Quantification of model uncertainty from data: input design, interpolation, and connection with robust control design specifications

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Abstract

Identification of linear models in view of robust control design requires the identification of a control-relevant nominal model, and a quantification of model uncertainty. In this paper a procedure is presented to quantify the model uncertainty of any prespecified nominal model, from a sequence of measurement data of input and output signals from a plant. By employing a non-parametric empirical transfer function estimate (ETFE), we are able to split the model uncertainty into three parts: the inherent uncertainty in the data due to data-imperfections, the unmodelled dynamics in the nominal model, and the uncertainty due to interpolation. A frequency-dependent hard error bound is constructed, and results are given for tightening the bound through input design. When the upper bound on the model uncertainty is too conservative, in view of the control design specifications, information is provided as to which additional experiments have to be performed in order to improve the bound.

1 Introduction

In the systems and control community there is a growing interest in merging the problems of system identification and (robust) control system design. This interest is based on the conviction that, in many situations, models obtained from process experiments will be used as a basis for control system design. On the other hand, in model-based robust control design, models and model uncertainties have to be available that are essentially provided by, or at least validated by, measurement data from the process.

Recently several approaches to the identification problem have been presented, considering the identification in view of the control design. By far the most attention is paid to the construction of so-called hard error bounds, see e.g. [6, 7, 8, 9, 10, 15]. In [4, 5] an identification procedure is presented that provides probabilistic (soft) error bounds.

In the references mentioned, there is a strong connection between the identification of nominal models and the quantification of model uncertainty. Only identification methods for nominal models are selected for which (H_∞) error bounds can be derived. This seems to exclude many methods and model structures that could be useful but are rather intractable when it comes to deriving error bounds. When discussing the suitability of models as a basis for control system design, the availability of reliable hard error bounds certainly is important in order to obtain robust stability, and possibly also robust performance. However the nominal model that is used as a basis for the design, will determine the nominal performance of the control system, and one will definitely not be willing to implement a control system when the nominal performance does not meet the specifications. As a result, the identification of nominal models, apart from the quantification of model uncertianty, is an important issue in identification for control design, see e.g. [1, 13, 14].

In addition to this reasoning, in this paper we will deal with the following problem: given a prespecified nominal model G_{nom} for an unknown linear plant G_o, with G_{nom} and G_o rational transfer functions, can we construct an error bound for

$$| G_o(e^{j\omega}) - G_{nom}(e^{j\omega}) | \qquad (1)$$

based on noise corrupted measurements from input and output samples of the plant ? Note that the nominal model may be available from any (control-relevant) identification procedure. The problem is going to be tackled, through the construction of an intermediate data representation in the frequency domain, leading to the inequality:

$$| G_o(e^{j\omega_k}) - G_{nom}(e^{j\omega_k}) | \leq$$
$$\leq | G_o(e^{j\omega_k}) - \hat{G}(e^{j\omega_k}) | + | \hat{G}(e^{j\omega_k}) - G_{nom}(e^{j\omega_k}) | \qquad (2)$$

with $\hat{G}(e^{j\omega_k})$ an -intermediate- representation of the measurement data in the frequency domain. This means that $\hat{G}(e^{j\omega_k})$ basically constitutes a finite number of complex points on the unit circle, obtained from the Discrete Fourier Transformation (DFT) of the time-domain data. The first term on the right hand side of (2) can be considered to reflect inherent uncertainty in the data at ω_k, whereas the second term is related to the quality of the nominal model, e.g. determined by unmodelled dynamics. Having constructed a data representation $\hat{G}(e^{j\omega_k})$, the second term can be calculated exactly. Hence, to give an upper bound for $| G_o(e^{j\omega_k}) - G_{nom}(e^{j\omega_k}) |$, the problem is to construct an upper bound for $| G_o(e^{j\omega_k}) - \hat{G}(e^{j\omega_k}) |$. Note however that (2) is only defined at the finite number of frequency points ω_k, while our aim is to bound the model error for all $\omega \in [0, 2\pi)$. The second problem therefore is to bound the model error for all $\omega \in [0, 2\pi)$ using the error bounds at ω_k. These two problems will be the main topics of this paper.

Related work has been published in [10, 9] where error bounds for $| G_o(e^{j\omega_k}) - \hat{G}(e^{j\omega_k}) |$ have been obtained at a finite number of frequency points. In [10] this has been done by employing the Emperical Transfer Function Estimate (ETFE, see [11]), and in [9] through sinewave excitation and actually measuring the frequency response in a finite number of points. In [6, 8] the frequency domain estimate and discrete error bound are used to obtain a model in H_∞ and a continuous error bound (valid on the whole unit circle). It is tried to keep the H_∞ error small by using an intermediate high order L_∞ model and Nehari approximation, obtaining a Finite Impulse Response (FIR) model.

In section 3 of this paper also the ETFE is used to obtain a nonparametric frequency domain estimate $\hat{G}(e^{j\omega_k})$, and a discrete error bound. In contrast with [6, 8, 9] this error bound is frequency-dependent, which makes it more informative than a simple H_∞-bound. Moreover it does not require the frequency points of the discrete estimate to be equidistantially distributed over the unit circle. This paves the way for designing specific input signals in order to improve the estimates, and tightening the bound. Additionally a frequency-dependent continuous error bound is constructed in section 4 by interpolation of the discrete bound, using smoothness conditions on the system. In section 5 it is shown how robust control design specifications can advocate new experiments in order to reduce model uncertainty in specific frequency ranges. Finally, in section 6, a simulation example is given to illustrate the merits of the procedure proposed.

Due to space constraints all proofs are omitted, they can be found in [2].

2 Preliminaries

It is assumed that the plant, and the measurement data that is obtained from this plant, allow a description:

$$y(t) = G_o(q)u(t) + v(t) \tag{3}$$

with $y(t)$ the output signal, $u(t)$ the input signal, $v(t)$ an additive output noise, q^{-1} the delay operator, and G_o a proper transfer function that is time-invariant and exponentially stable. The transfer function can be written in its Laurent expansion around $z = \infty$, as

$$G_o(z) = \sum_{k=0}^{\infty} g_o(k) z^{-k} \tag{4}$$

with $g_o(k)$ the impulse response of the plant.
Throughout the paper we will consider discrete time intervals for input and output signals denoted by $T^N := [0, N-1]$, $T^N_{N_s} := [N_s, N + N_s - 1]$ with N and N_s appropriate integers. We will denote

$$\bar{u} = \sup_{t \in T^{N+N_s}} |u(t)| = \bar{u}$$

For a signal $x(t)$, defined on T^N, we will denote the N-point Discrete Fourier Transform (DFT) and its inverse by:

$$X(\tfrac{2\pi k}{N}) = \sum_{t=0}^{N-1} x(t) e^{-j\frac{2\pi k}{N}t} \quad \text{for } k \in T^N \tag{5}$$

$$x(t) = \frac{1}{N} \sum_{k=0}^{N-1} X(\tfrac{2\pi k}{N}) e^{j\frac{2\pi k}{N}t} \quad \text{for } t \in T^N \tag{6}$$

For a signal $x(t)$ that is defined on the interval $T^N_{N_s}$, $N_s > 0$, we will denote the N-point DFT of a shifted version of the signal x, shifted over N_s time instants, by

$$X^s(\tfrac{2\pi k}{N}) = \sum_{t=0}^{N-1} x(t + N_s) e^{-j\frac{2\pi k}{N}t} \quad \text{for } k \in T^N \tag{7}$$

$$x(t) = \frac{1}{N} \sum_{k=0}^{N-1} X^s(\tfrac{2\pi k}{N}) e^{j\frac{2\pi k}{N}(t-N_s)} \quad \text{for } t \in T^N_{N_s} \tag{8}$$

Note that this reflects the N-point DFT of a signal, of which the first N_s time instants are discarded.
Throughout this paper we will adopt a number of additional assumptions on the system and the generated data.

Assumption 2.1 *There exists a finite*

i. $\bar{u}^p$, *such that* $|u(t)| \le \bar{u}^p$ *for* $t < 0$;

ii. *pair* $\{M, \rho\} \in \mathbb{R}$, $\rho > 1$, *such that* $|g_o(k)| \le M\rho^{-k}$, *for* $k \in \mathbb{Z}_+$;

iii. *upper bound on the DFT of the output noise:* $|V^s(\frac{2\pi k}{N})| \le \overline{V}^s(\frac{2\pi k}{N})$, *for* $k \in T^N$.

3 Discrete error bound.

3.1 Motivation.

The motivation to consider the ETFE is that we want $\hat{G}(e^{j\omega_k})$ to be an intermediate data representation in the frequency domain. The ETFE is the quotient of the DFT of the output signal and the DFT of the input signal, and in discrete Fourier transforming a signal no information is lost or added, the mapping from time to frequency domain is one to one. The motivation to look at input design is that the ETFE for an arbitrary input signal is in general not satisfactory. We will try to improve the quality of the frequency domain data by input design.

3.2 Results.

A nonparametric frequency domain discrete upper bound on the additive error for the ETFE will be presented in this section. Errors due to unknown initial and final conditions of the system and additive noise on the output are taken into account. We will use a partly periodic input signal for excitation, and we will discard the first part of the signals in the estimation.

Definition 3.1 *A partly periodic signal x is a signal having the first part equal to the last part:* $x = [x_1\ x_2\ x_1]$.

The length of x_1 will be denoted by N_s. Only the last part $[x_2\ x_1]$ will be used in the identification and has length N. The total length of the signal x now is $N_s + N$. We will show that the value of N_s influences the error due to initial and final conditions in the estimate. Note that the largest possible value of N_s is N.

Theorem 3.2 *Consider a SISO system, satisfying the assumptions stated in section 2. Using a partly periodic input signal,* $N_s \in T^{N+1}$, *and the estimate*

$$\hat{G}^s(\tfrac{2\pi\ell}{N}) = \frac{Y^s(\frac{2\pi\ell}{N})}{U^s(\frac{2\pi\ell}{N})} \quad \text{for} \quad \ell = \{\ell \in T^N \mid U^s(\tfrac{2\pi\ell}{N}) \neq 0\}$$

the following error bound is satisfied

$$|G_o(\tfrac{2\pi\ell}{N}) - \hat{G}^s(\tfrac{2\pi\ell}{N})| \le \alpha(\tfrac{2\pi\ell}{N})$$

with

$$\alpha(\tfrac{2\pi\ell}{N}) = \frac{\bar{u}^P + \bar{u}}{|U^s(\frac{2\pi\ell}{N})|} \frac{M\rho(1-\rho^{-N})}{(\rho-1)^2} \rho^{-N_s} + \frac{\overline{V}^s(\frac{2\pi\ell}{N})}{|U^s(\frac{2\pi\ell}{N})|}$$

The first term on the right hand side of the error bound given in the theorem is the error due to the effects of initial and final conditions of the system, i.e. the effects of the unknown signals outside the measurement interval. This error converges exponentially with N_s, as ρ^{-N_s}. The properties of $|U^s(\frac{2\pi\ell}{N})|$ of course depend on the specific choice of the input signal $u(t)$ for $t \in T^N_{N_s}$. For a random signal the expectation of the magnitude of the N point DFT, as defined in (5) and (7), is approximately proportional to $\sqrt{N}$, see [11, lemma 6.2]. Hence, if the input is random for $t \in T^N_{N_s}$, the error due to the effects of initial and final conditions approximately converges as $\rho^{-N_s}/\sqrt{N}$.

The second term on the right hand side of the error bound given in theorem 3.2 is the error due to the additive noise on the output. If the input is random for $t \in T^N_{N_s}$, this error does not converge at all, it is just the noise to signal ratio in the frequency domain. It is possible to obtain convergence for the error due to the noise by choosing the input signal to be periodic. The highest rate of convergence is obtained by an input signal having an integer number of periods in the interval

$T_{N_s}^N$. Let N_0 denote the length of one period of the input signal and let the interval $T_{N_s}^N$ contain exactly k_0 periods, so that $N = k_0 N_0$. In this case $U^s(\frac{2\pi k}{N}) = 0$ if k/k_0 is not an integer, see [11, example 2.2]. Therefore only $U^s(\frac{2\pi\ell}{N_0})$ is not identically equal to zero. It is now straightforward to show that the DFT over k_0 periods of a periodic signal is exactly k_0 times as large as the DFT over one period. In conclusion, $|U^s(\frac{2\pi\ell}{N_0})|$ is exactly proportional to N if $N = k_0 N_0$ with $k_0 \in \mathbb{Z}$.

Corollary 3.3 *Consider a SISO system, satisfying the assumptions stated in section 2. Using a partly periodic input signal having an integer number of periods in the interval $T_{N_s}^N$, $N_s \in T^{N+1}$, and the estimate*

$$\hat{G}^s(\tfrac{2\pi\ell}{N_0}) = \frac{Y^s(\frac{2\pi\ell}{N_0})}{U^s(\frac{2\pi\ell}{N_0})} \quad \textit{for} \quad \ell = \{\ell \in T^{N_0} \mid U^s(\tfrac{2\pi\ell}{N_0}) \neq 0\}$$

the following error bound is satisfied

$$|G_o(\tfrac{2\pi\ell}{N_0}) - \hat{G}^s(\tfrac{2\pi\ell}{N_0})| \leq \alpha(\tfrac{2\pi\ell}{N_0})$$

with

$$\alpha(\tfrac{2\pi\ell}{N_0}) = \frac{\bar{u}^P + \bar{u}}{|U^s(\frac{2\pi\ell}{N_0})|} \, \frac{M\rho(1-\rho^{-N})}{(\rho-1)^2} \, \rho^{-N_s} + \frac{\overline{V}^s(\frac{2\pi\ell}{N_0})}{|U^s(\frac{2\pi\ell}{N_0})|}$$

The error bound given in the corollary goes to zero if N_s and N are going to infinity, N_0 is constant, and the noise $v(t)$ does not contain a periodic component. The error due to the effects of initial and final conditions converges as ρ^{-N_s}/N. The error due to the additive noise on the output converges approximately as $1/\sqrt{N}$, if $v(t)$ is a random signal. The expectation of the magnitude of the N point DFT of a random signal is approximately proportional to $\sqrt{N}$, while the magnitude of the N point DFT of the periodic input is exactly proportional to N, see [11]. The price for this convergence is that less points of transfer function are estimated (N_0 instead of $N = k_0 N_0$).

3.3 Remarks.

A partly periodic signal can be seen as a generalization of a sinewave input. This generalization is profitable because sinewave testing (sinewave excitation and actually measuring the frequency response in a finite number of frequency points) is time consuming. For each new sinewave input one must wait until the system has reached its steady state response. A partly periodic signal can consist of N sinewaves, but one has to wait only one single time for the effects of initial and final conditions to vanish.

For $N_s = 0$ the ETFE as defined in [11] arises. In this case the error due to initial and final conditions converges only as $1/\sqrt{N}$ if $u(t)$ is a random signal for $t \in T^N$, as was also shown in [11]. The error due to initial and final conditions now will in general dominate the error bound. Note however that for $N_s = 0$ the input signal is completely free. The choice for $N_s > 0$ hence is a choice to restrict the input signal in order to be able to obtain a tight error bound for the nominal model.

By designing an appropriate input signal (i.e. by choosing $|U^s(\frac{2\pi\ell}{N})|$) one can of course shape the error due to the noise. An input signal having a DFT with desired magnitude and satisfactory time domain properties can be easily designed, see [12].

Finally we note that the extension to the MIMO case of theorem 3.2 has been made by the authors. To be able to do this, the Discrete Fourier Transforms of the different input signals have to satisfy an orthogonality condition.

4 Continuous error bound.

4.1 Motivation.

We now have an upper bound $\alpha(\omega_k)$ on the error $|G_o(e^{j\omega_k}) - \hat{G}(e^{j\omega_k})|$. This error bound is only defined at a finite number of frequency points $\omega_k \in \Omega$, with $\Omega := \{\omega_k \in \mathbb{R} \cap [0, 2\pi) \mid |U^s(e^{j\omega_k})| \neq 0\}$. This is due to the fact that $\hat{G}(e^{j\omega_k})$ is only defined at a finite number of frequency points when N, the number of datapoints used in the estimate, is finite. The aim is to find an upper bound $\delta(\omega)$ such that

$$|G_o(e^{j\omega}) - G_{nom}(e^{j\omega})| \leq \delta(\omega)$$

for all frequencies in the interval $[0, 2\pi)$. It is straightforward to give a discrete upper bound $\delta(\omega_k)$. From the knowledge of G_{nom} we can exactly calculate $\beta(\omega_k) := |\hat{G}(e^{j\omega_k}) - G_{nom}(e^{j\omega_k})|$. From inequality (2) it now follows that $\delta(\omega_k)$ can be set to $\delta(\omega_k) = \alpha(\omega_k) + \beta(\omega_k)$. Hence the problem is to find the behaviour of $\delta(\omega)$ between the estimated frequency points for the prespecified nominal model. As argued in section 3.1, the data does essentially not contain more information about the transfer function of the system than is captured by the discrete estimate $\hat{G}(e^{j\omega_k})$. Therefore, assumptions about the system must be used to be able to bound the error at frequencies $\omega \neq \omega_k$. We will use smoothness assumptions on the system, and we will interpolate the discrete error bound $\delta(\omega_k)$ using these smoothness properties.

Note that we do not interpolate $\alpha(\omega_k)$, as is done in [8]. To be able to interpolate $\alpha(\omega_k)$, one first has to interpolate the discrete estimate $\hat{G}(e^{j\omega_k})$. However, in doing this, an intermediate model is constructed that is not based on the data. Therefore we take the approach of interpolating the error bound $\delta(\omega_k)$.

4.2 Bounds on derivatives.

Smoothness properties of the system in the form of upper bounds on the derivatives of $G_o(e^{j\omega})$ with respect to the frequency, can be obtained from the assumed upper bound on the impulse response.

Proposition 4.1 *For a SISO system with $|g_o(m)| \leq M\rho^{-m}$ there holds*

$$\left|\frac{d\,G_o(e^{j\omega})}{d\omega}\right| \leq \frac{M\rho}{(\rho-1)^2} \qquad \textit{and} \qquad \left|\frac{d^2\,G_o(e^{j\omega})}{d\omega^2}\right| \leq \frac{M\rho(\rho+1)}{(\rho-1)^3}$$

Note that the upper bounds on the derivatives are not obtained in the same way as in [8], the M and ρ used in this paper are different from the ones used in [8]. It is the authors' opinion that it is easier in practice to obtain a good upper bound on the impulse response of the system, than to obtain a margin of relative stability of the system together with the infinity norm of the system over the circle in the complex plane with radius equal to this margin.

To be able to bound the derivatives of the magnitude of the error system $|G_o(e^{j\omega}) - G_{nom}(e^{j\omega})|$ we need the following proposition.

Proposition 4.2 *For a SISO system there holds, for $k = 1$ and $k = 2$*

$$\left|\frac{d^k}{d\omega^k}|G_o(e^{j\omega}) - G_{nom}(e^{j\omega})|\right| \leq$$

$$\leq \left|\frac{d^k\,G_o(e^{j\omega})}{d\omega^k}\right| + \left|\frac{d^k\,G_{nom}(e^{j\omega})}{d\omega^k}\right| \qquad (9)$$

An upper bound for (9) can be calculated using proposition 4.1 and the knowledge of $G_{nom}(e^{j\omega})$.

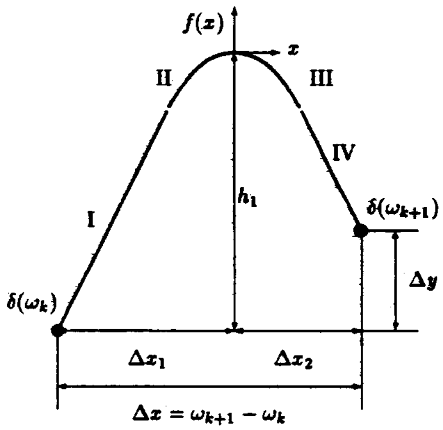

Fig. 1: The interpolating function $f(x)$ for the discrete error bound.

4.3 Interpolation.

In this section we will address the problem of calculating an upper bound on the error $|G_o(e^{j\omega}) - G_{nom}(e^{j\omega})|$ between the frequency points ω_k where an upper bound $\delta(\omega_k)$ is known. Hence, we have to find the highest possible value $\delta(\omega)$ of this error for each frequency ω between two given points, say $\delta(\omega_k)$ and $\delta(\omega_{k+1})$. We are able to bound this error by taking into account the bounds on the first and second derivatives of $|G_o(e^{j\omega}) - G_{nom}(e^{j\omega})|$ that were derived in section 4.1, say γ_1 and γ_2 respectively. The maximum value of the error $\delta(\omega)$ now arises by interpolating the discrete error bound $\delta(\omega_k)$ using the function $f(x)$ depicted in figure 1. To explain the construction of this function $f(x)$, assume that there is a maximum between the two frequency points. Starting at the maximum ($x = 0$, $f(x) = 0$ and $df(x)/dx = 0$) we want $f(x)$, in a smooth way, to decrease as fast as possible: the faster $f(x)$ decreases, the higher the maximum lies above the two given points $\delta(\omega_k)$. Hence we use a function having a constant second derivative equal to the bound γ_2 on this derivative. In this way parts II and III of the error bound are constructed. The absolute value of the first derivative of this function will clearly increase with the distance $|x|$ to the maximum. At $|x| = \gamma_1/\gamma_2$ the first derivative becomes equal to the bound γ_1 on this derivative. Hence, from thereon we use a function having a constant first derivative equal to the bound γ_1. In this way part I or IV of the error bound is constructed. The function constructed in this way is unique and given by

$$\begin{aligned} f(x) &= -\tfrac{\gamma_2}{2}x^2 && \text{for} \quad |x| \leq \tfrac{\gamma_1}{\gamma_2} \\ &= -\gamma_1|x| + \tfrac{\gamma_1^2}{2\gamma_2} && \text{for} \quad |x| > \tfrac{\gamma_1}{\gamma_2} \end{aligned} \qquad (10)$$

The function $f(x)$ given in (10) directly gives the value of $\delta(\omega)$

$$\delta(\omega) = \delta(\omega_k) - f(\Delta x_1) + f(x) \quad \text{for} \quad \omega \in [\omega_k, \omega_{k+1}] \qquad (11)$$

However, in (11) the values of Δx_1 and x are unknown, because the location of the maximum is as yet unknown. Analytic expressions for the location of the maximum can be given by specifying Δx_1 or Δx_2, as a function of γ_1, γ_2, $\delta(\omega_k)$ and $\delta(\omega_{k+1})$. One has to distinguish several cases, depending on which part of the interpolating function $f(x)$ actually is used. It is e.g. possible that γ_1, γ_2, $\delta(\omega_k)$ and $\delta(\omega_{k+1})$ are such that the interpolating function $f(x)$ reduces to part I. In all, there are eleven possibilities: only part I, only part II, part I and II, etc.

Algorithm 4.3 *All possibilities of the function $f(x)$ given in (10) to interpolate two points are given below, as a function of Δx, Δy, γ_1 and γ_2.*
A maximum occurs if

$$|\Delta y| < \gamma_1 \Delta x - \frac{\gamma_1^2}{2\gamma_2} \quad \textit{and} \quad \Delta x \geq \frac{\gamma_1}{\gamma_2}$$

or if

$$|\Delta y| < \frac{\gamma_2}{2}\Delta x^2 \quad \textit{and} \quad \Delta x \leq \frac{\gamma_1}{\gamma_2}$$

If a maximum occurs we can distinguish the following four cases.

1. *If $\Delta x_1 \geq \gamma_1/\gamma_2$ and $\Delta x_2 \geq \gamma_1/\gamma_2$ then $\Delta x_1 = \frac{\Delta y + \gamma_1 \Delta x}{2\gamma_1}$. All four parts of $f(x)$, as depicted in figure 1, are used.*
2. *If $\Delta x_1 \geq \gamma_1/\gamma_2$ and $\Delta x_2 < \gamma_1/\gamma_2$ then $\Delta x_1 = \frac{\gamma_1}{\gamma_2} + \Delta x - \sqrt{\frac{2}{\gamma_2}(\gamma_1\Delta x - \Delta y)}$. Parts I, II and III of $f(x)$ are used.*
3. *If $\Delta x_1 < \gamma_1/\gamma_2$ and $\Delta x_2 \geq \gamma_1/\gamma_2$ then $\Delta x_1 = \sqrt{\frac{2}{\gamma_2}(\gamma_1\Delta x + \Delta y)} - \frac{\gamma_1}{\gamma_2}$. Parts II, III and IV of $f(x)$ are used.*
4. *If $\Delta x_1 < \gamma_1/\gamma_2$ and $\Delta x_2 < \gamma_1/\gamma_2$ then $\Delta x_1 = \frac{\Delta y}{\gamma_2 \Delta x} + \frac{\Delta x}{2}$. Parts II and III of $f(x)$ are used.*

The maximum height h_1 above $\delta(\omega_k)$ is given by $h_1 = -f(\Delta x_1)$, where $f(x)$ is given in (10).
If no maximum occurs we can distinguish the following seven cases.

1. *If $\gamma_1\Delta x - \frac{\gamma_2}{2}\Delta x^2 \leq \Delta y < \gamma_1 \Delta x$ then $\Delta x_1 = \frac{\gamma_1}{\gamma_2} + \Delta x - \sqrt{\frac{2}{\gamma_2}(\gamma_1\Delta x - \Delta y)}$. Note that $\Delta x_1 \geq \Delta x$. Parts I and II of $f(x)$ are used.*
2. *If $\frac{\gamma_2}{2}\Delta x^2 \leq \Delta y < \gamma_1\Delta x - \frac{\gamma_2}{2}\Delta x^2$ then $\Delta x_1 = \frac{\Delta x}{2} + \frac{\Delta y}{\gamma_2 \Delta x}$. Note that $\Delta x_1 \geq \Delta x$. Only part II of $f(x)$ is used.*
3. *If $\gamma_1\Delta x - \frac{\gamma_2}{2}\Delta x^2 \leq -\Delta y < \gamma_1 \Delta x$ then $\Delta x_2 = \frac{\gamma_1}{\gamma_2} + \Delta x - \sqrt{\frac{2}{\gamma_2}(\gamma_1\Delta x + \Delta y)}$. Note that $\Delta x_2 \geq \Delta x$. Parts III and IV of $f(x)$ are used.*
4. *If $\frac{\gamma_2}{2}\Delta x^2 \leq -\Delta y < \gamma_1\Delta x - \frac{\gamma_2}{2}\Delta x^2$ then $\Delta x_2 = \frac{\Delta x}{2} - \frac{\Delta y}{\gamma_2 \Delta x}$. Note that $\Delta x_2 \geq \Delta x$. Only part III of $f(x)$ is used.*
5. *If $\Delta y = \gamma_1 \Delta x$ then $\Delta x_1 = \frac{\gamma_1}{\gamma_2} + \Delta x$. Only part I of $f(x)$ is used.*
6. *If $\Delta y = -\gamma_1 \Delta x$ then $\Delta x_2 = \frac{\gamma_1}{\gamma_2} + \Delta x$. Only part IV of $f(x)$ is used.*
7. *If $|\Delta y| > \gamma_1 \Delta x$ then the estimated point of the discrete estimate with the highest error bound must not be used. Interpolation from neighbouring points, although over a greater distance, gives a lower error bound. This situation can also arise when $|\Delta y| \leq \gamma_1 \Delta x$, see figure 2.*

Now we are able to give an upper bound for the difference between the system and the nominal model over the whole frequency interval.

Note that, as opposed to [8, 6], algoritm 4.3 allows for a discrete error bound $\alpha(\omega_k)$ that is frequency dependent, and it yields a continuous error bound $\delta(\omega)$ that is frequency dependent. Moreover, the discrete frequency points are not required to be equidistant.

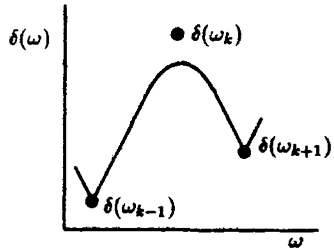

Fig. 2: A situation in which the point $\delta(\omega_k)$ must not be used.

4.4 Remarks.

Taking a closer look at the results up till now, we can summarize as follows. In section 3 a bound $\alpha(\omega_k)$ has been derived

$$|G_o(e^{j\omega_k}) - \hat{G}(e^{j\omega_k})| \leq \alpha(\omega_k) \quad (12)$$

for all ω_k in a set $\Omega \subset \mathbf{R} \cap [0, 2\pi)$ containing a finite number ($\leq N$) of elements. Since the nominal model is assumed to be known, the error

$$\beta(\omega_k) := |\hat{G}(e^{j\omega_k}) - G_{nom}(e^{j\omega_k})| \quad (13)$$

can be calculated exactly for all $\omega_k \in \Omega$. In this section 4, a continuous bound $\delta(\omega)$ has been derived

$$|G_o(e^{j\omega}) - G_{nom}(e^{j\omega})| \leq \delta(\omega) \quad (14)$$

with $\delta(\omega_k) = \alpha(\omega_k) + \beta(\omega_k)$ for $\omega_k \in \Omega$. In the nonparametric discrete estimate, cf. (12), no error due to under modelling is present, i.e. no error due to approximation is made, because complete freedom exist for each frequency point to fit $G_o(e^{j\omega_k})$. The approximation error therefore is completely due to the nominal model, cf. (13).

In the procedure presented, the determination of the nominal model and the determination of the error bound clearly are completely separated. We addressed the problem of determining the error bound. The problem of determining, from the discrete estimate, a nominal model such that the error bound is as low as possible is addressed in [8, 6]. Methods for tuning the nominal model to nominal control design specifications are discussed in [1, 13, 14].

5 Relation with control design specifications.

To show the applicability of the approach presented in this paper to robust control design, we will consider the following situation. In order to verify desired robustness properties of a designed controller for the system, an allowable error bound is specified for the difference between G_o and G_{nom}

$$|G_o(e^{j\omega}) - G_{nom}(e^{j\omega})| \leq \delta_a(\omega)$$

The allowable error $\delta_a(\omega)$ is a function of the nominal model, the designed controller and the robust control design specifications. Given measurement data from the system, it now has to be verified whether a specific nominal model lies within the specified error bound. If not, there should be determined which action should be taken in order to solve the problem: either constructing a new nominal model, or performing new experiments to reduce the uncertainty. This can be done by comparing the allowable error $\delta_a(\omega)$ with the actual error bound $\delta(\omega)$. For those values of ω where $\delta(\omega) > \delta_a(\omega)$ we can analyse $\delta(\omega)$ and evaluate its different components.

At the finite number of frequency points $\omega_k \in \Omega$, we have $\delta(\omega_k) = \alpha(\omega_k) + \beta(\omega_k)$. Therefore we know that

1. when $\alpha(\omega_k) >> \beta(\omega_k)$, the uncertainty is mainly due to the inherent uncertainty in the data, i.e. effects of initial and final conditions, bad signal-to-noise ratio and/or restricted length of the data set. Actions to be taken to improve the bound include: increasing N_s, increasing the power of the input signal, and increasing N. In the case of periodic input signals, the signal-to-noise ratio in the frequency domain is proportional to $\sqrt{N/N_0}$. Consequently the error bound can also be improved by decreasing N_0.

2. when $\alpha(\omega_k) << \beta(\omega_k)$, the uncertainty is mainly due to a bad nominal model (approximation error). A straightforward action is then to choose a new nominal model, that is better able to represent the system dynamics in the specific frequency range.

In between the finite number of frequency points $\omega_k \in \Omega$, say for $\omega_k < \omega < \omega_{k+1}$, the error bound $\delta(\omega)$ is determined through interpolation between the adjacent points $\delta(\omega_k)$, $\delta(\omega_{k+1})$. Therefore

3. when $\delta(\omega) >> max(\delta(\omega_k), \delta(\omega_{k+1}))$, the uncertainty is mainly due to the interpolation step. Note that uncertainty due to interpolation is strongly determined by the distance between two adjacent discrete frequency points. Consequently new experiments should be performed with a smaller distance between the discrete frequency points in the specific frequency region.

Note that it is possible, at each frequency, to determine whether the main source of the actual error is the inherent uncertainty in the data, the nominal model, or the interpolation step caused by the absence of data due to the specific excitation of the system. Also it is possible to decrease the contribution of these different error sources almost independently. Now it is possible to iteratively decrease the error bound, until the level of the allowable error is reached, successively by input design and additional experiments, and by tuning the nominal model. Using this procedure we can determine whether or not specific robust control design specifications can be met. Note that the error bound $\alpha(\omega_k)$ is essentially frequency dependent and that the frequency points $\omega_k \in \Omega$ need not be positioned equidistantially over the frequency axis. In comparison with the existing methods (see e.g. [8, 6]), this creates a lot of freedom to shape the error bound into an allowable form, which from a control point of view definitely should be frequency-dependent.

6 Example.

To illustrate the merits of the procedure a simulation was made with a fifth order system who's impulse response satisfies $M = 2.5$ and $\rho = 1.25$, and a third order nominal model. There was 10 percent (in amplitude) noise on the output. The upper bound $\overline{V}^s(\omega_k)$ was set to $\overline{V}^s(\omega_k) = 2 \cdot \max_{\omega_k} |V^s(\omega_k)|$. The input signal was chosen to obey $\bar{u}^p = 2$ and $\bar{u} = 1$. We used 1074 points with $N = 1024$, $N_0 = 256$ and $N_s = 50$. The magnitude of the DFT of the input signal in the interval $T^N_{N_s}$, $|U^s(\omega_k)|$, is given in figure 3. Note that the frequencies ω_k are not equidistant. The input was designed to result in an error bound smaller than the allowable error by choosing the frequency grid and the magnitude of $U^s(\omega_k)$. In figure 4 the allowable error $\delta_a(\omega)$, the error bound $\delta(\omega)$ and the error due to approximation $\beta(\omega_k)$ are given. The inherent uncertainty in

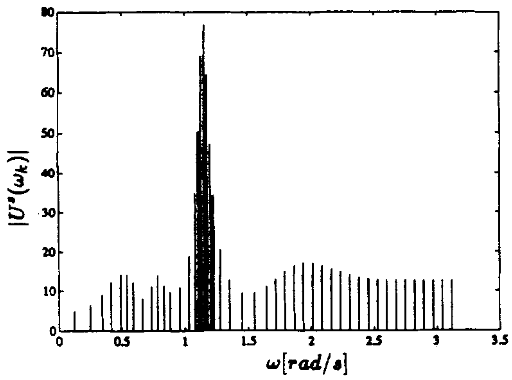

Fig. 3: $|U^*(\omega_k)|$, the magnitude of the DFT of the input signal in $T^N_{N_s}$.

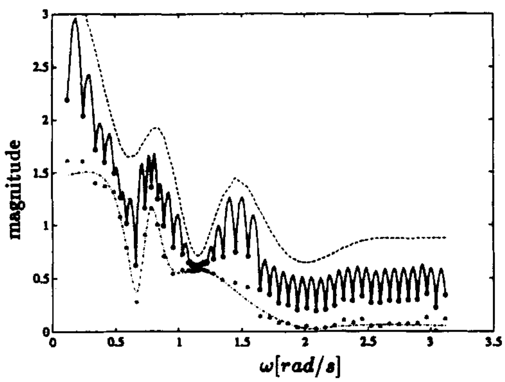

Fig. 4: The error bounds and the true error: $\delta_a(\omega)$ (dash), $\delta(\omega)$ (solid), $\delta(\omega_k)$ (o), $\beta(\omega_k)$ (*), $|G_o(\omega) - G_{nom}(\omega)|$ (dashdot).

the data $\alpha(\omega_k)$ equals $\delta(\omega_k) - \beta(\omega_k)$. The error due to interpolation is indicated by the curves between the points $\delta(\omega_k)$. Note that $\beta(\omega_k)$ provides a good indication of the true approximation error, and that the error bound $\delta(\omega)$ can be made almost equal to the true approximation error by input design. Comparing $\beta(\omega_k)$ and $\delta(\omega)$, it follows that in the frequency interval $\omega \in [1.1, 1.3]$ the error due to approximation clearly dominates, whereas for $\omega \in [2, \pi]$ the inherent uncertainty in the data and the error due to interpolation clearly dominate.

7 Conclusions

In this paper a procedure is presented to quantify the model uncertainty of any prespecified nominal model, given a sequence of measurement data from a plant. The empirical transfer function estimate (ETFE) is used to construct a -nonparametric-estimate of the transfer function in a discrete number of frequency points, together with an upper bound on the error. Through interpolation, this error bound is transformed to a bound which is available on a continuous frequency interval. A frequency dependent upper bound is obtained, which is much more tailored to the needs of a robust control design scheme, than an H_∞-bound. In order to obtain a tight error bound, a special input signal is proposed (partly periodic) which has advantages over -classical- sinewave experiments.

The estimated upper bound for the model error of a prespecified nominal model can be split into three parts: one part due to the inherent uncertainty in the data, a second part due interpolation, and a third part due to imperfections of the nominal model. These three components can be tuned almost independently, by experiment design and by choosing an appropriate nominal model. When the error bound is too conservative in relation with control design specifications, information is provided as to which action should be taken (new experiments or alternative nominal model) in order to satisfy the design requirements.

In the procedure presented an upper bound on the DFT of the noise is assumed to be known. Current investigation is aimed at relaxing this assumption, using a probabilistic description for the influence of the noise, see [3].

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