spectrum $\Phi_v(\omega)$, i.e. the noise on different node signals being uncorrelated. This leads to a multiple input single output identification setup. The sufficient condition on the experimental data to warrant consistency of the direct method was formulated in terms of a positive definite spectral density:

$$\Phi_\eta(\omega) > 0 \quad \text{for a sufficient number of frequencies } \omega,$$

where η is a vector signal composed of the output node signal w_j and all predictor input node signals $w_{\mathcal{D}}$. If $\mathcal{D}$ is chosen to be equal to $\mathcal{W}_j$, as in Van den Hof et al. (2013), (i.e. all in neighbours of w_j are used as predictor input), then this data-informativity condition concerns all chosen in-neighbours. It has been shown in later work, Dankers et al. (2016), that this situation can be relaxed to selecting $w_{\mathcal{D}}$ such that all parallel paths and loops around the output are blocked by a measured signal. It appears that this condition can exactly be formulated by the construction of a disconnecting set as in the previous section, where we can select $w_{\mathcal{D}} := w_{\mathcal{W}_d} \cup \{w_i\}$ as the set of predictor inputs. This leads to a data-informativity condition for the direct method that is based on $\eta = w_{\mathcal{W}_d\cup\{w_i,w_j\}}$.[1]

It can be verified that the spectrum condition is equivalent to requiring that there exist $|\mathcal{W}_d| + 1$ vertex disjoint paths

[1] Note that in the reduced input case, $\mathcal{D} \neq \mathcal{W}_j$, the data-informativity conditions do not guarantee the absence of confounding variables, which is required for consistent module estimates.

from the external signals (r and e) in $\mathcal{U}$ to the predictor input signals $w_{\mathcal{D}}$ together with a persistence of excitation condition on the external excitation signals in $\mathcal{U}$. Note that this excitation condition is a sufficient condition for any type of linear dynamics to be estimated, i.e. for any order of the underlying system. As such it is a data-informativity condition that is sufficient generically. For lower order models less conservative conditions can be derived, but in general are less insightful, see e.g. Gevers and Bazanella (2015).

4.2 The indirect method

For the indirect method the data-informativity conditions for consistency are closely related. In this situation we also choose $w_{\mathcal{D}} := w_{\mathcal{W}_d} \cup \{w_i\}$. According to the conditions for identifiability, there are $|\mathcal{U}|$ external signals that should provide excitation for estimating the matrices $\bar{T}_{\mathcal{Y}}$ and $T_{\mathcal{D}}$. In the most common situations this will require $|\mathcal{D}|$ independent measured excitation signals r that are persistently exciting. This implies that if the conditions for single module identifiability are satisfied, then the additional data-informativity condition for consistency of the indirect method, is that all external signals in $\mathcal{U}$ are measured excitation signals r that are persistently exciting, so that they can serve as inputs in the estimation problems (4)-(5).

4.3 Comparison

Comparing the data-informativity conditions of the two methods confirms the known difference that the direct method benefits from noise excitation of the network (it exploits both r and ed components in $\mathcal{U}$), while the indirect method relies on excitation through measured excitation signals r, thus requiring more "expensive" experiments.

5. EXTENSION TOWARDS A NON-DIAGONAL DISTURBANCE SPECTRUM

In the situation of networks where the disturbance spectrum is allowed to be non-diagonal, the situation becomes more involved. The single module identifiability result stays invariant, but the data-informativity results for the different estimation methods will change, in particular for the direct method.

For the direct method, the identification setup now becomes a MIMO setup, where correlations between disturbance signals are treated by including multivariate noise models, as in (3), and where the predicted node signals $w_{\mathcal{Y}}$ and the predictor inputs $w_{\mathcal{D}}$ might have common node signals $w_{\mathcal{Q}}$ that appear in both signal vectors (Ramaswamy and Van den Hof (2019)). The data-informativity condition for consistent estimation results of the direct method now become (Ramaswamy and Van den Hof (2019)):

$$\Phi_{\kappa}(\omega) > 0 \quad \text{for a sufficient number of frequencies } \omega,$$

where $\kappa(t) := \left[w_{\mathcal{D}}^{\top}(t)\ \xi_{\mathcal{Q}}^{\top}(t)\ w_o(t)\right]^{\top}$, $\xi_{\mathcal{Q}}$ is the vector of innovation signals related to node signals $w_{\mathcal{Q}}$ when all non-measured signals are removed from the network (immersed), and w_o is the output signal of the target module G_{ji} to be identified, if this signal is not included in $w_{\mathcal{Q}}$; otherwise it is void.

Denote with w_m the set of measured node signals $w_{\mathcal{D}} \cup w_{\mathcal{Y}}$ having n_m components.

Similar as for the situation in Section 4, the data-informativity condition can now be formulated in terms of path-based conditions that depend on the presence of external excitation signals, that have particular path-based properties in the topology of the network model set, that guarantee that the data required for the direct identification method is sufficiently informative.

In the considered situation the path-based conditions amount to having n_m vertex disjoints paths from external excitation and noise signals, excluding $\xi_{\mathcal{Q}}$, to the node signals w_m. So apparently the consequence is that when node signals appear in both input and output of the estimation problem, then the corresponding disturbance signal that connects to that node can not serve as an external signal that contributes to data-informativity. This also leads to the conclusion that in considered situation there need to be at least $|\mathcal{Q}|$ external excitation signals r to provide data-informativity.

These path-based conditions then merge the single module identifiability conditions presented in Section 3 that are method-independent, with additional conditions for data-informativity that are induced by the choice of identification method.

For the indirect identification method, the setup and results remain the same as for the situation with diagonal disturbance spectrum, as presented in Section 4.

6. CONCLUSIONS

Identifying a local module that is embedded in a dynamic network, requires several conditions to be satisfied, among which the two most important ones: network identifiability and data-informativity. Network identifiability conditions are typically independent of the identification method chosen, while the additional data-informativity requirements are dependent on the identification method. Direct and indirect identification approaches lead to different conditions in this respect. Both concepts (identifiabiality and data-informativity) can be cast in a generic context in terms of path-based conditions formulated on the external signals that are present in the network (excitation signals and noises).

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