Identifiability of linear dynamic networks through switching modules ⋆

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Abstract: Identifiability of linear dynamic networks typically depends on the presence and location of external (excitation or disturbance) signals, in relation to the topology of the parametrized network model set. For closed-loop identification, it is known that switching (non-parameterized) controllers can also provide excitation, thereby rendering the model set identifiable. In this paper, we derive verifiable conditions for network identifiability of the non-switching modules in presence of (non-parameterized) switching modules. These conditions generalize the classical result in closed-loop identification towards network identification. Furthermore, verifiable path-based conditions for identifiability in a generic sense are developed on the graph of the network model set.

Keywords: dynamic networks, system identification, switching systems, identifiability

1. INTRODUCTION

Technological progress towards large-scale interconnections of dynamical systems requires monitoring, control and optimization techniques to operate safely and efficiently. An attractive approach to achieve this is by using model-based techniques, for which a representation in terms of interconnected modules is advantageous. The vast availability of data, as a result of the ubiquitous presence of sensors in modern systems, motivates research of identification in linear dynamic networks. Different type of identification problems can be formulated, including topology detection, see e.g., Materassi and Innocenti (2010); Chiuso and Pillonetto (2012), identification of a single module, see e.g., Van den Hof et al. (2013); Ramaswamy et al. (2018); Dankers (2014), or identification of the full network dynamics, see e.g., Goncalves and Warnick (2008); Weerts et al. (2018b).

Before performing the actual identification, it is important to know whether a unique model can be retrieved from the identification setup, which is captured by the notion of identifiability. The derived conditions for identifiability depend on the network interconnection structure, the selection of measured nodes and the locations where external signals enter the network. These conditions are typically verified on the rank of a particular transfer matrix of the network, which, for the full network problem, is repeated for every node in the network, see e.g., Weerts et al. (2018a) for the situation that all nodes are measured and some nodes are excited. Another concept called generic identifiability is independent of numerical values of transfer matrices and relies solely on the network interconnection structure. This notion allows the rank conditions to be translated to graph-based conditions, see e.g., Bazanella et al. (2017); Hendrickx et al. (2019) for the (dual) problem where all nodes are excited and some are measured.

It is known from identifiability results for closed-loop systems, see e.g., Söderström et al. (1976), that besides excitation from external signals, identifiability of linear time-invariant plants can also be achieved by introducing time-varying or switching controllers in the system. This gives rise to the thought that, possibly additional to the presence of external signals, switching modules can be used for establishing identifiability of non-switching modules. This leads to the following research question: "under which conditions is additional excitation provided by switching modules such that identifiability of the remaining non-switching modules is ensured?"

In this paper, verifiable conditions are derived for identifiability of non-switching modules in dynamic networks in presence of switching modules and for the situation that all node signals can be measured (full-measurement case). The analysis is based on rank conditions on a specific part of transfer matrices from external signals to internal node signals that were derived for non-switching networks, see e.g., Weerts et al. (2018a). The switching modules lead to different transfer matrices for each network mode, which gives additional information on the non-switching parts that can be extracted, as has been done for the closed-loop setup with switching controllers, see e.g., Söderström et al. (1976). The presented results can be seen as a generalization of Weerts et al. (2018a) and Söderström et al. (1976). Furthermore, the developed rank conditions are connected to path-based conditions using the generic identifiability setting in a similar way as in Bazanella et al. (2017); Hendrickx et al. (2019).

Nomenclature: Denote $\mathbb{N}$ as the set of natural number, $\mathbb{R}$ as the set of real numbers, and $\mathbb{R}(z)$ is the rational function

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field over $\mathbb{R}$ with variable z. v_i denotes the i-th element of a vector v, and A_{ij} denotes the (i, j)-th entry of a matrix A. The cardinality of a set $\mathcal{V}$ is given by $|\mathcal{V}|$.

2. PROBLEM STATEMENT

In this section, a dynamic network model and an associated model set is formulated in which switching modules can be captured. The model set is based on the separation of parametrized non-switching modules and nonparametrized switching modules. Following this setup, network identifiability can be defined for networks with switching modules.

2.1 Module switching dynamic network setup

Following the dynamic network setup proposed in Van den Hof et al. (2013); Weerts et al. (2018a), a dynamic network consists of L scalar *internal variables*, *nodes* or *vertices* $w_j, j \in \{1, \ldots, L\}$, and K *external variables* $r_k, k \in \{1, \ldots, K\}$ that can be manipulated by the user. Each internal variable is described as:

$$w_j(t) = \sum_{i=1}^{L} G_{ji}(q)w_i(t) + \sum_{k=1}^{K} R_{jk}(q)r_k(t) + v_j(t) \qquad (1)$$

where q^{-1} is the delay operator, i.e., $q^{-1}w_j(t) = w_j(t-1)$ and $t \in \mathbb{N}$ denotes time. The *modules* G_{ji} contain the dynamic relationship between the nodes, excluding *self-loops*, i.e., $G_{jj} = 0$ for all j. Unmeasured *process noise* variables v_j are included, where the vector process $v = [v_1 \ldots v_L]^\top$ is modelled as a stationary stochastic process with rational spectral density $\Phi_v(\omega)$, such that there exists a p-dimensional (zero-mean) white noise process $e := [e_1 \ldots e_p]^\top$, with $p \leq L$ and covariance matrix $\Lambda > 0$ such that

$$v(t) = H(q)e(t).$$

The combination of all the L nodes can be written in terms of the full network expression

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G_{12} & \cdots & G_{1L} \\ G_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & G_{L-1L} \\ G_{L1} & \cdots & G_{LL-1} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + R \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_K \end{bmatrix} + H \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_p \end{bmatrix}, \qquad (2)$$

where dependence on q is omitted for compactness of notation. The compact form of (2) is given by

$$w = Gw + Rr + He. \qquad (3)$$

A *network model* that includes these concepts and the considered properties in this work is defined below.

Definition 1. (Network model). A network model of L nodes, and K external excitation signals, with a noise process of rank $p \leq L$ is defined by the quadruple:

$$M = (G, R, H, \Lambda)$$

with

- $G \in \mathbb{R}^{L \times L}(z)$, diagonal entries 0, all modules proper and stable; [1]
- $R \in \mathbb{R}^{L \times K}(z)$, proper;
- $H \in \mathbb{R}^{L \times p}(z)$, stable, with a left stable inverse and satisfying $H(q) = \begin{bmatrix} H_a \\ H_b \end{bmatrix}$ with H_a square, proper, monic, stable and minimum-phase;
- $\Lambda \in \mathbb{R}^{p \times p}$, $\Lambda > 0$;
- The network is well-posed [2], see e.g., Dankers (2014), with $(I - G)^{-1}$ proper and stable.

In this definition, the flexibility of a rank-reduced noise process ($p < L$), i.e., a non-square noise model H, is allowed, see e.g., Weerts et al. (2018a).

In order to capture switching modules in the defined network model, a *multimode network model* is defined, which contains $m \in \mathbb{N}$ different *network modes*, where the current network mode is denoted by ℓ.

Definition 2. (Multimode network model). A multimode dynamic network with m modes is defined as a finite set of network models M_ℓ, i.e.,

$$\mathbb{M} := \{M_\ell\}_{\ell \in \{1, \ldots, m\}}, \qquad (4)$$

with $M_\ell = (G_\ell, R_\ell, H_\ell, \Lambda_\ell)$.

In this paper we will not consider the switching mechanism as such, but we will assume that in each mode the transfer functions between measured signals can be estimated from data. In this multimode network model, the modules that remain the same for every mode are called *mode-invariant.* Since these links may contain different dynamics or modules for some network modes, it is more convenient to introduce some graph-based concepts that specify the structure and distinguish between mode-invariant and switching links. In this setting the nodes are referred to as *vertices* and links as *edges*, for which the distinction is defined below.

Definition 3. (Multimode network graph). A multimode network graph is a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, which captures the structure of multimode dynamic network $\mathbb{M}$, with vertex set $\mathcal{V} := \{1, \ldots, L\}$ and the set of edges by

$$\mathcal{E} := \{(i, j) \in \mathcal{V} \times \mathcal{V} \mid G_{ji,\ell} \neq 0 \text{ for some } \ell \in \{1, \ldots, m\}\}, \qquad (5)$$

where the mode-invariant edges are defined by

$$\mathcal{E}_{inv} := \{(i, j) \in \mathcal{E} \mid G_{ji,\ell_1} = G_{ji,\ell_2} \ \forall\, \ell_1, \ell_2 \in \{1, \ldots, m\}\}, \qquad (6)$$

and the set of switching edges as $\mathcal{E}_s = \mathcal{E} \backslash \mathcal{E}_{inv}$.

This multimode network graph is helpful in explicitly specifying the mode-invariant and switching parts in the multimode dynamic network. This is used in the definition of a *module switching network model* that is given below.

Definition 4. (Module switching network model). A module switching dynamic network with m modes, is a multimode dynamic network $\mathbb{M}$, with $M_\ell = (G_\ell, R_\ell, H_\ell, \Lambda_\ell) \in \mathbb{M}$ that satisfies

$$\begin{aligned} R_\ell &= R, & \forall \ell \in \{1, \ldots, m\} \\ H_\ell &= H, & \forall \ell \in \{1, \ldots, m\} \\ \Lambda_\ell &= \Lambda, & \forall \ell \in \{1, \ldots, m\} \\ G_\ell &= G^{inv} + G^s_\ell, & \end{aligned} \qquad (7)$$

[1] The assumption of having all modules stable is made in order to guarantee that T_{we} is a stable spectral factor of the noise process that affects the node variables.

[2] This implies that all principal minors of $(I - G(\infty))^{-1}$ are nonzero.

where for each edge either the element in G^{inv} or G^s_ℓ is 0, i.e., for $(i,j) \notin \mathcal{E}_{inv}$, it holds that $G^{inv}_{ji} = 0$ and for $(i,j) \notin \mathcal{E}_s$, it holds that $G^s_{ji,\ell} = 0$ for all $\ell \in \{1,\dots,m\}$.

2.2 Full network identifiability problem

The considered problem in this paper is identifiability of the full network, in which switching modules are present with known dynamics. Therefore, in any parametrized model set for the network only the mode-invariant modules in $\mathcal{E}_{inv}$ need to be parametrized. The restriction in parametrization leads to the following definition of the model set that is considered for the identifiability problem.

Definition 5. (Module switching network model set). A module switching network model set $\mathcal{M}$ for a module switching network $\mathbb{M}$ with m modes, is defined as

$$\mathcal{M} := \{\mathbb{M}(\theta), \theta \in \Theta\} \tag{8}$$

where $\mathbb{M}(\theta) = \{M_\ell(\theta)\}_{\ell\in\{1,\dots,m\}}$ and

$$M_\ell(\theta) = (G^{inv}(\theta) + G^s_\ell, R(\theta), H(\theta), \Lambda(\theta)) \tag{9}$$

with $G^{inv}_{ji}(\theta), (i,j) \in \mathcal{E}_{inv}$ and Θ a finite dimensional parameter space, $\Theta \subset \mathbb{R}^{n_\theta}$.

The objective for this model set is to assess whether the structural properties in terms of location and presence of excitation signals and topology of the network can distinguish different candidate models. In other words, the objective is to assess whether a unique model exists, without taking into account whether the data is informative enough. To do so, the concepts of identifiability for non-switching networks in Weerts et al. (2018a) are generalized to include excitation through known switching modules. For this, the transfer matrices from external signals r and e to internal nodes w for all the network modes ℓ are used, which for each mode are given by

$$T_\ell(\theta) := (I - G_\ell(\theta))^{-1} U(\theta) \tag{10}$$

with

$$U(\theta) := [R(\theta)\ H(\theta)]\,. \tag{11}$$

These transfer matrices contain all the structural information used by most identification methods under mild conditions, see e.g., Weerts et al. (2018a). These are, either absence of direct feedthrough terms or of algebraic loops, which occur when all modules in a loop contain direct feedthrough terms. By this result, the following definition of network identifiability for the module switching network model set is formulated, which concerns uniqueness of elements in the model and disregards uniqueness of parameters.

Definition 6. (Module switching network identifiability). The module switching network model set $\mathcal{M}$ is network identifiable from (r,w) at $\mathbb{M}_0 := \mathbb{M}(\theta_0)$ if for all models $\mathbb{M}(\theta_1) \in \mathcal{M}$, it holds that

$$T_\ell(\theta_1) = T_\ell(\theta_0)\ \forall \ell \in \{1,\dots,m\} \Rightarrow \mathbb{M}(\theta_1) = \mathbb{M}(\theta_0). \tag{12}$$

We call $\mathcal{M}$ globally network identifiable from (r,w) if (12) holds for all $\mathbb{M}_0 \in \mathcal{M}$. and we call $\mathcal{M}$ generically network identifiable from (r,w) if (12) holds for almost all $\mathbb{M}_0 \in \mathcal{M}$.

The concept of generic identifiability follows from the concept introduced in Bazanella et al. (2017); Hendrickx et al. (2019) for non-switching networks. The above definitions differ from the ones in literature by allowing information contained in transfer matrices for all network modes together to draw conclusions on the existence of a unique model. All the transfer matrices act as additional equations to solve for the unknown elements, which is the main aspect that will be exploited in this paper.

3. RANK CONDITIONS FOR NETWORK IDENTIFIABILITY

In this section, verifiable conditions for identifiability of the defined model set are derived on the basis of transfer matrices, as a natural approach arising from Definition 6. These conditions generalize the developed rank conditions in Weerts et al. (2018a), where switching modules are not considered, and the identifiability results on a closed-loop setup with a switching controller in Söderström et al. (1976). The relation of the developed conditions to these methods will also be studied.

3.1 Identifiability of module switching networks

As in Weerts et al. (2018a) we start from evaluating (10) in the form $(I - G_\ell(\theta))T_\ell(\theta) = U(\theta)$, before analyzing under which conditions $G_\ell(\theta)$ and $U(\theta)$ are uniquely retrieved from $T_\ell(\theta)$. Suppose that row j of $(I - G_\ell(\theta))$ has α_j parametrized transfer functions, and, similarly, each row j of $U(\theta)$ has β_j parametrized transfer functions. The objective is to isolate these parametrized transfer functions for each row and assess whether they can be uniquely retrieved from the transfer matrices. This is done through permutation of both matrices, for which permutation matrices $P_j \in \mathbb{N}^{L\times L}$ and $Q_j \in \mathbb{N}^{K+p\times K+p}$ are introduced. These gather all parametrized entries in row j of $(I - G_\ell(\theta))$ on the left hand side, and all parametrized entries in the considered row of $U(\theta)$ on the right hand side, i.e.,

$$(I - G_\ell(\theta))_{j\star} P_j = \begin{bmatrix}(I - G(\theta))^{(1)}_{j\star} & (I - G_\ell)^{(2)}_{j\star}\end{bmatrix} \tag{13}$$

$$\begin{aligned} U(\theta)_{j\star} Q_j &= \begin{bmatrix} U^{(1)}_{j\star} & U(\theta)^{(2)}_{j\star}\end{bmatrix} \\ &= \begin{bmatrix} U^{(1)}_{j\star} & 0\end{bmatrix} + U(\theta)^{(2)}_{j\star} V_j \end{aligned} \tag{14}$$

with $V_j = \begin{bmatrix} 0 & I_{\beta_j}\end{bmatrix}$ of appropriate dimensions. These permutation matrices allow to formulate a particular sub-matrix of the transfer matrix (10). This sub-matrix consists of the mapping for network mode ℓ from all the external excitation signals to the input nodes of the parametrized modules that connect to node j, which is given by

$$\breve{T}_{j,\ell}(\theta) = \begin{bmatrix} I_{\alpha_j} & 0\end{bmatrix} P_j^{-1} T_\ell(\theta) Q_j. \tag{15}$$

This matrix may contain different dynamics for some network modes, which is information that is helpful in uniquely retrieving the parametrized modules. Therefore, this matrix is central in formulating the following result for identifiability of module switching network model sets.

Theorem 7. The module switching network model set $\mathcal{M}$

I) is network identifiable from (r,w) at $\mathbb{M}(\theta_0)$ if for each row j the matrix $\begin{bmatrix} \breve{T}_{j,1}(\theta_0) & \dots & \breve{T}_{j,m}(\theta_0) \\ V_j & \dots & V_j \end{bmatrix}$ has full row rank for $\theta_0 \in \Theta$;

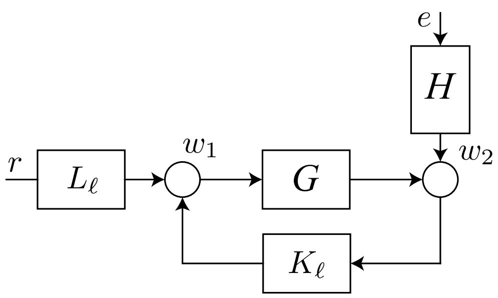

Fig. 1. A multimode closed-loop setup where the controller K_ℓ and the prefilter L_ℓ switch between different modes.

II) is globally network identifiable from (r, w) if for each row j the matrix $\begin{bmatrix} \breve{T}_{j,1}(\theta) & \dots & \breve{T}_{j,m}(\theta) \\ V_j & \dots & V_j \end{bmatrix}$ has full row rank for all $\theta \in \Theta$.

III) is generically network identifiable from (r, w) if for each row j the matrix $\begin{bmatrix} \breve{T}_{j,1}(\theta) & \dots & \breve{T}_{j,m}(\theta) \\ V_j & \dots & V_j \end{bmatrix}$ has full row rank for almost all $\theta \in \Theta$.

Proof. The proof is given in the Appendix.

Theorem 7 effectively generalizes two results. Namely, the presented derivation to reach the matrix (15) corresponds to the results in Weerts et al. (2018a) for one network mode, whereas juxtaposition of all the transfer matrices corresponds to the closed-loop results presented in Söderström et al. (1976). The relation between these methods and the presented method is studied further below.

3.2 Non-switching networks

In order to show that the presented method generalizes the results in Weerts et al. (2018a), we evaluate the rank condition in Theorem 7 for a non-switching network, i.e., $m = 1$. We can write

$$\begin{bmatrix} \breve{T}_j(\theta) \\ V_j \end{bmatrix} = \begin{bmatrix} \breve{T}_j^{(1)}(\theta) & \breve{T}_j^{(2)}(\theta) \\ 0 & I_{\beta_j} \end{bmatrix}$$

from which it follows directly that this matrix has full row rank if the submatrix $\breve{T}_j^{(1)}(\theta)$ has full row rank, which is exactly the condition as presented in Weerts et al. (2018a), i.e. the matrix composed of the first $K + p - \beta_j$ columns of $\breve{T}_j$.

3.3 Switching controllers in closed-loop

The relation between the rank condition in Theorem 7 and the classical identifiability condition from Söderström et al. (1976) for switching controllers in a closed-loop system configuration, can be evaluated by considering the simple (feedback) network depicted in Figure 1, corresponding to the equation

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 & K_\ell \\ G(\theta) & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} L_\ell & 0 \\ 0 & H(\theta) \end{bmatrix} \begin{bmatrix} r \\ e \end{bmatrix}. \quad (16)$$

This network has a transfer matrix, given by (10),

$$T_\ell(\theta) = \begin{bmatrix} (I - K_\ell G(\theta))^{-1} L_\ell & K_\ell (I - G(\theta) K_\ell)^{-1} H(\theta) \\ (I - G(\theta) K_\ell)^{-1} G L_\ell & (I - G(\theta) K_\ell)^{-1} H(\theta) \end{bmatrix}. \quad (17)$$

The analysis presented in this paper is performed for every node j, while there is no parametrized module connected to node 1, i.e., there are no parametrized entries in row 1 of (16). Therefore, it suffices to analyze the situation for node 2 only, which has the convenient structure that the permutation matrices are given by $P_2 = Q_2 = I_2$, since the parametrized entries are already correctly ordered in (16). Then, application of the conditions in Theorem 7 leads to verifying that the following matrix has to be full row rank

$$\begin{bmatrix} \breve{T}_{j,1} & \cdots & \breve{T}_{j,m} \\ V_j & \cdots & V_j \end{bmatrix} = \begin{bmatrix} \hat{S}_1 L_1 & K_1 \hat{S}_1 \hat{H} & \cdots & \hat{S}_m L_m & K_m \hat{S}_m \hat{H} \\ 0 & I & \cdots & 0 & I \end{bmatrix} \quad (18)$$

with $\hat{S}_\ell = (I - \hat{G} K_\ell)^{-1}$ where the elements that depend on θ are denoted by $\hat{(\cdot)}$ for brevity. Now, the following decomposition can be made

$$\begin{bmatrix} \hat{S}_1 L_1 & K_1 \hat{S}_1 \hat{H} & \cdots & \hat{S}_m L_m & K_m \hat{S}_m \hat{H} \\ 0 & I & \cdots & 0 & I \end{bmatrix} = \Xi_1 \Xi_2 \Xi_3, \quad (19)$$

where

$$\Xi_1 = \begin{bmatrix} 0 & I \\ \hat{H}^{-1} & -\hat{H}^{-1}\hat{G} \end{bmatrix}, \Xi_2 = \begin{bmatrix} I & \cdots & I & 0 & \cdots & 0 \\ K_1 & \cdots & K_m & L_1 & \cdots & L_m \end{bmatrix},$$

$$\Xi_3 = \begin{bmatrix} \mathrm{diag}(\hat{P}_1 L_1, \ldots, \hat{P}_m L_m) & \mathrm{diag}(\hat{S}_1 \hat{H}, \ldots, \hat{S}_m \hat{H}) \\ I & 0 \end{bmatrix}$$

with $\hat{P}_\ell = (I - \hat{G} K_\ell)^{-1} \hat{G}$ and where $\mathrm{diag}(\hat{P}_1 L_1, \ldots, \hat{P}_m L_m)$ is the block diagonal matrix with $\hat{P}_1 L_1, \ldots, \hat{P}_m L_m$ on the diagonal entries, and $\mathrm{diag}(\hat{S}_1 \hat{H}, \ldots, \hat{S}_m \hat{H})$ is defined similarly. Since, the matrices Ξ_1 and Ξ_3 are always full rank, due to the well-posedness of the network and the monicity of the noise model H, the rank condition in (18) depends on the row rank of Ξ_2. This rank condition corresponds exactly to the condition of Söderström et al. (1976), which shows that the conditions of Theorem 7 generalize this result.

Remark 8. The defined module switching network model does not include external signals that enter nodes through switching modules, like L_ℓ in the closed-loop case. This is not a principle limitation but rather a choice for simplifying exposition. Furthermore, mode switching in L_ℓ does not affect the row rank of Ξ_2 and therefore does not affect the identifiability result.

4. PATH-BASED CONDITIONS

The developed rank conditions of Theorem 7 will typically require knowledge of the different transfer functions in every mode of the network, thereby making them unattractive for actual computations. However, following the reasoning introduced in Bazanella et al. (2017); Hendrickx et al. (2019) for non-switching networks, in the generic situation the rank evaluation can be replaced by path-based conditions on the graph of the network models. In this section we will extend this reasoning to the situation of having switching modules in networks.

Let us recall graph $\mathcal{G}$ as defined in Section 2, for which an extended graph can be formulated that includes the correlation structure in H and R from (2), as in Cheng et al. (2019). This is defined below.

Definition 9. (Extended graph (Cheng et al., 2019)). Consider a directed dynamic network (2). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be its underlying graph. An extended graph $\hat{\mathcal{G}} = (\hat{\mathcal{V}}, \hat{\mathcal{E}})$ is defined as

$$\hat{\mathcal{V}} := \mathcal{V} \cup \{L+1, L+2, \ldots, L+K+p\} \tag{20}$$

$$\hat{\mathcal{E}} := \mathcal{E} \cup \{(i,j) \mid i \in \hat{\mathcal{V}}, j \in \mathcal{V}, U_{j,i-L}(\theta) \neq 0\}, \tag{21}$$

with $L = |\mathcal{V}|$ the number of nodes, K the number of external signals $r(t)$ and p the number of noise signals $e(t)$.

Here, the additional vertices in $\hat{\mathcal{V}}$ with respect to $\mathcal{V}$ consist of all external signals $e(t)$ and $r(t)$, and additional edges in $\hat{\mathcal{E}}$ reflect the correlation structure of these signals. Therefore, the extended graph captures the structure of graph $\mathcal{G}$ and correlation structure of external signals simultaneously. This allows to formulate the set of parametrized in-neighbors of node j as

$$\mathcal{W}_j := \{i \in \hat{\mathcal{V}} \,|(i,j) \in \mathcal{E}_{inv} \text{ or } U_{j,i-L} \text{ is parametrized}\}. \tag{22}$$

Let $\mathcal{U}$ be the set of all external signals $r(t)$ and $e(t)$ entering the network.

For formulating the path-based conditions for generic identifiability, we will use a technical result for the situation of non-switching networks, following Hendrickx et al. (2019); Cheng et al. (2019). In order to properly deal with possibly parametrized entries in $G_{inv}(\theta)$ and $U(\theta)$, we follow the line of reasoning of Cheng et al. (2019).

Assumption 10. (Cheng et al. (2019)). In model set $\mathcal{M}$,

- all parametrized transfer functions are parametrized independently;
- each row and column of $U(\theta)$ contains either a single nonzero (parametrized or nonparametrized) entry or only multiple nonzero parametrized entries.

Then, generic identifiability can be verified using the following lemma, which was inspired by results in, e.g., Hendrickx et al. (2019). This result is formulated using the concept of *maximum number of vertex-disjoint paths* from a set $\mathcal{A}$ to $\mathcal{B}$, denoted by $b_{\mathcal{A}\to\mathcal{B}}$, which corresponds to the maximum number of paths from the set of nodes $\mathcal{A}$ to the set of nodes $\mathcal{B}$ that do not share any nodes or vertices.

Lemma 11. (Cheng et al. (2019)). For $m = 1$, a network model set $\mathcal{M}$ that satisfies Assumption 10 is generically identifiable from (r, w) if in its extended graph $\hat{\mathcal{G}}$

$$b_{\mathcal{U}\to\mathcal{W}_j} = |\mathcal{W}_j| \tag{23}$$

holds for all $j \in \mathcal{V}$.

There also exist necessary conditions of this result, under additional conditions, see e.g., Cheng et al. (2019). This path-based condition has also been connected to the rank conditions in Weerts et al. (2018a), see e.g., Van der Woude (1991); Hendrickx et al. (2019). Application of this relation in the presented framework in this paper for the situation $m = 1$ leads to

$$b_{\mathcal{U}\to\mathcal{W}_j} = \text{rank } T_{\mathcal{W}_j\mathcal{U},1} \tag{24}$$

with

$$T_{\mathcal{W}_j\mathcal{U},\ell} = \begin{bmatrix} \breve{T}_{j,\ell} \\ V_j \end{bmatrix} \tag{25}$$

for $\ell \in \{1, \ldots, m\}$, which indicates the mapping from external signals at the nodes $\mathcal{U}$ to the internal set of nodes $\mathcal{W}_j$. This close relation will be used to establish path-based conditions for general module switching networks.

In the path-based conditions developed below, the partitioning of the edge set of Definition 3 is used. The particular set of interest is the set of switching edges $\mathcal{E}_s$, which is used to specify the output nodes of these edges as *stimulated.* These stimulated nodes can be defined by the following set

$$\mathcal{S} := \{j \in \mathcal{V} \mid (i,j) \in \mathcal{E}_s \wedge b_{\mathcal{U}\to\{w_i\}} \neq 0\}, \tag{26}$$

where the second condition ensures that at least one path from an external signal to the input node of the switching edge exists. Without an excitation signal reaching the input, the switching module would not appear in any transfer matrix in Theorem 7 and this situation is therefore excluded. The set of stimulated nodes by switching modules plays a key role, since the excitation that these switching modules produce is similar to the effect of excitation by external signals.

To ensure that these stimulated nodes $\mathcal{S}$ provide additional excitation, the following Assumption is formulated on the transfer matrix $T_{\mathcal{S}\mathcal{U},\ell}$, which is composed by selecting the corresponding rows to $\mathcal{S}$ from T_ℓ in (10).

Assumption 12. The matrix

$$\begin{bmatrix} T_{\mathcal{S}\mathcal{U},1} & \cdots & T_{\mathcal{S}\mathcal{U},m} \\ I & \cdots & I \end{bmatrix} \tag{27}$$

is full row rank.

The assumption implies that sufficient network modes are available and that there are no particular dynamics and switching sequences that would result in loss of rank. In other words, the condition requires a particular form of independence of the dynamics in a sufficient number of switching modes.

The role in terms of excitation that the stimulated nodes $\mathcal{S}$ play is formulated in the following main result.

Theorem 13. Consider a module switching dynamic network model set $\mathcal{M}$ with $\mathcal{S}$, given in (26), for which Assumptions 10 and 12 are satisfied. Then, $\mathcal{M}$ is generically identifiable from (r, w) if in its extended graph $\hat{\mathcal{G}}$

$$b_{(\mathcal{U}\cup\mathcal{S})\to\mathcal{W}_j} = |\mathcal{W}_j| \tag{28}$$

holds for every node $j \in \mathcal{V}$.

Proof. The proof is given in the Appendix.

The result of Theorem 13 shows that switching modules provide excitation in a similar way as external signals, with the main difference that switching modules require input from an external excitation signal. This formulation allows for easier interpretation on the influence of switching modules on identifiability of dynamic networks than conditions in Theorem 7, while in addition efficient existing algorithms like the Ford-Fulkerson algorithm can be used for evaluation, see e.g., Ford and Fulkerson (1956).

5. EXAMPLE

In this section, the presented theory will be illustrated using a simple example. We will consider the network depicted in Figure 2, which includes a known switching module with two different operating modes. The network can be described as follows

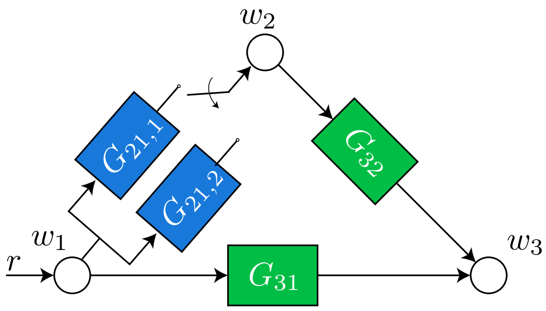

Fig. 2. A module switching network example with one switching module (blue) that establishes identifiability of the two parametrized modules (green).

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ G_{21,\ell} & 0 & 0 \\ G_{31}(\theta) & G_{32}(\theta) & 0 \end{bmatrix}}_{G_\ell(\theta)} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{U} r, \tag{29}$$

where $\ell \in \{1, 2\}$ and $G_{21,1} \neq G_{21,2}$. Since all of the parametrized modules are connected to node $j = 3$, it suffices to analyze the conditions of Theorem 7 and Theorem 13 for this node only.

Let us start by constructing the matrix $\breve{T}_{3,\ell}$, corresponding to the procedure described in Section 3.1. This starts by formulation of the permutation matrices P_3 and Q_3, such that all the $\alpha_3 = 2$ parametrized entries of $(I - G_\ell(\theta))$ are on the left side, whereas $\beta_3 = 0$, since there are no parametrized entries in $U(\theta)$. The matrices from (29) are already correctly ordered, so $P_3 = I_3$ and $Q_3 = 1$. Using these ingredients the transfer matrix $\breve{T}_{3,\ell}$ is given by

$$\breve{T}_{3,\ell} = [I_{\alpha_3} \ 0]\, T_\ell = \begin{bmatrix} 1 \\ G_{21,\ell} \end{bmatrix}, \tag{30}$$

with T_ℓ given by (10) and where V_3 does not exist, since $\beta_3 = 0$. Clearly, for $m = 1$, the conditions of Theorem 7 would not hold, since the matrix in (30) has more rows than columns. However, application of Theorem 7 for $m = 2$ yields

$$\text{rank}\left[\breve{T}_{3,1} \ \breve{T}_{3,2}\right] = \text{rank}\begin{bmatrix} 1 & 1 \\ G_{21,1} & G_{21,2} \end{bmatrix} = 2, \tag{31}$$

which does satisfy the conditions and renders the model set identifiable.

Identifiability of the model set can also be verified using the path-based conditions of Theorem 13, where it suffices again to consider node $j = 3$ only. The set of inputs to parametrized modules that connect to node 3 is given by $\mathcal{W}_3 = \{1, 2\}$ and the set of nodes excited by an external signal is given by $\mathcal{U} = \{1\}$. Without including excitation by switching modules, this graph has only one path from $\mathcal{U}$ to $\mathcal{W}_3$, which does not satisfy the conditions in Lemma 11. Let us now include switching edges, for which the set of stimulated nodes by switching modules is given by $\mathcal{S} = \{2\}$, corresponding to (26). Since the paths $\mathcal{U} \to \mathcal{W}_3$ and $\mathcal{S} \to \mathcal{W}_3$ are vertex-disjoint, the conditions of Theorem 13 are satisfied and therefore the model set is generically identifiable.

6. CONCLUSION

In this paper, it has been shown that switching modules provide additional excitation in dynamic networks, such that remaining mode-invariant modules are identifiable. Verifiable conditions in terms of a transfer matrix composed from transfer matrices for all network modes are formulated. These conditions generalize existing results on network identifiability and identifiability of a plant in closed-loop with a switching controller, for which their relation is explicitly shown. These rank conditions are translated to path-based conditions for the notion of generic identifiability, which uses a standard graph formulation and therefore allows usage of standard algorithms to be evaluated.

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Appendix A

Proof of Theorem 7 This proof is inspired by Weerts et al. (2018a), where the left hand side of implication (12) can be written for a single mode ℓ as

$$(I - G_\ell(\theta))T_\ell = U(\theta) \tag{A.1}$$

where shorthand notation is used for $T_\ell = T_\ell(\theta_0)$, $G_\ell(\theta) = G_\ell(\theta_1)$ and $U(\theta) = U(\theta_1)$. The following reasoning is then applied to every row of (A.1). Consider row j, for which permutation matrices P_j and Q_j, defined in (13)-(14), are applied to obtain:

$$(I - G_\ell(\theta))_{j\star} P_j P_j^{-1} T_\ell Q_j = U(\theta)_{j\star} Q_j, \tag{A.2}$$

which leads to

$$-G(\theta)^{(1)}_{j\star}\breve{T}_{j,\ell} + (I - G_\ell)^{(2)}_{j\star}\bar{T}_{j,\ell} = \begin{bmatrix} U^{(1)}_{j\star} & 0 \end{bmatrix} + U(\theta)^{(2)}_{j\star} V_j \tag{A.3}$$

with $-G(\theta)^{(1)}_{j\star} = (I - G(\theta))^{(1)}_{j\star}$ and $P_j^{-1} T_\ell Q_j = \begin{bmatrix} \breve{T}_{j,\ell} \\ \bar{T}_{j,\ell} \end{bmatrix}$. This equation can be rewritten into

$$\begin{bmatrix} G(\theta)^{(1)}_{j\star} & U(\theta)^{(2)}_{j\star} \end{bmatrix} \begin{bmatrix} \breve{T}_{j,\ell} \\ V_j \end{bmatrix} = \rho_\ell, \tag{A.4}$$

with $\rho_\ell = (I - G_\ell)^{(2)}_{j\star}\bar{T}_{j,\ell} - \begin{bmatrix} U^{(1)}_{j\star} & 0 \end{bmatrix}$, which has to hold for all $\ell \in \{1, \dots, m\}$. All of these equations can be grouped for all $\ell \in \{1, \dots, m\}$ as follows

$$\begin{bmatrix} G(\theta)^{(1)}_{j\star} & U(\theta)^{(2)}_{j\star} \end{bmatrix} \begin{bmatrix} \breve{T}_{j,1} & \cdots & \breve{T}_{j,m} \\ V_j & \cdots & V_j \end{bmatrix} = \begin{bmatrix} \rho_1 & \cdots & \rho_m \end{bmatrix}. \tag{A.5}$$

In (A.5) all the parametrized entries are contained in the left block $\begin{bmatrix} G(\theta)^{(1)}_{j\star} & U(\theta)^{(2)}_{j\star} \end{bmatrix}$, whereas the other parts of the equation are independent of θ. Therefore, all of the parametrized elements are uniquely determined if the matrix $\begin{bmatrix} \breve{T}_{j,1} & \cdots & \breve{T}_{j,m} \\ V_j & \cdots & V_j \end{bmatrix}$ has full row rank. A solution $G_\ell(\theta_0)$ and $U(\theta_0)$ exists to the equation (A.1) by Definition 6 of identifiability. Since the solution to the equation (A.5) is unique and $G_\ell(\theta_0)$ and $U(\theta_0)$ is a possible solution, the validity of (12) is proven.

The proof for situation II) is that the provided conditions should apply for all models $\theta \in \Theta$, meaning that one model for which the given transfer matrix loses rank, the model set is not identifiable. However, for the situation III), the provided conditions should hold for almost all $\theta \in \Theta$, meaning that a finite set of models may be excluded. ■

Proof of Theorem 13 The proof is based on showing that the hypothesis of the theorem guarantees that the rank condition of Theorem 7.III) holds. In the first step of the proof, we show that the transfer matrix can be decomposed into a part that is dependent on the network mode ℓ and an independent part in the following form

$$T_{\mathcal{W}_j\mathcal{U},\ell} = \bar{T}_{\mathcal{W}_j\bar{\mathcal{D}}_j}\bar{T}_{\bar{\mathcal{D}}_j\mathcal{U}} + \tilde{T}_{\mathcal{W}_j\mathcal{S}}T_{\mathcal{S}\mathcal{U},\ell},$$

where the subscript $(\cdot)_\ell$ denotes dependence on the network mode ℓ and where $T_{\mathcal{W}_j\mathcal{U},\ell}$ is related to the rank conditions of Theorem 7.III) by (25). In the second step of the proof, we establish that these rank conditions can equivalently be formulated in terms of transfer matrices that are independent of the network modes, using the decomposition of the first step. In the final step, we show that the theorem hypothesis is equivalent to the established rank condition in the second step. This rank condition has been shown to be equivalent to rank condition of Theorem 7.III) and therefore this step completes the proof.

Before going into the first step, we introduce a partitioning of the set of nodes $\mathcal{V}$ into three disjoint sets $\mathcal{A}_j$, $\mathcal{B}_j$ and $\mathcal{D}_j$ for each node $j \in \mathcal{V}$. To do this, let $\mathcal{D}_j$ denote the *minimum disconnecting set* of the paths $\mathcal{U} \to \mathcal{W}_j$, where a disconnecting set refers to a set of nodes for which removal would eliminate all paths from $\mathcal{U}$ to $\mathcal{W}_j$. In addition, this set should be of minimal cardinality, see e.g., Schrijver (2003) for details. This disconnecting set allows us to define the following disjoint sets:

- $\mathcal{D}_j$ is the minimum disconnecting set for node j;
- $\mathcal{A}_j$ is the set of nodes to which a path from $\mathcal{U}$ exists that does not intersect any node in $\mathcal{D}_j$;
- $\mathcal{B}_j$ is given by $\mathcal{B}_j = \mathcal{V}\backslash(\mathcal{D}_j \cup \mathcal{A}_j)$, i.e., the set of remaining nodes.

Note that $\mathcal{W}_j \subseteq (\mathcal{D}_j \cup \mathcal{B}_j)$ and that the set of output nodes of switching modules $\mathcal{S}$ are included in these sets.

To address the the set of output nodes of switching modules $\mathcal{S}$, we define a new partitioning of $\mathcal{V}$ dependent on $\mathcal{A}_j$, $\mathcal{B}_j$ and $\mathcal{D}_j$, that explicitly includes $\mathcal{S}$. The partitioning is of the following disjoint sets

- $\mathcal{S}$
- $\bar{\mathcal{A}}_j = \mathcal{A}_j\backslash\mathcal{S}$
- $\bar{\mathcal{B}}_j = \mathcal{B}_j\backslash\mathcal{S}$
- $\bar{\mathcal{D}}_j = \mathcal{D}_j\backslash\mathcal{S}$

The notation $\bar{(\cdot)}$ indicates the absence of nodes that are output of a switching module. Furthermore, the following reasoning will be applied for every node $j \in \mathcal{V}$, so the notation indicating dependence on the node j will be omitted for readability.

Step 1:
To decompose the switching and mode-invariant parts of $T_{\mathcal{W}_j\mathcal{U},\ell}$, we are first going to represent T_ℓ on the basis of (10), while using the partitioning of $\mathcal{V}$ that is specified above. With $(I - G_\ell)T_\ell = U$ we decompose:

$$T_\ell = \begin{bmatrix} T_{\bar{\mathcal{B}}\mathcal{U},\ell} \\ T_{\mathcal{S}\mathcal{U},\ell} \\ T_{\bar{\mathcal{D}}\mathcal{U},\ell} \\ T_{\bar{\mathcal{A}}\mathcal{U},\ell} \end{bmatrix}, \; U = \begin{bmatrix} 0 \\ U_{\mathcal{S}\mathcal{U}} \\ U_{\bar{\mathcal{D}}\mathcal{U}} \\ U_{\bar{\mathcal{A}}\mathcal{U}} \end{bmatrix},$$
$$G_\ell = \begin{bmatrix} G_{\bar{\mathcal{B}}\bar{\mathcal{B}}} & G_{\bar{\mathcal{B}}\mathcal{S}} & G_{\bar{\mathcal{B}}\bar{\mathcal{D}}} & 0 \\ G_{\mathcal{S}\bar{\mathcal{B}},\ell} & G_{\mathcal{S}\mathcal{S},\ell} & G_{\mathcal{S}\bar{\mathcal{D}},\ell} & G_{\mathcal{S}\bar{\mathcal{A}},\ell} \\ G_{\bar{\mathcal{D}}\bar{\mathcal{B}}} & G_{\bar{\mathcal{D}}\mathcal{S}} & G_{\bar{\mathcal{D}}\bar{\mathcal{D}}} & G_{\bar{\mathcal{D}}\bar{\mathcal{A}}} \\ G_{\bar{\mathcal{A}}\bar{\mathcal{B}}} & G_{\bar{\mathcal{A}}\mathcal{S}} & G_{\bar{\mathcal{A}}\bar{\mathcal{D}}} & G_{\bar{\mathcal{A}}\bar{\mathcal{A}}} \end{bmatrix}, \tag{A.6}$$

where $G_{\bar{\mathcal{B}}\bar{\mathcal{A}}} = 0$ and $U_{\bar{\mathcal{B}}\mathcal{U}} = 0$ by definition, as otherwise $\mathcal{D}$ would not be a disconnecting set. Then, due to the relation $(I - G_\ell)T_\ell = U$, we can rewrite the equation in the first block row into

$$T_{\bar{\mathcal{B}}\mathcal{U},\ell} = (I - G_{\bar{\mathcal{B}}\bar{\mathcal{B}}})^{-1}(G_{\bar{\mathcal{B}}\mathcal{S}}T_{\mathcal{S}\mathcal{U},\ell} + G_{\bar{\mathcal{B}}\bar{\mathcal{D}}}T_{\bar{\mathcal{D}}\mathcal{U},\ell}). \tag{A.7}$$

Similarly the equation of the last block row can be manipulated as follows

$$T_{\bar{\mathcal{A}}\mathcal{U},\ell} = (I - G_{\bar{\mathcal{A}}\bar{\mathcal{A}}})^{-1}(G_{\bar{\mathcal{A}}\bar{\mathcal{B}}}T_{\bar{\mathcal{B}}\mathcal{U},\ell} + G_{\bar{\mathcal{A}}\bar{\mathcal{D}}}T_{\bar{\mathcal{D}}\mathcal{U},\ell} + U_{\bar{\mathcal{A}}\mathcal{U}} + G_{\bar{\mathcal{D}}\mathcal{S}}T_{\mathcal{S}\mathcal{U},\ell}) = T_{\bar{\mathcal{A}}\bar{\mathcal{D}}}T_{\bar{\mathcal{D}}\mathcal{U},\ell} + T_{\bar{\mathcal{A}}\mathcal{S}}T_{\mathcal{S}\mathcal{U},\ell} + (I - G_{\bar{\mathcal{A}}\bar{\mathcal{A}}})^{-1}U_{\bar{\mathcal{A}}\mathcal{U}} \quad \text{(A.8)}$$

with $T_{\bar{\mathcal{A}}\bar{\mathcal{D}}} = (I - G_{\bar{\mathcal{A}}\bar{\mathcal{A}}})^{-1}(G_{\bar{\mathcal{A}}\bar{\mathcal{D}}} + G_{\bar{\mathcal{A}}\bar{\mathcal{B}}}(I - G_{\bar{\mathcal{B}}\bar{\mathcal{B}}})^{-1}G_{\bar{\mathcal{B}}\bar{\mathcal{D}}})$ and $T_{\bar{\mathcal{A}}\mathcal{S}} = (I - G_{\bar{\mathcal{A}}\bar{\mathcal{A}}})^{-1}(G_{\bar{\mathcal{A}}\mathcal{S}} + G_{\bar{\mathcal{A}}\bar{\mathcal{B}}}(I - G_{\bar{\mathcal{B}}\bar{\mathcal{B}}})^{-1}G_{\bar{\mathcal{B}}\mathcal{S}})$ that are independent of the network mode ℓ and where in $T_{\bar{\mathcal{B}}\mathcal{U},\ell}$ is substituted.

Then, the third block row of $(I - G_\ell)T_\ell = U$ can be simplified by substitution of $T_{\bar{\mathcal{B}}\mathcal{U},\ell}$ and $T_{\bar{\mathcal{A}}\mathcal{U},\ell}$, given in (A.7) and (A.8), respectively, into

$$T_{\bar{\mathcal{D}}\mathcal{U},\ell} = (I - G_{\bar{\mathcal{D}}\bar{\mathcal{D}}})^{-1}(G_{\bar{\mathcal{D}}\bar{\mathcal{B}}}T_{\bar{\mathcal{B}}\mathcal{U},\ell} + G_{\bar{\mathcal{D}}\bar{\mathcal{A}}}T_{\bar{\mathcal{A}}\mathcal{U},\ell} + U_{\bar{\mathcal{D}}\mathcal{U}} + G_{\bar{\mathcal{D}}\mathcal{S}}T_{\mathcal{S}\mathcal{U},\ell}) = \bar{T}_{\bar{\mathcal{D}}\mathcal{U}} + \tilde{T}_{\bar{\mathcal{D}}\mathcal{S}}T_{\mathcal{S}\mathcal{U},\ell} \quad \text{(A.9)}$$

with

$$\bar{T}_{\bar{\mathcal{D}}\mathcal{U}} = (I - G_{\bar{\mathcal{D}}\bar{\mathcal{D}}} - G_{\bar{\mathcal{D}}\bar{\mathcal{A}}}T_{\bar{\mathcal{A}}\bar{\mathcal{D}}} - G_{\bar{\mathcal{D}}\bar{\mathcal{B}}}(I - G_{\bar{\mathcal{B}}\bar{\mathcal{B}}})^{-1}G_{\bar{\mathcal{B}}\bar{\mathcal{D}}})^{-1} (U_{\bar{\mathcal{D}}\mathcal{U}} + (I - G_{\bar{\mathcal{A}}\bar{\mathcal{A}}})^{-1}U_{\bar{\mathcal{A}}\mathcal{U}}) \quad \text{(A.10)}$$

the transfer matrix $T_{\bar{\mathcal{D}}\mathcal{U},\ell}$ that would occur when nodes in $\mathcal{S}$ are removed, so independent of the network mode ℓ, and

$$\tilde{T}_{\bar{\mathcal{D}}\mathcal{S}} = (I - G_{\bar{\mathcal{D}}\bar{\mathcal{D}}} - G_{\bar{\mathcal{D}}\bar{\mathcal{A}}}T_{\bar{\mathcal{A}}\bar{\mathcal{D}}} - G_{\bar{\mathcal{D}}\bar{\mathcal{B}}}(I - G_{\bar{\mathcal{B}}\bar{\mathcal{B}}})^{-1}G_{\bar{\mathcal{B}}\bar{\mathcal{D}}})^{-1} (G_{\bar{\mathcal{D}}\bar{\mathcal{A}}}T_{\bar{\mathcal{A}}\mathcal{S}} + G_{\bar{\mathcal{D}}\mathcal{S}} + G_{\bar{\mathcal{D}}\bar{\mathcal{B}}}(I - G_{\bar{\mathcal{B}}\bar{\mathcal{B}}})^{-1}G_{\bar{\mathcal{B}}\mathcal{S}}) \quad \text{(A.11)}$$

which in product with $T_{\mathcal{S}\mathcal{U},\ell}$ covers the excluded part by $\bar{T}_{\bar{\mathcal{D}}\mathcal{U}}$. Similarly, using the decomposition in (A.9), $T_{\bar{\mathcal{B}}\mathcal{U},\ell}$ in (A.7) can be rewritten into

$$T_{\bar{\mathcal{B}}\mathcal{U},\ell} = (I - G_{\bar{\mathcal{B}}\bar{\mathcal{B}}})^{-1}(G_{\bar{\mathcal{B}}\bar{\mathcal{D}}}\bar{T}_{\bar{\mathcal{D}}\mathcal{U}} + (G_{\bar{\mathcal{B}}\mathcal{S}} + G_{\bar{\mathcal{B}}\bar{\mathcal{D}}}\tilde{T}_{\bar{\mathcal{D}}\mathcal{S}})T_{\mathcal{S}\mathcal{U},\ell}) = \bar{T}_{\bar{\mathcal{B}}\bar{\mathcal{D}}}\bar{T}_{\bar{\mathcal{D}}\mathcal{U}} + \tilde{T}_{\bar{\mathcal{B}}\mathcal{S}}T_{\mathcal{S}\mathcal{U},\ell} \quad \text{(A.12)}$$

with $\tilde{T}_{\bar{\mathcal{B}}\mathcal{S}} = (I - G_{\bar{\mathcal{B}}\bar{\mathcal{B}}})^{-1}(G_{\bar{\mathcal{B}}\mathcal{S}} + G_{\bar{\mathcal{B}}\bar{\mathcal{D}}}\tilde{T}_{\bar{\mathcal{D}}\mathcal{S}})$ and $\bar{T}_{\bar{\mathcal{B}}\bar{\mathcal{D}}} = (I - G_{\bar{\mathcal{B}}\bar{\mathcal{B}}})^{-1}G_{\bar{\mathcal{B}}\bar{\mathcal{D}}}$.

The set of in-neighbors $\mathcal{W} \subseteq \bar{\mathcal{B}} \cup \mathcal{S} \cup \bar{\mathcal{D}}$, for which the required transfer matrix for the conditions of Theorem 7 can be retrieved by using a selection matrix $C \in \{0,1\}^{|\mathcal{W}| \times L}$ which has for each row one nonzero entry at the column corresponding to the nodes that are in $\mathcal{W}$ from the sets $\bar{\mathcal{B}}$, $\mathcal{S}$ and $\bar{\mathcal{D}}$. By substitution of (A.9) and (A.12) this leads to

$$T_{\mathcal{W}\mathcal{U},\ell} = C \begin{bmatrix} T_{\bar{\mathcal{B}}\mathcal{U},\ell} \\ T_{\mathcal{S}\mathcal{U},\ell} \\ T_{\bar{\mathcal{D}}\mathcal{U},\ell} \end{bmatrix} = C \left(\begin{bmatrix} \bar{T}_{\bar{\mathcal{B}}\bar{\mathcal{D}}} \\ 0 \\ I \end{bmatrix} \bar{T}_{\bar{\mathcal{D}}\mathcal{U}} + \begin{bmatrix} \tilde{T}_{\bar{\mathcal{B}}\mathcal{S}} \\ I \\ \tilde{T}_{\bar{\mathcal{D}}\mathcal{S}} \end{bmatrix} T_{\mathcal{S}\mathcal{U},\ell} \right) = \bar{T}_{\mathcal{W}\bar{\mathcal{D}}}\bar{T}_{\bar{\mathcal{D}}\mathcal{U}} + \tilde{T}_{\mathcal{W}\mathcal{S}}T_{\mathcal{S}\mathcal{U},\ell} \quad \text{(A.13)}$$

with $\bar{T}_{\mathcal{W}\bar{\mathcal{D}}} = C\left[\bar{T}_{\bar{\mathcal{B}}\bar{\mathcal{D}}}^\top \ 0 \ I\right]^\top$ the mapping from external signals on $\bar{\mathcal{D}}$ to $\mathcal{W}$ excluding nodes in $\mathcal{S}$ and $\tilde{T}_{\mathcal{W}\mathcal{S}} = C\left[\tilde{T}_{\bar{\mathcal{B}}\mathcal{S}}^\top \ I \ \tilde{T}_{\bar{\mathcal{D}}\mathcal{S}}^\top\right]^\top$ the mapping from external signals on $\mathcal{S}$ to $\mathcal{W}$. The result in (A.13) decomposes $T_{\mathcal{W}\mathcal{U},\ell}$ into a mode-invariant and switching term, which completes the first step of the proof.

Step 2:
To show that the rank conditions of Theorem 7.III) can equivalently be formulated on transfer matrices independent of the network mode, we start by relating these rank conditions to the transfer matrix $T_{\mathcal{W}\mathcal{U},\ell}$ by (25) as follows

$$\operatorname{rank}\begin{bmatrix} \breve{T}_{j,1} \dots \breve{T}_{j,m} \\ V_j \ \dots \ V_j \end{bmatrix} = \operatorname{rank}\left[T_{\mathcal{W}\mathcal{U},1} \dots T_{\mathcal{W}\mathcal{U},m}\right]. \quad \text{(A.14)}$$

Substitution of $T_{\mathcal{W}\mathcal{U},\ell}$ by (A.13) yields

$$\operatorname{rank}\left[T_{\mathcal{W}\mathcal{U},1} \dots T_{\mathcal{W}\mathcal{U},m}\right] = \operatorname{rank}\left[\tilde{T}_{\mathcal{W}\mathcal{S}} \ \bar{T}_{\mathcal{W}\bar{\mathcal{D}}}\bar{T}_{\bar{\mathcal{D}}\mathcal{U}}\right] \begin{bmatrix} T_{\mathcal{S}\mathcal{U},1} & \cdots & T_{\mathcal{S}\mathcal{U},m} \\ I & \dots & I \end{bmatrix}, \quad \text{(A.15)}$$

where the left matrix is independent of the network mode ℓ. By Assumption 12 the following relation holds

$$\operatorname{rank}\left[\tilde{T}_{\mathcal{W}\mathcal{S}} \ \bar{T}_{\mathcal{W}\bar{\mathcal{D}}}\bar{T}_{\bar{\mathcal{D}}\mathcal{U}}\right] \begin{bmatrix} T_{\mathcal{S}\mathcal{U},1} & \cdots & T_{\mathcal{S}\mathcal{U},m} \\ I & \dots & I \end{bmatrix} = \operatorname{rank}\left[\tilde{T}_{\mathcal{W}\mathcal{S}} \ \bar{T}_{\mathcal{W}\bar{\mathcal{D}}}\bar{T}_{\bar{\mathcal{D}}\mathcal{U}}\right]. \quad \text{(A.16)}$$

This means that the row rank is only dependent on the left-most matrix in (A.16), which does not depend on different network modes. This completes the second step of the proof.

Step 3:
In the last step we show that the hypothesis of the theorem holds, which states that evaluating the rank condition in Theorem 7 for generic identifiability is equivalent to evaluating the rank of the same network structure, where the switching modules are represented by virtual excitation signals on the output nodes, i.e., the set $\mathcal{S}$. To do so, we define the transfer matrix $T_{\mathcal{W}(\mathcal{U}\cup\mathcal{S}),\ell}$ as the mapping from $\mathcal{U}$ and $\mathcal{S}$ to $\mathcal{W}$ that is given by

$$T_{\mathcal{W}(\mathcal{U}\cup\mathcal{S}),\ell} := \left[T_{\mathcal{W}\mathcal{U},\ell} \ T_{\mathcal{W}\mathcal{S},\ell}\right], \quad \text{(A.17)}$$

with

$$T_{\mathcal{W}\mathcal{S},\ell} := C(I - G_\ell)^{-1}U_{\mathcal{S}}, \quad \text{(A.18)}$$

with $U_{\mathcal{S}} := [0 \ I \ 0 \ 0]^\top$, where the structure of G_ℓ and U is equivalent to (A.6). Consequently, the matrix $U_{\mathcal{S}}$ can be interpreted as the U matrix that virtually applies excitation signals to all nodes in $\mathcal{S}$. This is a virtual situation, since the signals to do so are missing, but structurally, it is equivalent. The difference in structure of $U_{\mathcal{S}}$ with respect to U in (A.6) leads through following the procedure in step 1 with $U_{\bar{\mathcal{D}}\mathcal{U}} = 0$, $U_{\bar{\mathcal{A}}\mathcal{U}} = 0$ and $U_{\mathcal{S}\mathcal{U}} = I$

$$T_{\mathcal{W}(\mathcal{U}\cup\mathcal{S}),\ell} = \tilde{T}_{\mathcal{W}\mathcal{S}}T_{\mathcal{S}\mathcal{S},\ell} \quad \text{(A.19)}$$

with the feedback terms included in

$$T_{\mathcal{S}\mathcal{S},\ell} = (I - G_{\mathcal{S}\mathcal{S},\ell})^{-1}\left(I + \left[G_{\mathcal{S}\bar{\mathcal{A}},\ell} \, G_{\mathcal{S}\bar{\mathcal{D}},\ell} \, G_{\mathcal{S}\bar{\mathcal{B}},\ell}\right] \begin{bmatrix} T_{\bar{\mathcal{A}}\mathcal{S},\ell} \\ T_{\bar{\mathcal{D}}\mathcal{S},\ell} \\ T_{\bar{\mathcal{B}}\mathcal{S},\ell} \end{bmatrix}\right), \quad \text{(A.20)}$$

the result is similar as (A.13), but $\bar{T}_{\bar{\mathcal{D}}\mathcal{U}} = 0$ in this case, which can be seen in (A.9).

To relate the transfer matrix $T_{\mathcal{W}(\mathcal{U}\cup\mathcal{S}),\ell}$ to path-based conditions, we show first that the rank of this transfer matrix is equivalent to the rank of the module switching dynamic network. This is done as follows

$$\begin{aligned} &\operatorname{rank}\left[T_{\mathcal{W}\mathcal{U},\ell} \ T_{\mathcal{W}\mathcal{S},\ell}\right] \\ &= \operatorname{rank}\left[\bar{T}_{\mathcal{W}\bar{\mathcal{D}}}\bar{T}_{\bar{\mathcal{D}}\mathcal{U}} + \tilde{T}_{\mathcal{W}\mathcal{S}}T_{\mathcal{S}\mathcal{U},\ell} \ \tilde{T}_{\mathcal{W}\mathcal{S}}T_{\mathcal{S}\mathcal{S},\ell}\right] \\ &= \operatorname{rank}\left[\tilde{T}_{\mathcal{W}\mathcal{S}} \ \bar{T}_{\mathcal{W}\bar{\mathcal{D}}}\bar{T}_{\bar{\mathcal{D}}\mathcal{U}}\right] \begin{bmatrix} T_{\mathcal{S}\mathcal{U},\ell} & T_{\mathcal{S}\mathcal{S},\ell} \\ I & 0 \end{bmatrix} \\ &= \operatorname{rank}\left[\tilde{T}_{\mathcal{W}\mathcal{S}} \ \bar{T}_{\mathcal{W}\bar{\mathcal{D}}}\bar{T}_{\bar{\mathcal{D}}\mathcal{U}}\right], \end{aligned} \quad \text{(A.21)}$$

where $T_{\mathcal{S}\mathcal{S},\ell}$ is full row rank for all ℓ, due to the identity term in (A.20), which leads to the right-most matrix to be full row rank and allows us to make the last implication.

This step has shown that the rank of the transfer matrix $T_{\mathcal{W}(\mathcal{U}\cup\mathcal{S}),\ell}$ depends on the same matrix as the rank conditions in Theorem 7, which has been derived in Step 2. This along with Assumption 10 allows us to apply Lemma 11 that is based on the $T_{\mathcal{W}(\mathcal{U}\cup\mathcal{S}),\ell}$ matrix. This completes the proof.

■