The Hambo transform: a signal and system transform induced by generalized orthonormal basis functions [3]

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Abstract

In this paper a signal and system transformation is analyzed that is induced by a recently introduced generalized orthonormal basis for $\mathcal{H}_2$-systems and ℓ_2-signals. This basis is very flexible and generalizes the pulse, Laguerre and Kautz bases. The corresponding system and signal transformations generalize the Fourier and z-transforms; they are characterized in terms of state space forms, and interesting properties of the representations in the transform domain are shown. The transformations facilitate the asymptotic analysis of related system identification algorithms, in which the basis functions are employed as linear model parametrizations; additionally they are shown to provide powerful results in system approximation.

Key words: Orthogonal basis functions; Laguerre functions; discrete-time systems; Fourier transform; system approximation; network synthesis.

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[3] Report N-512, Mechanical Engineering Systems and Control Group, Delft University of Technology, April 3, 1998. Paper submitted to Automatica on March 20, 1998. The original version of this paper was presented at the 13th IFAC World Congress, 30 June - 4 July 1996, San Francisco, CA, USA.

1 Introduction

The idea of decomposing representations of linear time-invariant dynamical systems and related input/output signals, e.g. with respect to their power density spectra, in terms of orthogonal components other than the standard Fourier series, dates back to the work of Lee and Wiener in the thirties, as reviewed in Lee (1960). Laguerre functions have been very popular in this respect, mainly because of the fact that their frequency response is rational (Gottlieb, 1938; Schetzen, 1970).
In an attempt to find more general classes of orthogonal basis functions with this same property, Kautz (1954) formulated a general class of functions, composed of damped exponentials, to be used for signal decomposition. An interesting account of this work is also given in Huggins (1956), where the idea of using nonsinusoidal spectral representations is discussed.
In the seventies and eighties, particularly Laguerre functions were often applied in problems of network synthesis, system approximation and identification (King and Paraskevopoulos, 1979; Nurges 1987). In some cases a system transformation in terms of the Laguerre basis functions has been considered here, mainly for the purpose of data reduction. Later, in Wahlberg (1991, 1994a,1994b) Laguerre functions and so-called two-parameter Kautz functions have been used in the identification of the expansion coefficients of approximate models by simple linear regression methods.
Extending this work further, Heuberger (1991) has developed a theory on the construction of orthogonal basis functions, based on balanced realizations of inner (all-pass) transfer functions, see Heuberger et al. (1995). Here a concatenation of similar all-pass functions is used to generate orthogonal basis functions. This situation has been shown to generalize the Laguerre and two-parameter Kautz case, and has led to a generalization of the identification results of Wahlberg, see Van den Hof et al. (1995).
A further generalization of this situation is presented in Ninness and Gustafsson (1997), where concatenations of freely chosen all-pass sections are considered as basis-generators.
These recently developed basis functions have been shown to have attractive properties in several respects. First the use as linear model parametrizations in system identification problems has been shown to be attractive; this is due to the fact that smartly chosen basis functions can provide a fast rate of convergence of the corresponding series expansion, thus leading to linear model parametrizations with a limited number of parameters. Statistical properties of time domain (least-squares) identification methods have been analysed in Van den Hof et al. (1995) and Ninness et al. (1997), and frequency-domain methods in Ninness and Gómez (1996), Schipp et al. (1996) and De Vries and Van den Hof (1998). Additionally, the linear model parametrizations are indispensible in explicitly quantifying (probabilistic and/or worst-case) model error bounds, see e.g. De Vries (1994) and Hakvoort and Van den Hof (1997).

Besides the use of these functions for identification purposes, also the problem of system approximation has been addressed, see particularly Wahlberg and Mäkilä (1996), Oliveira e Silva (1995, 1996), Den Brinker et al. (1996), Schipp and Bokor (1997).

The particular basis functions of Heuberger et al. (1995) give rise to a general theory on dynamical signals and systems transformations induced by these so-called Hambo basis functions. This is caused by the simple group structure that is involved in these functions. Properties of these transformations have been crucial in the development of the identification results in Van den Hof et al. (1995). There it has been shown that the corresponding least-squares problem of identifying expansion coefficients, as in the usual situation of e.g. FIR models, is governed by a (block) Toeplitz systems of equations, which facilitates analysis considerably.

The transformation results have also been shown to be very helpful in the problem of approximating a given system $P(z) = \sum_{k=1}^{\infty} c_k F_k(z)$ by its finite expanion sequence $\hat{P}_n(z) = \sum_{k=1}^{n} c_k F_k(z)$, where $\{F_k(z)\}_{k=1,2,\cdots}$ is a sequence of (orthonormal) basis functions in $\mathcal{H}_2$. For a given value of n the approximation error

$$P(z) - \sum_{k=1}^{n} c_k F_k(z)$$

is clearly dependent on the character of the basis functions F_k. The transform theory that will be presented in this paper, will be applied to specify the approximation error, and the role of the basis functions in this approximation.

The sequel of the paper is constructed as follows. First in section 2 the considered basis functions will be specified and reviewed. Simple constructions will be shown based on balanced realizations of all-pass functions. After defining the resulting transformations on ℓ_2-signals, structural properties of the basis functions will be analysed in section 3. Based on these structural properties, the signal transforms can be brought back to simple expressions, as shown in section 4. The dynamical system transform is then further analysed and properties are given in section 5. Consequences for the system approximation problem are subject of discussion in section 6.

Throughout the paper the following notaton will be adopted:

$(\cdot)^T$	Transpose of a matrix.
$\mathbb{R}^{p\times m}$	set of real-valued matrices with dimension $p \times m$.
$\ell_2[1,\infty)$	Space of squared summable sequences on the time interval of positive integers.
$\ell_2^{p\times m}[1,\infty)$	Space of matrix sequences $\{F_k \in \mathbb{R}^{p\times m}\}_{k=1,2,\cdots}$ such that $\sum_{k=1}^{\infty} tr(F_k^T F_k)$ is finite.
$\mathcal{H}_2^{p\times m}$	Set of real $p \times m$ matrix functions, analytic for $\|z\| \geq 1$, that are squared integrable on the unit circle.
$\mathcal{R}H_2^{p\times m}$	Set of real rational $p \times m$ matrix functions that are squared integrable on the unit circle.
$\mathbb{Z}_+$	Set of nonnegative integers.
$\otimes$	Kronecker matrix product.
e_i	i-th Euclidean basis vector in $\mathbb{R}^n$.
I_n	$n \times n$ Identity matrix.
$\delta(t)$	Unit pulse function, i.e. $\delta(t) = 1$, $t = 0$; $\delta(t) = 0$, $t \neq 0$.

Unless otherwise mentioned all systems in this paper are scalar systems.

2 Generalized orthonormal basis functions

We will consider the generalized orthonormal basis functions that were introduced in Heuberger et al. (1995), based on the preliminary work of Heuberger and Bosgra (1990) and Heuberger (1991). Extensions have been presented in Ninness and Gustafsson (1997). The basis functions are constructed from state trajectories related to balanced realizations of square inner functions (i.e. stable all-pass systems). A transfer function $G_b \in \mathcal{H}_2$ is inner if it satisfies $|G_b(e^{i\omega})| = 1$ for all $\omega \in [-\pi, \pi]$. It has been shown in Roberts and Mullis (1987) that inner functions can be realized by particular state space realizations that are so-called orthogonal, i.e. they satisfy $G_b(z) = D + C(zI - A)^{-1}B$ where

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \begin{bmatrix} A & B \\ C & D \end{bmatrix} = I.$$

From this orthogonality property it follows directly that the controllability gramian P and the observability gramian Q, which are defined as

$$
\begin{aligned}
P &= APA^T + BB^T \\
Q &= A^T QA + C^T C
\end{aligned}
$$

satisfy

$$P = Q = I$$

and so realizations with this property are balanced in the sense of Moore (1981).
The state sequence x defined by $x(t+1) = Ax(t) + Bu(t)$ for a stationary white noise input u will satisfy

$$E\{x(t)x^T(t)\} = P = I$$

and so the states of the all-pass function are orthogonal to one another, according to the inner product $< x_i, x_j >= E\{x_i(t)x_j(t)\}$. Similarly for a pulse input $u(t) = \delta(t)$, it follows that

$$\sum_{t=1}^{\infty} x(t)x^T(t) = P = I$$

showing orthonormality of the state components with respect to the ℓ_2 inner product: $< x_i, x_j >= \sum_{t=1}^{\infty} x_i(t)x_j(t)$.

A second result from Roberts and Mullis (1987) as indicated in Bodin and Wahlberg (1994) is that for two all-pass functions $G_i \in \mathcal{H}_2$, $(i = 1, 2)$, with corresponding balanced realizations (A_i, B_i, C_i, D_i), the product $G_2 G_1$ has a balanced realization (A, B, C, D) with

$$
\begin{aligned}
A &= \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} \\
C &= \begin{bmatrix} D_2 C1 & C_2 \end{bmatrix} \quad D = D_2 D_1.
\end{aligned}
$$

For any input signal and initial state, the state sequence $x(t)$ related to this realization, can be decomposed by $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where x_i is the state trajectory related to the realization of G_i separately. In other words, there exists a recursive structure, where concatenating all-pass functions provides an increasing number of state functions that are orthogonal to each other.

An all-pass transfer function can always be written in the form of a Blaschke product:

$$G_b(z) = \pm \prod_{i=1}^{n_b} \frac{1 - \xi_i^* z}{z - \xi_i}$$

with $\{\xi_i\}_{i=1,\cdots n_b}$ a sequence of poles, coming in complex conjugate pairs. The basic result leading to a sequence of basis functions for ℓ_2 can now be formulated as follows:

Proposition 1 *Consider a sequence of poles $\{\xi_1, \cdots, \xi_k\}$. Then for $k \to \infty$ then the states $\{x_i(t)\}_{i=1,\cdots k}$ of the concatenated balanced realization of the related inner functions, under pulse input excitation, constitute a complete orthonormal basis for the signal space $\ell_2[1, \infty)$, under the condition that*

$$\sum_{i=1}^{\infty} 1 - |\xi_i| = \infty.$$

PROOF. See e.g. Ninness and Gustafsson (1997). □

In Heuberger et al. (1995), a specific form of these basis functions is considered that exhibit an attractive shift structure. It is based on the concatenation of similar inner functions G_b, considering the balanced state sequences related to powers of G_b. To this end the following proposition is attractive.

Proposition 2 *(Roberts and Mullis, 1987; Heuberger et al., 1995.) Let G_b be a square inner transfer function with minimal balanced realization (A, B, C, D) having state dimension $n_b > 0$. Then for any $k > 1$ the realization (A_k, B_k, C_k, D_k) with*

$$A_k = \begin{bmatrix} A & 0 & \cdots & \cdot & 0 \\ BC & A & 0 & \cdot & 0 \\ BDC & BC & \cdot & \cdot & 0 \\ \vdots & \vdots & \cdot & \ddots & 0 \\ BD^{k-2}C & BD^{k-3}C & \cdots & BC & A \end{bmatrix} \qquad B_k = \begin{bmatrix} B \\ BD \\ BD^2 \\ \vdots \\ BD^{k-1} \end{bmatrix} \tag{1}$$

$$C_k = \begin{bmatrix} D^{k-1}C & D^{k-2}C & \cdots & DC & C \end{bmatrix} \qquad D_k = D^k \tag{2}$$

is a minimal balanced realization of G_b^k with state dimension $n_b \cdot k$. □

Given a balanced realization of G_b one can directly construct a balanced realization of G_b^k for any $k > 0$. However there is more to the structure of the realization shown above. By evaluating the realization it follows that it can be constructed by the following recursive mechanism:

$$A_k = \begin{bmatrix} A_{k-1} & 0 \\ BC_{k-1} & A \end{bmatrix} \qquad B_k = \begin{bmatrix} B_{k-1} \\ BD^{k-1} \end{bmatrix} \tag{3}$$

$$C_k = \begin{bmatrix} D^{k-1}C & C_{k-1} \end{bmatrix} \qquad D_k = D \cdot D_{k-1}. \tag{4}$$

By writing down the equation for the state trajectory related to the realization (A_k, B_k, C_k, D_k):

$$x_k(t+1) = A_k x_k(t) + B_k u(t) = \begin{bmatrix} A_{k-1} & 0 \\ BC_{k-1} & A \end{bmatrix} x_k(t) + \begin{bmatrix} B_{k-1} \\ BD^{k-1} \end{bmatrix} u(t) \tag{5}$$

it follows from the structure of A_k and B_k that for any input and initial state, $x_k(t)$ can be written as

$$x_k(t) = \begin{bmatrix} x_{k-1}(t) \\ \phi_k(t) \end{bmatrix}, \tag{6}$$

and thus

$$x_k(t) = \begin{bmatrix} \phi_1^T(t) & \phi_2^T(t) & \cdots & \phi_k^T(t) \end{bmatrix}^T. \tag{7}$$

By Proposition 1, this state sequence, under pulse input conditions, provides an orthonormal basis for $\ell_2[1, \infty)$ when $k \to \infty$. By construction, this particular state sequence satisfies

$$x_k(t+1) = \begin{bmatrix} v_1(t+1) \\ v_2(t+1) \\ \vdots \\ v_k(t+1) \end{bmatrix} = A_k \cdot \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_k(t) \end{bmatrix} + B_k \delta(t), \tag{8}$$

while $x_k(t) = 0$, $t \leq 0$.

It is a particular choice to structure this basis in terms of the n_b-dimensional components $v_k(t)$. This is motivated by the shift structure in the basis, as specified in the next Proposition.

Proposition 3 *Let G_b be a scalar inner function with McMillan degree $n_b > 0$, having a minimal balanced realization (A, B, C, D). Consider $v_k(t)$ as defined before. Then*

$$v_{k+1}(t) = G_b(q) I_{n_b} \cdot v_k(t) \quad k = 1, 2, \cdots \tag{9}$$

$$v_1(t) = A^{t-1} B \tag{10}$$

where the shift operator q operates on the time sequence v_k, and $v_k(t) = 0$ for $t \leq 0$. □

PROOF. By definition $v_1(t)$ is determined by $v_1(t+1) = Av_1(t) + B\delta(t)$ with $v_1(t) = 0$ for $t \leq 0$. This directly shows (10).
For any value of $k \geq 1$, $v_k(t)$ is determined by $v_k(t+1) = Av_k(t) + Bu_k(t)$ with $u_k(t) = G_b^{k-1}(q)\delta(t)$ and $v_k(t) = 0$, $t \leq 0$. Consequently, $v_{k+1}(t+1) = Av_{k+1}(t) + BG_b(q)u_k(t)$ which leads to $v_{k+1}(t) = (qI - A)^{-1}BG_b(q)u_k(t)$. This latter expression equals $G_b(q)(qI - A)^{-1}Bu_k(t) = G_b(q)v_k(t)$. □

Directly resulting from the basis for $\ell_2[1, \infty)$, a basis for the related Hilbert space of strictly proper stable systems in $\mathcal{H}_2$ follows, by considering the z-transforms of the $\ell_2[1, \infty)$-signals. Denoting

$$V_k(z) := \sum_{t=1}^{\infty} v_k(t) z^{-t} = (zI - A)^{-1} B G_b^{k-1}(z), \tag{11}$$

the components of the n_b-dimensional rational functions $V_k(z)$ will constitute an orthonormal basis for the set of strictly proper systems in $\mathcal{H}_2$.

Note that these basis functions exhibit the property that they can incorporate systems dynamics in a very general way. One can construct an inner function G_b from any given set of poles, and thus the resulting basis can incorporate dynamics of any complexity, combining e.g. both fast and slow dynamics in damped and resonant modes.

For specific choices of $G_b(z)$ well known classical basis functions can be generated.

(1) With $G_b(z) = z^{-1}$, having minimal balanced realization $(0, 1, 1, 0)$, the standard pulse basis $V_k(z) = z^{-k}$ results.

(2) Choosing a first order inner function $G_b(z) = \dfrac{1-az}{z-a}$, with some real-valued a, $|a| < 1$, and balanced realization

$$(A, B, C, D) = (a, \sqrt{1-a^2}, \sqrt{1-a^2}, -a) \tag{12}$$

the Laguerre basis results:

$$V_k(z) = \sqrt{1-a^2}\frac{(1-az)^{k-1}}{(z-a)^k}. \tag{13}$$

(3) Similarly the two-parameter Kautz functions (Kautz, 1954; Wahlberg, 1994a), originate from the choice of a second order inner function

$$G_b(z) = \frac{-cz^2 + b(c-1)z + 1}{z^2 + b(c-1)z - c} \tag{14}$$

with some real-valued b, c satisfying $|c|, |b| < 1$. see Heuberger et al. (1995). A balanced realization is given by

$$A = \begin{bmatrix} b & \sqrt{(1-b^2)} \\ c\sqrt{(1-b^2)} & -bc \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \sqrt{(1-c^2)} \end{bmatrix} \tag{15}$$

$$C = \begin{bmatrix} \sqrt{(1-b^2)(1-c^2)} & -b\sqrt{(1-c^2)} \end{bmatrix} \qquad D = -c. \tag{16}$$

Considering the general class of basis functions, for any strictly proper system $H(z) \in \mathcal{H}_2$ or signal $y(t) \in \ell_2[1, \infty)$ there exist unique series expansions:

$$H(z) = \sum_{k=1}^{\infty} L_k^T V_k(z) \qquad L_k \in \mathbb{R}^{n_b \times 1}, \tag{17}$$

$$y(t) = \sum_{k=1}^{\infty} \mathcal{Y}^T(k) v_k(t) \qquad \mathcal{Y}(k) \in \mathbb{R}^{n_b \times 1}. \tag{18}$$

These expressions are collected in the two networks that are sketched in Figures 1 and 2. Figure 1 denotes a dynamical system representation $y(t) = H(z)u(t)$ in terms of a series expansion of H as in (17). Figure 2 shows the related network for the series expansion of an ℓ_2-signal, exposing the basis functions $v_k(t)$ explicitly. In both networks, the arrows departing from mid-way block locations refer to balanced *state* readout.

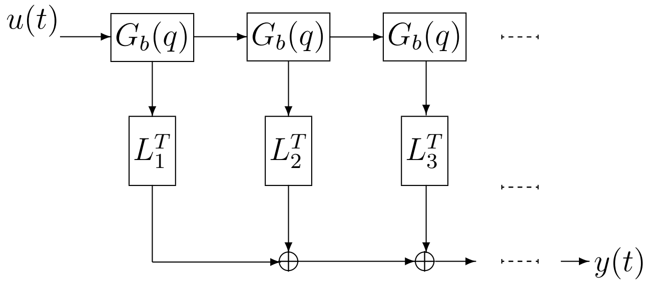


Fig. 1. Dynamical system representation employing the $\mathcal{H}_2$ basis functions.

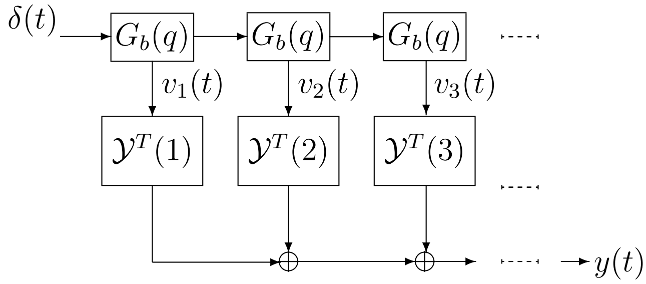


Fig. 2. Representation of ℓ_2-signal in terms of the ℓ_2 basis functions $v_k(t)$.

Signal transformations

The ℓ_2-basis functions presented in the previous section generate a signal transformation $\ell_2[1,\infty) \to \ell_2[1,\infty)$ as follows. When denoting the scalar valued functions $F_k(z)$ satisfying:

$$\begin{bmatrix} V_1(z) \\ V_2(z) \\ \vdots \\ V_k(z) \end{bmatrix} = \begin{bmatrix} F_1(z) \\ F_2(z) \\ \vdots \\ F_{kn_b}(z) \end{bmatrix}$$

and $f_j(t)$ defined by $F_j(z) = \sum_{t=1}^{\infty} f_j(t) z^{-t}$, then there exists a mapping $\ell_2[1,\infty) \to \ell_2[1,\infty)$ defined by $y \to \eta$ with $\eta(k) =< f_k, y >$.

A second transformation that will appear to be very attractive is the mapping $\ell_2[1,\infty) \to \ell_2^{n_b}[1,\infty)$ that is defined by the n_b-dimensional basis functions v_k according to $y \to \mathcal{Y}$ with $\mathcal{Y}(k) =< v_k, y >$. Due to the particular shift structure of the functions v_k, as indicated in Proposition 3, this latter transformations will appear to be very attractive. The shift structure will also enable the construction of a dynamical system transformation in terms of rational functions, as will be discussed in section 5.

In order to specify the transformations and to formulate their properties, first a number of structural properties of the basis functions have to be investigated further.

3 The dual bases in ℓ_2 and $\ell_2^{n_b}$

A basis for ℓ_2 that is dual to the basis presented in the previous section, is given in the next proposition.

Proposition 4 *Let G_b be a scalar inner function with McMillan degree $n_b > 0$, generating basis functions $f_k(t)$ as discussed before. Denote*

$$\gamma_k(t) := f_t(k) \quad \text{for } t, k = 1, ..., \infty. \tag{19}$$

Then the ℓ_2-signals γ_k constitute an orthonormal basis for $\ell_2[1, \infty)$, which is dual to the basis in the previous section, in the sense that for each $y \in \ell_2[1, \infty)$ there is a transform $\eta \in \ell_2[1, \infty)$ given by $\eta(k) =< y, f_k >$ and $y(k) = < \eta, \gamma_k >$, such that

$$\eta(t) = \sum_{k=1}^{\infty} y(k)\gamma_k(t). \tag{20}$$

PROOF. The fact that $\{\gamma_k(t)\}_{k=1,\cdots,\infty}$ is an orthonormal basis of $\ell_2[1, \infty)$ follows directly from Heuberger et al. (1995). It results from the fact that the controllability matrix of G_b^k not only has orthonormal rows, but for $k \to \infty$ also columns that are orthonormal and complete. With $\eta(k)$ defined by $\eta(k) =< y, f_k >$, it follows that $< \eta, \gamma_k >= \sum_{t,j=1}^{\infty} y(t)f_j(t)f_j(k) = y(k)$. □

For the original n_b-dimensional basis reflected by $v_k(t)$ we formulated a nice shift structure, as given in Proposition 3. We will next formulate a similar result for the dual basis. However we first need the following lemma.

Lemma 5 *Let G_b be a scalar inner function with McMillan degree $n_b > 0$, having a minimal balanced realization (A, B, C, D). Then (D, C, B, A) is a minimal balanced realization of the $n_b \times n_b$ inner function*

$$N(z) := A + B(z - D)^{-1}C \tag{21}$$

having McMillan degree equal to 1. □

PROOF. Provided (A, B, C, D) is a minimal realization, it represents an inner transfer function if and only if $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a unitary matrix (Roberts and Mullis, 1987; Heuberger et al. 1995). Since a unitary matrix remains unitary under row and column permutations, it follows that $\left[\begin{smallmatrix} D & C \\ B & A \end{smallmatrix}\right]$ is unitary. The realization (D, C, B, A) is minimal because $BC \neq 0_{n_b \times n_b}$, as either $B = 0$ or $C = 0$ would contradict the minimality property of (A, B, C, D); balancedness follows directly from the unitarity property, and stability is a result of $|D| < 1$. □

Proposition 6 *Let G_b be a scalar inner function with McMillan degree $n_b > 0$, having a minimal balanced realization (A, B, C, D). Consider $v_k(t)$ and $N(z)$ as defined before, and define*

$$w_k(t) := v_t(k). \tag{22}$$

Then

$$w_{k+1}(t) = N(q) \cdot w_k(t) \quad k = 1, 2, \cdots \tag{23}$$
$$w_1(t) = BD^{t-1} \tag{24}$$

where the shift operator q operates on the time sequence w_k, and $\psi_k(t) = 0$ for $t \leq 0$. □

PROOF. The proof follows from considering the last block row of equation (8). Using Proposition 2 this equation shows that for $k \geq 1$

$$\begin{aligned} v_{k+1}(t+1) = {} & [0_{n_b \times n_b} \; 0_{n_b \times n_b} \; \ldots \; I_{n_b}] A_{k+1} \cdot \\ & \cdot [v_1^T(t) \cdots v_{k+1}^T(t)]^T + BD^k \delta(t). \end{aligned} \tag{25}$$

Equivalently, by changing indexes,

$$w_{k+1}(t+1) = \sum_{i=1}^{t} BD^{i-1} C w_k(t+1-i) + A w_k(t+1) + BD^t \delta(k), \tag{26}$$

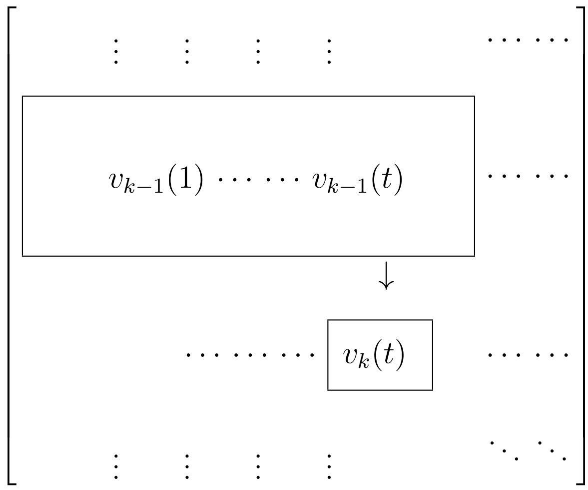
which proves the result. □

Both recursions show that G_b and N clearly play a dual role. They are simply related by ordering the state space realizations reversely. This duality can also be visualized by considering the controllability matrix of the balanced realization (A_n, B_n, C_n, D_n) of G_b^n, which contains the basis functions $\{v_k\}_{k=1,\cdots,n}$ in the following way:

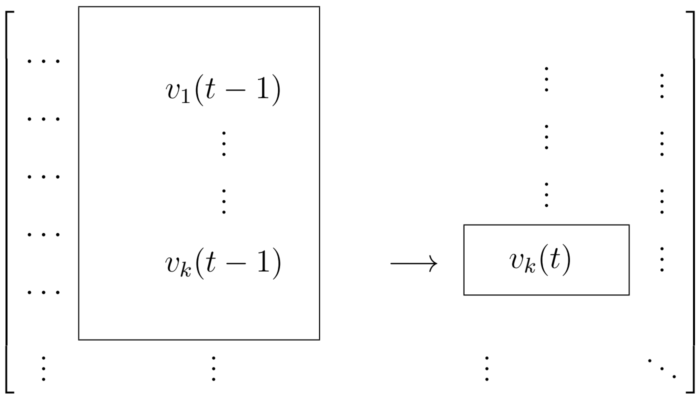
$$\begin{bmatrix} v_1(1) & v_1(2) & v_1(3) & \cdots & \cdots \\ v_2(1) & v_2(2) & v_2(3) & \cdots & \cdots \\ v_3(1) & v_3(2) & v_3(3) & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

In this matrix the element $v_k(t)$ can be written recursively in two different forms, indicated next.

- Determination of $v_k(t)$ by a recursion through $G_b(q)$, i.e. $v_k(t) = G_b(q) \cdot v_{k-1}(t)$, where the shift q^{-1} operates on the time index t:



- Determination of $v_k(t)$ by a recursion through $N(q)$, i.e. $v_k(t) = N(q) \cdot v_k(t-1)$ where q^{-1} operates on the sequence index k:



A similar duality that is present between G_b and N can be considered between the signal sequences $v_k(t)$ and $w_k(t)$. Whereas $v_k(t)$ originates from the balanced states of G_b^k, a different - but closely related - phenomenon occurs for $w_k(t)$. This is reflected in the following Proposition.

Proposition 7 *Consider the square inner function $N(z)$ with minimal balanced realization (D, C, B, A), and let $x_k(t)$ be the balanced state trajectory of N^k, $k \geq 1$. Then $w_k(t)$ is the output of the dynamical system $N^k(z)$ for zero input and initial state $x_k(0) = e_1$.* □

PROOF. For $k = 1$ it follows directly that pulse excitation of the first state leads to the output $y_1(t) = BD^{t-1}$ which equals $w_1(t)$. As by construction $y_{k+1}(t) = N(q)y_k(t)$ this proves the result. □

The construction of the $\ell_2^{n_b}[1, \infty)$-signals $w_k(t)$ is schematically depicted in the network of Figure 3. This network also shows how the transform $y \to \mathcal{Y}$ as discussed in the previous section can be effectively calculated. In this respect the network is dual to the network depicted in Figure 2.

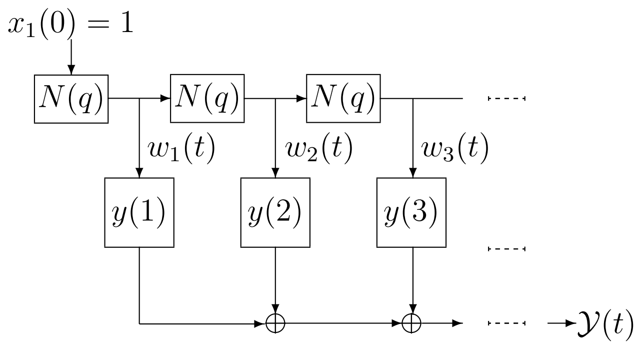


Fig. 3. Network showing the construction of $w_k(t)$ related to the dual basis, for calculation of the signal transform $y \to \mathcal{Y}$, $\mathcal{Y}(t) = \sum_{k=1}^{\infty} y(k)w_k(t)$.

Similar to (11) we will denote the z-transform of the functions $w_k(t)$ by

$$W_k(z) := \sum_{t=1}^{\infty} w_k(t)z^{-t} \tag{27}$$

while as a direct result of Proposition 6 it holds that

$$W_k(z) = N^{k-1}(z) \cdot W_1(z) \tag{28}$$

with $W_1(z) := (z - D)^{-1}B$.

It has to be noted that, whereas the scalar stable functions $\{F_k(z)\}_{i=1,\cdots,n_b}^{k=1,\cdots\infty}$ constructed from $V_k(z)$ constitute an orthonormal basis for the strictly proper systems in $\mathcal{H}_2$, the dual form of this basis is *not* given by the scalar components of $\{W_k(z)\}_{k=1,\cdots\infty}$. This is due to the fact that $\gamma_k(t)$ is not simply composed of the vector components of $w_k(t)$, but that a reshuffling of the components over time is required to arrive at the orthonormal (dual) basis for ℓ_2, indicated by

$$\begin{bmatrix} w_k(1) \\ w_k(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} \gamma_k(1) \\ \gamma_k(2) \\ \vdots \end{bmatrix}.$$

This phenomenon is illustrated in the following example.

Example 8 *Consider the inner function $G_b(z) = z^{-2}$, having state dimension $n_b = 2$. Then*

$$W_{k+1}(z) = \begin{bmatrix} z^{-1} & 0 \end{bmatrix}^T z^{-k/2} \tag{29}$$

$$W_{k+2}(z) = \begin{bmatrix} 0 & z^{-1} \end{bmatrix}^T z^{-k/2} \quad k = 0, 2, 4, \cdots \tag{30}$$

and hence $\{e_i^T W_k(z)\}_{i=1,\cdots,n_b}^{k=1,\cdots\infty}$ is given by $\{z^{-1}, 0, z^{-1}, 0, z^{-2}, 0, z^{-2}, \cdots\}$ which clearly is not a basis. □

As a result of this situation, the dual form of basis for the dynamical system representation as sketched in Figure 1 is not equally simple. It involves a time-reordering of signals, as reflected in the next result.

Proposition 9 *Let $\{W_k(z)\}$ be defined as before. Then $\{[z^{n_b-1} \ z^{n_b-2} \ \cdots \ 1]W_k^T(z^{n_b})\}_{k=1,\cdots}$ constitutes a basis for the strictly proper systems in $\mathcal{H}_2$. The corresponding basis functions satisfy*

$$[z^{n_b-1} \ z^{n_b-2} \ \cdots \ 1]W_k^T(z^{n_b}) = \sum_{t=1}^{\infty} \gamma_k(t) z^{-t}.$$

PROOF. By using $w_k^T(t) = [\gamma_k((t-1)n_b + 1) \ \cdots \ \gamma_k(tn_b)]$ it follows that the k-th function in the Proposition equals

$$\sum_{\ell=1}^{n_b} z^{n_b-\ell} \sum_{t=1}^{\infty} \gamma_k((t-1)n_b + \ell) z^{-tn_b}$$

which is equal to $\sum_{j=1}^{\infty} \gamma_k(j) z^{-j}$. Since the sequence of ℓ_2-functions $\{\gamma_k\}$ is known to be a basis for $\ell_2[1,\infty)$ this proves the result. □

The situation of the Proposition is depicted in the network of Figure 4, where the basis functions are interpreted as

$$W_k^T(z^{n_b})[z^{n_b-1}\ z^{n_b-2}\ \cdots\ 1]^T$$

and where use is made of the fact that $W_1^T(z^{n_b}) = (z^{n_b} - D)^{-1}B^T$ is the input to (balanced) state transfer of $N^T(z^{n_b})$. The network may give the impression that it is noncausal. However, since the input-to-state transfer of $N^T(z^{n_b})$ incorporates a time-delay of n_b steps, this shows that the network reflects a strictly proper transfer function.

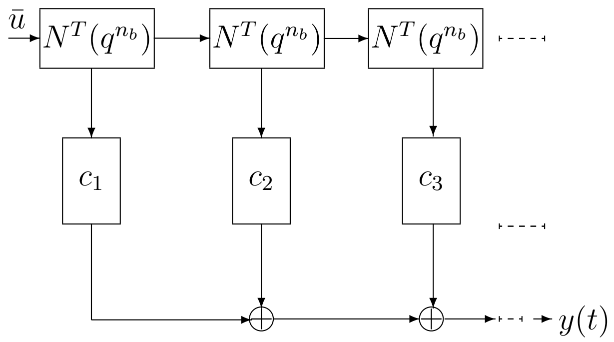


Fig. 4. Dynamical system representation employing the scalar ℓ_2 (dual) basis functions $\gamma_k(t)$, with $\bar{u} = [u(t+n_b-1)\cdots u(t)]^T$.

There exists also a second duality, in the sense that the set of functions $\{w_k(t)\}$ constitute a basis for $\ell_2^{n_b}$ and equivalently the set of functions $\{W_k(z)\}$ constitute a basis for $\mathcal{H}_2^{n_b}$. This is reflected in the next proposition.

Proposition 10 *Let G_b be a scalar inner function with McMillan degree $n_b > 0$, having a minimal balanced realization (A, B, C, D). Let $N(z)$, $W_k(z)$ and $w_k(t)$ as defined before. Then $\{w_k(t), k = 1, \cdots, \infty\}$ and $\{W_k(z), k = 1, \cdots, \infty\}$ constitute an orthonormal basis for $\ell_2^{n_b}[1,\infty)$ respectively the Hilbert space of strictly proper stable systems in $\mathcal{H}_2^{n_b}$.* □

PROOF. Orthonormality follows from:

$$< w_i, w_j >= \sum_t w_i^T(t) w_j(t) = \sum_s \gamma_i(s)\gamma_j(s) = \delta_{ij}.$$

because of Proposition 4. To show that $\{w_k(t)\}$ constitutes a basis, is it sufficient to show that for any $x \in \ell_2^{n_b}$ the Parseval identity holds:

$$\sum_k | < x, w_k > |^2 = \|x\|^2.$$

Now define $y \in \ell_2[1,\infty)$ by $[y(1)y(2)\cdots] = [x(1)^T x(2)^T \cdots]$. It is clear that $\|y\|^2 = \|x\|^2$. Now we can write:

$$\sum_k |\sum_t x^T(t) w_k(t)|^2 = \sum_l |\sum_s y(s)\gamma_l(s)|^2 = \|y\|^2 = \|x\|^2,$$

which proves the result. □

As a consequence, for any strictly proper system $L(z) \in \mathcal{H}_2^{n_b}$ or signal $\mathcal{Y}(t) \in \ell_2^{n_b}[1,\infty)$ there exist unique series expansions:

$$L(z) = \sum_{k=1}^{\infty} h_k W_k(z), \qquad h_k \in \mathbb{R}, \tag{31}$$

$$\mathcal{Y}(t) = \sum_{k=1}^{\infty} y(k) w_k(t), \qquad y(k) \in \mathbb{R}. \tag{32}$$

In fact, these expansions are exactly the inverses of the expansions given by the equations (17-18). This will further be formalized in section 4.

Example 11 *Applying again the situation of Example 8:*

$$W_{k+1}(z) = \begin{bmatrix} z^{-1} & 0 \end{bmatrix}^T z^{-k/2} \tag{33}$$

$$W_{k+2}(z) = \begin{bmatrix} 0 & z^{-1} \end{bmatrix}^T z^{-k/2} \quad k = 0, 2, 4, \cdots \tag{34}$$

however now applied to a strictly proper system $L(z) = [L_1(z)\ L_2(z)]^T \in \mathcal{H}_2^{n_b}$ *with* $L_j(z) = \sum_{k=1}^{\infty} l_{j,k} z^{-k}$*. Then by considering the vector-valued basis functions* $\{W_k(z)\}$*, it follows that*

$$L(z) = l_{1,1}W_0(z) + l_{2,1}W_1(z) + l_{1,2}W_2(z) + l_{2,2}W_3(z) + \cdots \tag{35}$$

$$= \sum_{k=1}^{\infty} l_{1,k}W_{2k-2}(z) + l_{2,k}W_{2k-1}(z). \tag{36}$$

This shows the -unique- basis expansions of the multivariable system. □

4 The Hambo signal transform

With the results of the previous section, we now have the basic ingredients for specifying and analysing the signal transforms that have been anounced in section 2.

Definition 12 *Let* $\{v_k(t)\}_{k=1,\cdots\infty}$ *be a sequence of* $\ell_2^{n_b}[1,\infty)$*-functions, being generated by an inner function* G_b *with McMillan degree* n_b *as presented in Section 2. Then we define the Hambo-transform as the mapping* H*:* $\ell_2^m[1,\infty) \rightarrow \mathcal{H}_2^{n_b\times m}$*, determined by*

$$\mathsf{H}(y) := \tilde{y}(\lambda) = \sum_{k=1}^{\infty} \mathcal{Y}(k)\lambda^{-k} \tag{37}$$

with the Hambo coefficients $\mathcal{Y}(k) \in \ell_2^{n_b\times m}[1,\infty)$*, determined by*

$$\mathcal{Y}(k) := \sum_{t=1}^{\infty} v_k(t) y^T(t). \tag{38}$$

Through this transformation, ℓ_2-signals are transformed to a transform domain, however signal vectors are transformed to matrices. This Hambo transform can be considered as a generalization of the Fourier or the z-transform, the latter of which for a signal $y \in \ell_2[1,\infty)$ is given by $y(z) = \sum_{t=1}^{\infty} y(t)z^{-t}$. This z-transform is generated by (37) employing the orthonormal (pulse) basis, $v_k(t) = \delta(k-t)$, corresponding to $G_b(z) = z^{-1}$.

Some basic properties of this signal transform are collected in the following Proposition.

Proposition 13 *The Hambo signal transform as defined in Definition 12 satisfies*

(1) $\tilde{y}(\lambda) = \sum_{k=1}^{\infty} W_k(\lambda) y^T(k)$.

(2) $\tilde{y}(\lambda) = [y^T(z)z]_{z^{-1}=N(\lambda)} \cdot [I_m \otimes W_1(\lambda)]$

(3) For scalar y $(m = 1)$*:*

$$\tilde{y}(\lambda) = [y(z)z]_{z^{-1}=N(\lambda)} \cdot W_1(\lambda) \tag{39}$$

$$= [y(z)]_{z^{-1}=N(\lambda)} \cdot \breve{W}_1(\lambda) \tag{40}$$

with $\breve{W}_1(\lambda) := N(\lambda)^{-1} W_1(\lambda) = [1 - \lambda D]^{-1} C^T$.

PROOF. Part (i) follows directly by substituting the definitions. For the proof of part (ii) substitute (28) into the result of part (i), i.e. $\tilde{y}(\lambda) = \sum_{k=1}^{\infty} N^{k-1}(\lambda)W_1(\lambda)x^T(k)$. When y is a scalar signal this reduces to $\tilde{y}(\lambda) = \sum_{k=1}^{\infty} y(k)N^{k-1}(\lambda) \cdot W_1(\lambda)$ which proves (39). The multidimensional result (ii) follows then from the fact that for $y = [y_1\ y_2]^T$, with y_1, y_2 scalar signals, part (i) of the Proposition shows that $\tilde{y} = [\tilde{y}_1\ \tilde{y}_2]$.
The similarity of (39) and (40) can be shown by verifying that $W_1(\lambda) = N(\lambda)\breve{W}_1(\lambda)$. Using state space forms for N and $\breve{W}_1$ their product satisfies $[A + B(\lambda - D)^{-1}C]C^T(1 - \lambda D)^{-1}$. Using $AC^T = -BD$ this expression is equal to $B[-D(\lambda - D) + CC^T](\lambda - D)^{-1}(1 - \lambda D)^{-1}$ and by using $CC^T + DD = 1$ it reduces to $B(\lambda - D)^{-1}$ which is equal to $W_1(\lambda)$. □

The Hambo transform also yields an inverse transform, formulated next.

Proposition 14 (Inverse Hambo transform) *The inverse Hambo transform* $\mathsf{H}^{-1} : \mathcal{H}_2^{n_b \times m} \to \ell_2^m$ *is defined by*

$$\mathsf{H}^{-1}(\tilde{y})(t) := \sum_{k=1}^{\infty} \mathcal{Y}^T(k) w_t(k) = y(t) \tag{41}$$

$$\text{with} \qquad \mathcal{Y}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{y}(e^{i\omega}) e^{ik\omega} d\omega. \tag{42}$$

PROOF. Equation (42) follows by inverse z-transform of (37). The validity of (41) follows from substitution of (38), leading to $\mathsf{H}^{-1}(\tilde{y})(t) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} y(\ell) v_k^T(\ell) w_t(k)$ which equals $\sum_{\ell=1}^{\infty} y(\ell) \sum_{k=1}^{\infty} w_\ell^T(k) w_t(k)$. Since $\sum_{k=1}^{\infty} w_\ell^T(k) w_t(k) = \sum_{k=1}^{\infty} \gamma_\ell(k)\gamma_t(k) = \delta(\ell - t)$ this shows the validity of (41). □

Dual to Proposition 13 the following results can be formulated for the inverse transform.

Proposition 15 *The Inverse Hambo transform as defined above satisfies*

(1) $y(z) = \sum_{k=1}^{\infty} \mathcal{Y}^T(k) V_k(z).$

(2)

$$y^T(z) = V_1^T(z) \cdot [\tilde{y}(\lambda)\lambda]_{\lambda^{-1} = G_b(z)}, \tag{43}$$

$$= \breve{V}_1^T(z) \cdot [\tilde{y}(\lambda)]_{\lambda^{-1} = G_b(z)} \tag{44}$$

with $\breve{V}_1(z) := G_b(z)^{-1} V_1(z) = [I - zA^T]^{-1} C^T.$

PROOF. Part (i) follows directly by substituting the definitions.
For the proof of Part (ii) substitute (11) into Part (i) showing that $y(z) = \sum_{k=1}^{\infty} V_1^T(z)G_b^{k-1}(z)\mathcal{Y}(k)$. The latter expression equals $V_1^T(z)\cdot\sum_{k=1}^{\infty}\mathcal{Y}(k)G_b^{k-1}(z)$ which proves (43). The similarity of (43) and (44) can be shown by verifying that $\breve{V}_1(z) = V_1(z)G_b^{-1}(z)$. With the inner property of G_b this is equivalent to $\breve{V}_1(z) = V_1(z)G_b(z^{-1})$. Using state space forms for G_b and V_1 it follows that

$$\breve{V}_1(z) = (zI - A)^{-1}B[D + B^T(zI - A^T)^{-1}C^T.$$

Using $BD = -AC^T$ and $BB^T = (I - AA^T)$ this expression leads to

$$\begin{aligned}\breve{V}_1(z) &= (zI - A)^{-1}[-A + (I - AA^T)z(I - zA^T)^{-1}]C^T && (45)\\ &= (zI - A)^{-1}[-A(I - zA^T) + (I - AA^T)z](I - zA^T)^{-1}C^T && (46)\\ &= (I - zA^T)^{-1}C^T && (47)\end{aligned}$$

which proves the result. □

One interesting phenomenon in both Hambo transform and inverse transform is that they essentially can be obtained by complex variable transformations. Further implications of these variable transformations will be discussed in the next section when analysing the related transform of dynamical systems.

Because of the fact that the introduced Hambo transform is isomorphic, quadratic signal properties, for signals $y_j, y_\ell \in \ell_2^m[1,\infty)$, simply satisfy

$$\begin{aligned}\sum_{t=1}^{\infty} y_j(t)y_\ell^T(t) &= \sum_{k=1}^{\infty} \mathcal{Y}_j^T(k)\mathcal{Y}_\ell(k)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{y}_j^T(e^{-i\omega})\tilde{x}_\ell(e^{i\omega})d\omega.\end{aligned}$$

The diagram shown in Figure 5 sketches the different transforms that have been considered for $\ell_2^m[1,\infty)$-signals.

5 The Hambo system transform

The Hambo transform of ℓ_2-signals, as introduced in the previous section, induces also a linear system transformation. This transformed system describes the dynamical relationship between (transformed) input and output signals.

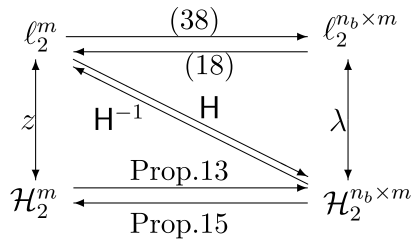


Fig. 5. Commuting diagram showing the Hambo-transform and the Inverse Hambo-transform, defined on ℓ_2-signals.

Proposition 16 *Let $P \in \mathcal{H}_2$ and let $u, y \in \ell_2$ such that $y(t) = P(q)u(t)$. Consider the Hambo transform of ℓ_2 signals as defined in definition 12. Then there exists a $\tilde{P} \in \mathcal{H}_2^{n_b \times n_b}$ satisfying*

$$\tilde{y}(\lambda) = \tilde{P}(\lambda)\tilde{u}(\lambda). \tag{48}$$

The mapping T: $\mathcal{H}_2 \to \mathcal{H}_2^{n_b \times n_b}$ defined by $\mathsf{T}(P) := \tilde{P}(\lambda)$ is referred to as the Hambo system-transform*, and the inverse mapping T^{-1}: $\mathcal{H}_2^{n_b \times n_b} \to \mathcal{H}_2$ given by $\mathsf{T}^{-1}(\tilde{P}) := P(z)$ is denoted as the* inverse Hambo system-transform. □

PROOF. The proof is by construction, and is given in the proof of the next Proposition.

Proposition 17 *Consider the situation of Proposition 16. Let P be written as $P(z) = \sum_{k=0}^{\infty} p_k z^{-k}$. Then the Hambo system-transform $\mathsf{T}(P)$ is determined by*

$$\tilde{P}(\lambda) = \sum_{k=0}^{\infty} p_k N(\lambda)^k, \tag{49}$$

or differently denoted: $\tilde{P}(\lambda) = P(z)|_{z^{-1}=N(\lambda)}$. □

PROOF. Writing $y(z) = P(z)u(z)$ it follows with Proposition 13b that $\tilde{y}(\lambda) = [P(z)u(z)z]_{z^{-1}=N(\lambda)} \cdot W_1(\lambda)$. It follows directly that $\tilde{y}(\lambda) = P(z)|_{z^{-1}=N(\lambda)} \cdot \tilde{u}(\lambda)$ which proves the result. □

The Hambo-transform of any system P can be obtained by a simple variable-transformation on the original transfer function, where the variable transformation concerned is given by $z^{-1} = N(\lambda)$.

In terms of the sequence of expansion coefficients, equation (48) shows that

$$\mathcal{Y}(k) = \tilde{P}(q)\mathcal{U}(k) \quad \text{for all } k \geq 1 \tag{50}$$

where $\mathcal{Y}$,$\mathcal{U}$ are $\ell_2^{n_b}[1,\infty)$ sequences induced by the Hambo transforms of y, u, and the shift operator q operates on the sequence index k. Note that this result generalizes the situation of a corresponding Laguerre transformation, where it concerns the variable-transformation $z = \dfrac{\lambda + a}{1 + a\lambda}$ (see also Wahlberg, 1991), or equivalently $z^{-1} = \dfrac{1 + a\lambda}{\lambda + a}$. The right hand side of this latter expression has a state space realization $(-a, \sqrt{1-a^2}, \sqrt{1-a^2}, a)$, being the reversed-order realization of G_b given in (12). However, unlike the Laguerre case, the Hambo-transformed system $\tilde{P}$ increases in input/output-dimension to $\tilde{P} \in \mathcal{H}_2^{n_b \times n_b}$, due to the fact that in our case the McMillan degree of the inner function that generates the basis is $n_b \geq 1$. Note again that $N(\lambda)$ is an $n_b \times n_b$ rational transfer function matrix of McMillan degree 1 (since D is scalar).

The previous Proposition considers scalar ℓ_2 signals and scalar systems. For some multivariable situations there exist straightforward extensions of this result.

Proposition 18 *Consider a scalar transfer function* $P \in \mathcal{H}_2$ *relating* m*-dimensional input and output signals* $y, u \in \ell_2^m[1,\infty)$*, according to*[4] $y(t) = [P(z)I_m]u(t)$*. Then* $\tilde{y}(\lambda) = \tilde{P}(\lambda)\tilde{u}(\lambda)$*, with* $\tilde{P}(\lambda)$ *as defined in* (49). □

PROOF. For $m = 1$ the result is shown in Proposition 16. If we write the relation between y and u componentwise, i.e. $y_i(t) = P(z)u_i(t)$ it follows from the mentioned Proposition that $\tilde{y}_i(\lambda) = \tilde{P}(\lambda)\tilde{u}_i(\lambda)$, where $\tilde{y}_i, \tilde{u}_i \in \mathbb{R}^{n_b \times 1}(\lambda)$. It follows directly that $\tilde{y}(\lambda) = [\tilde{y}_1(\lambda) \ \cdots \ \tilde{y}_m(\lambda)] =$
$\tilde{P}(\lambda)[\tilde{u}_1(\lambda) \ \cdots \ \tilde{u}_m(\lambda)] = \tilde{P}(\lambda)\tilde{u}(\lambda)$. □

The basis generating inner function G_b transforms itself to a simple shift in the Hambo-domain, as is reflected in the following Proposition.

Proposition 19 *Consider a scalar inner transfer function* $G_b(z)$ *generating an orthogonal basis as discussed before. Then* $\tilde{G}_b(\lambda) = \lambda^{-1}I_{n_b}$. □

PROOF. It follows from (9) that for all k, $G_b(q)I_{n_b}\phi_k(t) = \phi_{k+1}(t)$. With Proposition 18 it follows that $\tilde{\phi}_{k+1}(\lambda) = \tilde{G}_b(\lambda)\tilde{\phi}_k(\lambda)$. Using Proposition 13(i), it follows that for each k, $\tilde{\phi}_k(\lambda) = \sum_{t=1}^{\infty} W_t(\lambda)\phi_k^T(t)$. Substituting (27) now shows that $\tilde{\phi}_k(\lambda) = \sum_{t=1}^{\infty} I_{n_b}\delta(t-k)\lambda^{-t}$, and it follows that for all k, $I_{n_b}\lambda^{-k-1} = \tilde{G}_b(\lambda)I_{n_b}\lambda^{-k}$. Since this holds for all k it proves the result. □

[4] Since $P(z)$ is scalar we allow the notation $P(z)I_m$, which more formally should be denoted as $P(z) \otimes I_m$.

The complex (matrix) function $\tilde{P}(\lambda)$ provides an alternative representation of the dynamical system P. It will be shown that many of the system theoretic properties of $P(z)$ carry over to $\tilde{P}(\lambda)$. However there are also important differences that will be clarified in the sequel.

Many of the properties of the Hambo system-transform are derived from state-space realizations of the transformed system $\tilde{P}$. For this reason we will now present the Hambo system-transform in state space formulation.

Proposition 20 *Let G_b be an inner function with McMillan degree n_b and balanced realization (A, B, C, D), inducing a corresponding Hambo transform. Let $P \in \mathcal{H}_2$ be given by:*

$$P(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_{n_g} z^{-n_g}}{1 + a_1 z^{-1} + \cdots + a_{n_g} z^{-n_g}},$$

having McMillan degree n_g. Then $\tilde{P}(\lambda)$ has a minimal state-space realization $(A_{ort}, B_{ort}, C_{ort}, D_{ort})$ given by

$$\begin{bmatrix} A_{ort} & B_{ort} \\ C_{ort} & D_{ort} \end{bmatrix} = \begin{bmatrix} A_e - B_e(F_2 D_e)^{-1} F_2 C_e & B_e (F_2 D_e)^{-1} \\ F_1 C_e - F_1 D_e (F_2 D_e)^{-1} F_2 C_e & F_1 D_e (F_2 D_e)^{-1} \end{bmatrix}$$

with $A_e \in \mathbb{R}^{n_g \times n_g}$, $B_e \in \mathbb{R}^{n_g \times n_b}$, $C_e \in \mathbb{R}^{n_b(n_g+1) \times n_g}$, $D_e \in \mathbb{R}^{n_b(n_g+1) \times n_b}$ given by:

$$A_e = \begin{bmatrix} D & 0 & 0 & . & . \\ CB & D & 0 & . & . \\ . & . & . & . & . \\ CA^{n_g-2}B & . & . & CB & D \end{bmatrix}; \quad B_e = \begin{bmatrix} C \\ CA \\ . \\ CA^{n_g-1} \end{bmatrix}$$

$$C_e = \begin{bmatrix} 0 & 0 & 0 & . & 0 \\ B & 0 & 0 & . & . \\ AB & B & 0 & . & . \\ . & . & . & . & . \\ A^{n_g-1}B & . & . & AB & B \end{bmatrix}; \quad D_e = \begin{bmatrix} I \\ A \\ A^2 \\ . \\ A^{n_g} \end{bmatrix}$$

$$F_1 = [b_0 I_{n_b} \quad b_1 I_{n_b} \quad \cdots \quad b_{n_g} I_{n_b}] \quad \in \mathbb{R}^{n_b \times n_b(n_g+1)}$$
$$F_2 = [I_{n_b} \quad a_1 I_{n_b} \quad \cdots \quad a_{n_g} I_{n_b}] \quad \in \mathbb{R}^{n_b \times n_b(n_g+1)}.$$

PROOF. The transform can be written as $\tilde{P}(\lambda) = P_1(\lambda)P_2^{-1}(\lambda)$ where

$$P_1(\lambda) = b_0 \times I + b_1 \times N(\lambda) + \cdots + b_{n_g} \times N^{n_g}(\lambda)$$
$$P_2(z) = I + a_1 \times N(\lambda) + \cdots + a_{n_g} \times N^{n_g}(\lambda)$$

It is straightforward that $P_j(\lambda)$ has realization $(A_e, B_e, F_jC_e, F_jD_e)$. It can be readily verified (Heuberger, 1991) that $P_1(\lambda)P_2^{-1}(\lambda)$ is a fractional representation of $\tilde{P}$ and that $(A_{ort}, B_{ort}, C_{ort}, D_{ort})$ is a realization of $\tilde{P}$. Here the stability of P and G_b ensures the invertibility of F_2D_e.
Minimality of the realization follows from analyzing the situation $n_g = 1$. By direct calculations it can be shown that $\tilde{P}$ has McMillan degree smaller than 1 if and only if P has a pole-zero cancellation. By writing P as a product of first order terms, minimality follows under the condition that P has McMillan degree n_g. □

One of the direct consequences of this Proposition is:

Corollary 21 *Let $\tilde{P}$ be the Hambo system-transform of a scalar dynamical system $P \in \mathbb{R}\mathcal{H}_2$. Then $\tilde{P}$ and P have the same McMillan degree.* □

The poles and zeros of P and $\tilde{P}$ also have close relationships.

Proposition 22 *Let $\tilde{P}$ be the Hambo system-transform of a scalar dynamical system $P \in \mathbb{R}\mathcal{H}_2$, induced by the inner function G_b.*

(1) If $P(z)$ has a stable pole in $z = \alpha$, then $\tilde{P}(\lambda)$ has a stable pole in $\lambda = G_b(\alpha^{-1})$.
(2) If $P(z)$ has a zero in $z = \beta$, then $\tilde{P}(\lambda)$ has a zero in $\lambda = G_b(\beta^{-1})$. □

PROOF. For the case $n_g = 1$ with $P(z)$ having a pole in $z = \alpha$, it follows from the realization in Proposition 20 that $\tilde{P}$ has a pole in $D + \alpha C(I - \alpha A)^{-1}B$, which is equal to $G_b(\alpha^{-1})$. Since G_b is inner, its amplitude will be bounded by 1 outside the unit disc, and so stability is preserved. A similar result follows for the zeros. □

This leads to the following corollary.

Corollary 23 *Let $P \in \mathbb{R}\mathcal{H}_2$ have poles α_i, $i = 1, \cdots, n_g$, and let $G_b(z)$ have poles ρ_j, $j = 1, \cdots n_b$. Then $\tilde{P}$ will have poles*

$$\mu_i = \prod_{j=1}^{n_b} \frac{\alpha_i - \rho_j^*}{1 - \alpha_i \rho_j} \qquad i = 1, \cdots, n_g. \tag{51}$$

PROOF. The corollary follows directly from the previous Proposition by using the fact that $G_b(z)$ can be written as $G_b(z) = \prod_{j=1}^{n_b} \frac{1-\rho_j^* z}{z-\rho_j}$. □

Remark 24 *Note that if $G_b \in \mathbb{R}\mathcal{H}_2$ it's poles come in complex conjugate pairs and hence equation 51 can also be written as*

$$\mu_i = \prod_{j=1}^{n_b} \frac{\alpha_i - \rho_j}{1 - \alpha_i \rho_j} \qquad i = 1, \cdots, n_g. \tag{52}$$

Additionally we show that the Hambo system transform has certain contraction properties in the sense that under specific conditions the poles are being contracted by the transformation.

Corollary 25 *If G_b is strictly proper, then each pole α of P will be transformed to a pole $\mu(\alpha)$ of $\tilde{P}$, with $|\mu(\alpha)| \leq |\alpha|$.* □

PROOF. By Proposition 22 $\mu(\alpha) = G_b(\alpha^{-1})$ which equals $G_b(\alpha)^{-1}$, as G_b is inner. If G_b is strictly proper then $zG_b \in \mathcal{H}_2$ and inner, which implies that $|zG_b(z)| \geq 1$ for $|z| \leq 1$. Since $|\alpha| < 1$ it thus follows that $|\alpha G_b(\alpha)| \geq 1$ which proves $|G_b(\alpha^{-1})| = |G_b(\alpha)|^{-1} \leq |\alpha|$. □

Finally, we will specify how the inverse Hambo system transform can be calculated.

Proposition 26 *Let $\tilde{P} \in \mathcal{H}_2^{n_b \times n_b}$. Then the inverse Hambo system-transform $\mathsf{T}^{-1}(\tilde{P})$ is determined by*

$$P(z) = zV_1^T(z) \cdot \tilde{P}(\lambda)|_{\lambda^{-1}=G_b(z)} \cdot \frac{B}{1 - DG_b(z)} \tag{53}$$

$$= z\breve{V}_1^T(z) \cdot [\tilde{P}(\lambda)W_1(\lambda)]_{\lambda^{-1}=G_b(z)}. \tag{54}$$

PROOF. Let $p(t)$ be the pulse response of $P(z)$. Then $p(t-1) = P(q)\delta(t-1)$. Let $y(t) := p(t-1)$. Then , $\tilde{y}(\lambda) = \tilde{P}(\lambda)\tilde{\delta}(\lambda)$, with (by Proposition 13) $\tilde{\delta}(\lambda) = W_1(\lambda)$. By inverse signal transform, $y(z) = V_1^T(z) \cdot [\tilde{P}(\lambda)W_1(\lambda)\lambda]_{\lambda^{-1}=G_b(z)}$.
Using $P(z) = zy(z)$ then proves the result of (53).
Then (54) follows from the fact that

$$\frac{B}{1 - DG_b(z)} = G_b^{-1}(z) \cdot \frac{B}{\lambda - D}|_{\lambda^{-1}=G_b(z)} = G_b^{-1}(z) \cdot W_1(\lambda)_{\lambda^{-1}=G_b(z)}.$$

As a result, $P(z) = zV_1^T(z)G_b^{-1}(z) \cdot [\tilde{P}(\lambda)W_1(\lambda)]_{\lambda^{-1}=G_b(z)}$, which together with the definition of $\breve{V}_1$ proves (54). □

In correspondence with the forward transform, the inverse transform maps a system in $\mathcal{H}_2^{n_b \times n_b}$ back to a scalar system.

The Hambo system transform exhibits several more nice properties, as e.g. invariance properties of Hankel singular values and several norms as the $\mathcal{H}_\infty$-norm and the $\mathcal{H}_2$-norm. For more details the reader is referred to Heuberger (1991).

6 Hambo transform in system approximation

The most straightforward use of the signal and system transformations discussed in this paper, is in the area of system identification and system approximation. In Van den Hof et al. (1995) it has been shown how the Hambo transform can be used to specify the spectral density function that underlies the (block) Toeplitz matrix in a related least-squares identification problem. In that setting the expansion coefficients of a dynamical system in terms of a prespecified basis are identified from data.
Here attention will be given in particular to the approximation problem. Suppose we have been given a scalar stable and strictly proper dynamical system $P(z)$, then we can represent this system in the series expansion:

$$P(z) = \sum_{k=1}^{\infty} L_k^T V_k(z). \tag{55}$$

For an appropriate choice of the inner function G_b, the basis functions $V_k(z)$ should match the most dominant components of $P(z)$ such that the series expansion will have a high rate of convergence. In other words: for a given approximate model

$$\hat{P}_n(z) := \sum_{k=1}^{n} L_k^T V_k(z) \tag{56}$$

the approximation error $\|P - \hat{P}_n\|$ (in some norm), will be dependent on the choice of $V_k(z)$. It will be shown that the rate of convergence of this series expansion can be explicitly related to the dynamics that is present in P and G_b, by using the transform results discussed previously.

First we look at the classical way of explicitly writing the approximate model in any (orthonormal) series expansion, by using the so-called reproducing kernel (Davis, 1975).

Lemma 27 *Let any system* $P \in \mathcal{H}_2$ *be given as in* (55). *Then* $\hat{P}_n(z)$ *will satisfy*

$$\hat{P}_n(e^{i\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_0(e^{i\zeta}) \cdot K_n(\omega, \zeta) d\zeta$$
$$\textit{with} \quad K_n(\omega, \zeta) := \sum_{k=1}^{n} V_k^T(e^{i\omega}) V_k(e^{-i\zeta}). \tag{57}$$

PROOF. This classical result in Hilbert space operator theory follows directly by using the inner product expression for L_k:

$$L_k =< P_0, V_k >= \frac{1}{2\pi} \int_{-\pi}^{\pi} P_0(e^{i\omega}) V_k(e^{-i\omega}) d\omega \tag{58}$$

and substituting it in the expression for $\hat{P}_n(z)$. □

The role of the reproducing kernel $K_n(\omega, \zeta)$ (57) has also been discussed in Ninness et al. (1997) in the same approximation setting, and in De Vries and Van den Hof (1995) in the context of frequency domain identification algorithms.
Next we show how the reproducing kernel can be explicitly calculated. As discussed also in Ninness et al. (1997), in the classical approximation theory (Davis, 1975) such an explicit expression is referred to as a Christoffel-Darboux formula.

Lemma 28 (Christoffel-Darboux formula) *Let* G_b *be a scalar inner function, and let* V_1 *be as defined before. Then for any* $z_1, z_2 \in \mathbb{C}$ *with* $z_1 z_2 \neq 1$*:*

$$V_1^T(z_1) V_1(z_2) = \frac{G_b(z_1) G_b(z_2) - 1}{1 - z_1 z_2}. \tag{59}$$

PROOF. Using a balanced state space realization (A, B, C, D) of G_b it follows that $G_b(z_1)G_b(z_2) = [V_1^T(z_1)C^T + D^T][D + CV_1(z_2)]$. Writing out this product, and using the properties $C^T D = -A^T B$, $C^T C = I - A^T A$ and $D^T D = I - B^T B$, this leads to

$$G_b(z_1)G_b(z_2) = -V_1^T(z_1)A^TB - B^TAV_1(z_2) + V_1^T(z_1)(I - A^TA)V_1(z_2) + \\ +1 - B^TB.$$

Now substituting $B = (z_2I - A)V_1(z_2)$ and $B^T = V_1^T(z_1)(z_1I - A^T)$ shows that

$$\begin{aligned} G_b(z_1)G_b(z_2) - 1 &= V_1^T(z_1)\Big[-A^T(z_2I - A) - (z_1I - A^T)A + (I - A^TA) \\ &\qquad -(z_1I - A^T)(z_2I - A)\Big]V_1(z_2) \\ &= V_1^T(z_1)(1 - z_1z_2)V_1(z_2) \end{aligned}$$

which proves the result. □

For a simplified situation we now can formulate expressions for the approximation error $P(z) - \hat{P}_n(z)$, as formulated next.

Proposition 29 *Let P_0 (55) have all single poles $|p_j^0| < 1$, $j = 1, \cdots n_0$ so that it can be written in a partial fraction expansion*

$$P_0(z) = \sum_{j=1}^{n_0} \frac{b_j}{z - p_j^0},$$

and let G_b have poles $\{\xi_1, \cdots, \xi_{n_b}\}$. Then

$$P(z) - \hat{P}_n(z) = \sum_{j=1}^{n_0} \frac{b_j}{z - p_j^0}[G_b(\frac{1}{p_j^0})G_b(z)]^n \quad \textit{and} \tag{60}$$

$$|P(e^{i\omega}) - \hat{P}_n(e^{i\omega})| \leq \sum_{j=1}^{n_0} \left|\frac{b_j}{e^{i\omega} - p_j^0}\right| \prod_{k=1}^{n_b} \left|\frac{p_j^0 - \xi_k}{1 - \bar{\xi}_k p_j^0}\right|^n. \tag{61}$$

PROOF. Substituting the expression for P_0 into (58) shows that

$$L_k = \sum_{j=1}^{n_0} \frac{b_j}{p_j^0} V_k(\frac{1}{p_j^0}).$$

Consequently

$$P(z) - \hat{P}_n(z) = \sum_{j=1}^{n_0} \sum_{k=n+1}^{\infty} \frac{b_j}{p_j^0} V_k^T(\frac{1}{p_j^0})V_k(z)$$

$$
\begin{aligned}
&= \sum_{j=1}^{n_0} \frac{b_j}{p_j^0} V_1^T(\frac{1}{p_j^0}) V_!(z) \sum_{r=n}^{\infty} [G_b(\frac{1}{p_j^0}) G_b(z)]^r \\
&= \sum_{j=1}^{n_0} \frac{b_j}{p_j^0} V_1^T(\frac{1}{p_j^0}) V_1(z) \cdot \frac{[G_b(\frac{1}{p_j^0}) G_b(z)]^n}{1 - G_b(\frac{1}{p_j^0}) G_b(z)}.
\end{aligned} \tag{62}
$$

Using Lemma 28 for $z_1 = 1/p_j^0$ and $z_2 = z$ then leads to

$$
P(z) - \hat{P}_n(z) = \sum_{j=1}^{n_0} \frac{b_j}{p_j^0} \frac{[G_b(\frac{1}{p_j^0}) G_b(z)]^n}{\frac{z}{p_j^0} - 1}
$$

which proves (60). Inequality (61) then follows by using the inner property of $G_b = \pm \prod_{k=1}^{n_b} \frac{1-\xi_k^* z}{z - \xi_k}$. □

The characterization of the approximation error shows that whenever the sets of poles of the original system P_0 and of the basis-generating inner function G_b come close to each other, then the rate of convergence of the series expansion can drastically improve, thus enabling more accurate system approximations with fewer terms in the expansion. Because of the contraction property, as given in Corollary 25, the poles that govern the series expansion are guaranteed to be faster than the original ones (smaller amplitude) when G_b is strictly proper.

For the single pole situation of proposition 29 no transformation results were actually needed. The solution of the general (repeated pole) case however is more complex.

If $p(t)$ is the pulse response related to $P(z)$, then $p(t) = \sum_{k=1}^{\infty} L_k^T v_k(t)$, which implies with the signal transform definitions that

$$
\tilde{p}(\lambda) = \sum_{k=1}^{\infty} L_k \lambda^{-k}.
$$

In other words, the decay rate of the sequence $\{L_k\}_{k=1,\cdots}$ is governed by the dynamics that is present in $\tilde{p}(\lambda)$.

Proposition 30 *Let $P(z) \in \mathcal{H}_2$ be strictly proper and let $p \in \ell_2$ be the pulse response related to $P(z)$. Then*

$$
\tilde{p}(\lambda) = \tilde{P}(\lambda) \frac{C^T}{1 - \lambda D} \tag{63}
$$

and the sets of poles of $\tilde{p}$ and $\tilde{P}$ are equal. □

PROOF. By writing $p(t) = [P(q)q]\delta(t-1)$ we can apply the Hambo signal transform to both sides of the equation, showing that $\tilde{p}(\lambda) = \tilde{P}(\lambda)N^{-1}(\lambda)\frac{B}{\lambda - D}$. By writing $N(\lambda)C^T = AC^T + \frac{BCC^T}{\lambda - D}$ and using $AC^T = -BD$ (because of the fact that G_b is inner), it follows that $N(\lambda)C^T = B\frac{1-D\lambda}{\lambda - D}$. Inserting this into the expression for $\tilde{p}$ shows the result (63). Since P is strictly proper there exists a proper Q such that $P(z) = Q(z)z^{-1}$, and thus $\tilde{P}(\lambda) = \tilde{Q}(\lambda)N(\lambda)$. Then combining (63) with the above expression for $N(\lambda)C^T$ shows that for $D \neq 0$ the pole at $\lambda = D^{-1}$ is cancelled out in (63). □

This result leads to the following Proposition.

Proposition 31 *Let P have poles p_j^0, $j = 1, \cdots, n_0$, and let $G_b(z)$ have poles ξ_i, $i = 1, \cdots n_b$. Denote*

$$\mu := \max_j \prod_{j=1}^{n_b} \left| \frac{p_j^0 - \xi_i}{1 - p_j^0 \xi_i} \right|. \tag{64}$$

Then there exists a constant $c \in \mathbb{R}$ such that for all $\rho > \mu$

$$\|P - \hat{P}_n\|_2 \leq c \cdot \frac{\rho^{n+1}}{\sqrt{1-\rho^2}}. \tag{65}$$

PROOF. Since μ is the maximum amplitude of poles of $\sum_{k=1}^{\infty} L_k \lambda^{-k}$, there exist $\gamma_i \in \mathbb{R}$ such that for all $\rho > \mu$, $|L_{k,i}| \leq \gamma_i \rho^{k+1}$, with $L_{k,i}$ the i^{th}-element of L_k. As $\|P - \hat{P}_n\|_2^2 = \sum_{k=n+1}^{\infty} \sum_{i=1}^{n_b} L_{k,i}^2$, the result follows by substitution. □

In the general case, the convergence properties are similar to the simple (distinct pole) case. Although it has not been possible to specify the approximation error explicitly, relevant bounding of the error shows the same dynamical properties (indicated by the poles of the transformed system) as in the simpler case.

7 Conclusions

We have analyzed a signals and systems transform that is induced by a very general class of orthogonal functions. The basis functions are induced by the balanced states of scalar inner (stable all-pass) functions, and generalize the classical Laguerre and Kautz functions. The induced signals and systems transforms generalize the Fourier and z-transform to a multidimensional representation. The transforms have been analyzed in detail, providing insight into their structural properties, and showing that they can be interesting system theoretic tools. The benefit of the transformations in a related system approximation problem has been shown.

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