Proceedings of the 30th Conference
on Decision and Control
Brighton, England • December 1991

Approximate Identification with Linear Regression Models

Paul M.J. Van den Hof

Mechanical Engineering Systems and Control Group
Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands

Abstract

Linear regression models are analysed in the least squares identification of linear multivariable finite-dimensional models. Using the concept of system behaviour, the identification problem is formulated in a deterministic, signal-oriented framework, showing clearly the distinction between problems of identification (choice of model sets) and of parametrization. In order to obtain nontrivial identified models, the identification criterion should be able to distinguish between the different models in the set. This requirement of discriminability, puts restrictions on the model sets to be considered. Sets of sufficient conditions are formulated, in terms of the polynomial representations of the models, while noting that the identified models finally obtained, are essentially dependent on the restrictions that have been chosen. The problem discussed is shown to be closely related to the problem of constructing identifiable parametrizations for model sets described in (forward or backward) polynomial forms.

1 Introduction

The use of equation error models, often also denoted by linear regression models, is widespread in issues of modelling and identification of dynamical systems. The essential characteristic of the linear regression model is that a residual component e is defined which is a linear function of the unknown model coefficients. In the SISO (single input single output) situation we can write:

$$\begin{aligned} e(t) \;=\; & a_0 y(t) + a_1 y(t-1) + .. + a_{n_a} y(t-n_a) - \\ & b_0 u(t) - b_1 u(t-1) - b_{n_b} u(t-n_b) \end{aligned} \qquad (1)$$

with $y(t)$ the output signal, and $u(t)$ the input signal of the model, and $a_0, a_1, .., a_{n_a}, b_0, .., b_{n_b}$ the unknown parameters. With the restriction $a_0 = 1$, the above model is known as the ARX model (Ljung, 1987).

The use of these kinds of models in estimation and identification problems is essentially based on the argument that a least squares identification criterion, is an optimization problem that is analytically solvable.

However note that, as in the ARX-case with $a_0 = 1$, we have to restrict the general model (1) in some way, in order to prevent the resulting parameter estimation problem from only having the trivial solution $a_0 = .. = a_{n_a} = b_0 = .. = b_{n_b} = 0$. In some areas of application it is well known that the - through least squares - identified model, is essentially dependent on the choice of this restriction. Very often, as e.g. in statistical estimation, this restriction is not even recognized as a choice, but fixed a priori. In a statistical framework, the additional assumption is generally made that the residual signal $e(t)$ is a white noise sequence, which causes the identified model to be invariant under different choices of restrictions, as e.g. $a_0 = 1$, or $b_0 = 1$ in (1). However, in many situations it appears to be rather unrealistic to assume that we can construct models on the basis of measured data, in such a way that the residual signal (or error signal) is a white noise sequence. Therefore in approximate identification, the problem as indicated becomes relevant.

Linear regression models have been analysed before in the context of approximate identification of dynamical systems. In [8, 9] properties have been shown of least-squares identified equation error models in an approximative sense, formulated in terms of the Markov parameters of the models. For single input, single output systems, a frequency domain formulation of properties in the approximative situation is given in [16], while [3] gives frequency domain results for the consequences of choosing different coefficient restrictions.

In this paper the mechanism of restricting the models will be analysed in the multivariable situation. It will be investigated in which sense models of the type discussed above have to be restricted, in order to serve as appropriate models to be used in least-squares-based identification.

In sections three and four we will present some basic concepts concerning the notions of models and their use in (approximate) identification. We will adopt part of the signal-based framework of Willems [17, 18] for this discussion, as this framework is especially appropriate for formulating the problem of identification as well as of clearly distinguishing the problems of identification and parametrization. This refers to a clear distinction between choices that really effect the identified models (choice of the model set), and choices that only refer to matters of representation (choice of parametrization). The choice for a restriction is generalized into the requirement that a model set should be discriminable by an identification criterion, and several sets of sufficient conditions for this discriminability are presented in section 5.

The construction of discriminable model sets appears to have close connections to the problem of constructing identifiable parametrizations of dynamical systems, and of constructing sets of canonical forms for polynomial matrices. Albeit that in the present case we essentially do not have a parametrization problem, but a problem of constructing model sets.

Preliminary work on the subject of this paper has been published in [11, 12]. For the proofs of the results, the reader is referred to [15].

2 Notation

In order to be able to deal with both forward and backward time shift operations in one model representation, we have to consider polynomial matrices in two indeterminates, (sometimes called binomials). To this end, consider a polynomial matrix $T \in \mathbb{R}^{p \times q}[z, z^{-1}]$. We will denote:

$$\begin{aligned} T_{i*} &:= \text{the } i^{th} \text{ row of } T(z, z^{-1}), \;\; i = 1, .., p; \\ T_{*j} &:= \text{the } j^{th} \text{ column of } T(z, z^{-1}), \;\; j = 1, .., q; \\ \nu_i^{(u)}(T) &:= \text{the maximum power of } z \text{ in } T_{i*}(z, z^{-1}); \\ & \quad \text{upper row degree;} \\ \nu_i^{(l)}(T) &:= \text{the minimum power of } z \text{ in } T_{i*}(z, z^{-1}); \\ & \quad \text{lower row degree;} \\ \mu_i^{(u)}(T) &:= \text{the maximum power of } z \text{ in } T_{*i}(z, z^{-1}); \end{aligned}$$

upper column degree;

$\mu_i^{(l)}(T)$:= the minimum power of z in $T_{*i}(z, z^{-1})$; lower column degree;

Γ_{hr} := the leading row coefficient matrix of T, i.e. the coefficient matrix related to the highest row degree terms in T;

Γ_{lr} := the trailing row coefficient matrix of T, i.e. the coefficient matrix related to the lowest row degree terms in T;

Γ_{hc} := the leading column coefficient matrix of T;

Γ_{lc} := the trailing column coefficient matrix of T;

$\Gamma_{c,(n_1,..,n_q)}(T)$:= the column coefficient matrix of T related to the column degrees $n_1, .., n_q$.

Note that the integer indices $\nu_i^{(u)}$, $\nu_i^{(l)}$, $\mu_i^{(u)}$ and $\mu_i^{(l)}$ are either positive or negative and that $\nu_i^{(u)} \geq \nu_i^{(l)}$, and $\mu_i^{(u)} \geq \mu_i^{(l)}$.
The notation $M_1 = \tilde{M}_p(T_1)$ refers to the dynamical system M_1 that is being induced by the polynomial matrix (autoregressive) representation T_1 (see section 3). Further we will use $\mathbf{Z}$ for the set of integer numbers, $\mathbb{R}(z)$ for the field of rational functions, $\mathbb{R}[z, z^{-1}]$ for the ring of polynomials in two indeterminates. $T \in \mathbb{R}^{p\times p}[z, z^{-1}]$ is unimodular if its inverse is polynomial, i.e. $det\, T = cz^d$, with $c \neq 0$ and $d \in \mathbf{Z}$. For polynomial or rational matrices T, the notation det T, and rank T, will refer to the determinant and the rank of T taken over the field of rational functions $\mathbb{R}(z)$, while $\|T\|_2$ refers to the L_2-norm of T.

3 Dynamical systems and equation error models

In correspondence with [17] we will refer to a linear dynamical system, and consequently also to a model of a system, as a triplet $(T, W, \mathcal{B})$, with T the time set on which the signals that are related to the system will be defined; W the signal set, i.e. the vector space in which the signals take on their values; and $\mathcal{B}$ the behaviour of the system, i.e. the set of signal-trajectories that are admissible by the system. In this paper we will restrict attention the linear, time-invariant and finite-dimensional discrete-time systems ($T = \mathbf{Z}$). For a dynamical system having m inputs and p outputs the signal set W will generally be fixed to $W = \mathbb{R}^{p+m}$. Having specified T and W the essential characteristic of a dynamical system is reflected by its behaviour $\mathcal{B}$.
A more extensive discussion on these concepts related to the common input-output description of dynamical systems, can also be found in [10, 14].

For the purpose of approximate identification on the basis of measured input-output data we will consider models that have three different types of prespecified external signals: a p-dimensional output signal y, an m-dimensional input signal u, and a p-dimensional residual signal e. In an identification context, (part of) the signals u and y will be measured. The residual signal is not actually measured, but acts as a basis for a measure of fit between a measured data sequence $\{u, y\}$ and a model (cf. (1)).
The class of models that we will be dealing with in this paper, will be restricted to the class of input-output-processing residual ($i/o/pr$) models, see [10, 11, 15], denoted by $\hat{\Sigma}_{p,m}$, which means that the signal set $W = \mathbb{R}^{p+m+p}$ and that each model from this set can be represented by a polynomial description:

$$P(\sigma, \sigma^{-1})y - Q(\sigma, \sigma^{-1})u - R(\sigma, \sigma^{-1})e = 0 \qquad (2)$$

with $T = [\ P \mid -Q \mid -R\ ]$ a full row rank polynomial matrix, $T \in \mathbb{R}^{p\times(p+m+p)}[z, z^{-1}]$ with $det\ P \neq 0$, and $det\ R \neq 0$; the *shift operators* σ, σ^{-1} are defined by: $(\sigma w)(t) = w(t+1)$, $t \in \mathbf{Z}$ and $(\sigma^{-1}w)(t) = w(t-1)$, $t \in \mathbf{Z}$.
The combined set of signals (y, u, e) will be denoted by w. The behaviour of the model, defined by (2), is denoted by

$$\mathcal{B}(M) = \{w \in (\mathbb{R}^{p+m+p})^{\mathbf{Z}} \mid T(\sigma, \sigma^{-1})w = 0\}$$

Two models are equal, i.e. their behaviours are the same, if and only if the corresponding polynomial matrices T are related through unimodular premultiplication.
Note that this type of model is strongly related to the (stochastic) models that show up in prediction error identification methods.
Taking a closer look at the polynomial matrix T that generates a dynamical system in the class $\hat{\Sigma}_{p,m}$, it follows that T induces two rational matrices, that can be considered as transfer functions of the corresponding system. These two transfer functions are denoted by:

$$H_y(z) = [\ H_{yu}(z) \mid H_{ye}(z)\ ] := [\ P^{-1}Q \mid P^{-1}R\ ], \text{ and} \qquad (3)$$

$$H_e(z) = [\ H_{ey}(z) \mid H_{eu}(z)\ ] := [\ R^{-1}P \mid -R^{-1}Q\ ] \qquad (4)$$

The behaviour of M is completely characterized by either one of the two transfer functions H_y, H_e, if and only if M is controllable (Willems, 1988), *i.e.* if T is left coprime with respect to $\mathbb{R}[z, z^{-1}]$, or in other words: $rank_{\mathbf{C}} T(\lambda, \lambda^{-1}) = p$ for all $\lambda \in \mathbf{C}\backslash\{0\}$. The controllable part $\mathcal{B}_c(M)$ of a system behaviour $\mathcal{B}(M)$ is determined by the behaviour that is induced by the polynomial matrix T' that is left coprime with respect to $\mathbb{R}[z, z^{-1}]$ and that generates the transfer functions of M according to (3), (4).
The i/o-part of an ($i/o/pr$)-model will also be specified as

$$\mathcal{B}^{io}(M) := \{v \in \mathbb{R}^{(p+m)\mathbf{Z}} \mid (v, e) \in \mathcal{B}(M),\ e = 0\},\ \text{with } v = (y, u).$$

If an ($i/o/pr$)-model M is defined by the behaviour $\mathcal{B}(M)$, then its i/o-part M^{io} is defined by $\mathcal{B}^{io}(M)$. For evaluation of the i/o-part of a ($i/o/pr$)-model, the residual component in the model is simply discarded. We will also refer to the so called i/o-transfer-equivalence relation, $M_1 \overset{i/o}{\sim} M_2$, being defined by $\mathcal{B}_c(M_1^{io}) = \mathcal{B}_c(M_2^{io})$, which is equivalent with $H_{yu}^{(1)} = H_{yu}^{(2)}$.
Restrictions of signal variables and behaviours to the time set $\mathbf{Z}_+$ will be denoted by w^+ and $\mathcal{B}^+$.

Definition 3.1 *A linear regression (equation error) model is an ($i/o/pr$)-model that satisfies the additional property that the residual e is observable from (y,u), or in other words:*

$$\{(y, u, e_1) \in \mathcal{B}(M) \wedge (y, u, e_2) \in \mathcal{B}(M)\} \Rightarrow \{e_1 = e_2\}$$

The notion of observability of signals is defined in [17].
The definition implies that in the polynomial representation, the matrix R is restricted to be unimodular. Since unimodular premultiplication does not change the behaviour, this means that a linear regression model can always be represented by a full row rank polynomial matrix $T = [\ P \mid -Q \mid -I_p\ ]$, with I_p the $p \times p$ identity matrix. Moreover it follows that any equation error model is a controllable ($i/o/pr$)-model.

4 Modelling on the basis of data

In the problem of modelling dynamical systems on the basis of input-output data, three central aspects will be distinguished: (1) The set of models that is considered; (2) a parametrization, representing the models in the model set with (real valued) parameters,

and (3) an identification criterion that selects "best" or "optimal" models from the set of models, given the measured data.
Given the measurement data, the models that are finally obtained as a result of the modelling procedure should be determined by the set of models taken into account and by the identification criterion, and should not be dependent on other choices, like e.g. the parametrization of the model set. The parametrization acts as a tool for representing the models by real-valued parameter values in order to apply identification algorithms. In this paper we will consider as a set of models $\mathcal{M}$ any collection of (i/o/pr)-models with a fixed number of inputs m and outputs p, and having the additional property of being of the equation error form. A parametrization of a model set $\mathcal{M}$ is defined as a surjective mapping $\tilde{M} : \Theta \to \mathcal{M}$, with $\Theta \subset \mathbb{R}^d$ the parameter set, and $\mathcal{M}$ the parametrized set of models. Parametrizations will be considered in terms of the polynomial representations discussed before, with parameters being defined through the coefficients of the polynomials.
We will now focus on a formal definition of an identification criterion. Let there be given a measured data sequence v^+, and any set of models $\mathcal{M}$.
In this paper a restriction will be made to identification criteria that are based on the residual function:

$$\ell(e^+) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} e^T(t)e(t) \tag{5}$$

The identification criterion takes residual signals that are compatible with the measured data v^+ for a model M, as "measures of fit" between the data and the model. The residual function is examined on the residual signals that, together with the measured data sequence v^+, constitute an admissible trajectory $w^+ \in \mathcal{B}^+(M)$ for $M \in \mathcal{M}$. In formal terms: given a measured data sequence v^+, then each model $M \in \mathcal{M}$ generates a set of admissible residual signals, defined by:

$$\mathcal{E}(v^+, M) := \{e^+ \mid (v^+, e^+) \in \mathcal{B}^+(M)\}. \tag{6}$$

The residual function ℓ is evaluated on the set $\mathcal{E}$ for each $M \in \mathcal{M}$ in order to determine the set of identified models, by:

$$\begin{aligned} J(v^+, \mathcal{M}) &= \arg_{M\in\mathcal{M}} \min_{e^+\in\mathcal{E}(v^+,M)} \ell(e^+) && (7)\\ &:= \{\hat{M} \in \mathcal{M} \mid \exists \hat{e}^+ \in \mathcal{E}(v^+, \hat{M}),\ \ell(\hat{e}^+) \le \ell(e^+) \text{ for all} \\ &\qquad e^+ \in \mathcal{E}(v^+, M) \text{ and } M \in \mathcal{M}\} && (8)\end{aligned}$$

Actually the identification criterion is defined as a selection rule. Given a time series and a model set, the criterion J selects one or more "optimal" models from the model set. This formulation is a very general one. It provides a means for discussing fundamental properties of model sets, parametrizations and identification criteria, as e.g. identifiability and discriminability (see next part) at a higher level of abstractness and in a more generalized way, than is usual in the identification and parametrization literature.

Now the question arises if any combination of choices for model set and identification criterion leads to a useful identification problem. Apparently restriction of the model set is required in order to come up with a sensible identification problem, but it is also necessary in order to guarantee that all models in the model set can be distinguished by the identification criterion. The ability to distinguish the different models in a model set during identification experiments will now be formalized through the notion of discriminability, as was introduced in [10].

Definition 4.1 *An model M is called discriminable within $\mathcal{M}$ by an identification criterion J, defined on $\mathcal{M}$, if there exists a data sequence v^+ such that $J(v^+, \mathcal{M}) = \{M\}$.* □

When all models in a model set $\mathcal{M}$ are discriminable, the model set will be called discriminable by J, and J will be called *discriminating* on $\mathcal{M}$. An identification criterion that is discriminating on a model set can distinguish between the different models in this set. In general terms it can be stated that if a model set is not discriminable by J, discriminability can be obtained by making a restriction to a discriminable subset $\mathcal{M}_1 \subset \mathcal{M}$. The main goal of this paper is to discuss how to construct model sets that have this discriminability property.

Remark 4.2 *The concept of discriminability is closely related to the more conventional concept of system identifiability. However system identifiability very often refers to consistency properties of identification methods. In order to stress that we consider a different situation, we have chosen to denote the property of discriminability.*

Remark 4.3 *The definition of a discriminable model as presented in definition 4.1 also shows the aspect of experimental conditions under which the model is discriminable. In* [13] *discriminability under closed loop experimental conditions is discussed.*

It has to be stressed that the concept of discriminability, as discussed, is a property of a model set in conjunction with an identification criterion; it is not related to any parametrization of the set of (i/o/pr)-models.

5 Discriminability of equation error models through LS-identification

5.1 General results

In the next two theorems we will formulate two different sets of sufficient conditions for discriminability of model sets.

Theorem 5.1 *Let $\mathcal{M}$ be a set of controllable (i/o/pr)-models, $\mathcal{M} \subset \hat{\Sigma}_{p,m}$. If for all $M_1, M_2 \in \mathcal{M}$:*

$$M_1 \overset{i}{\underset{io}{\sim}} M_2 \quad \Rightarrow \quad M_1 = M_2 \tag{9}$$

then $\mathcal{M}$ is discriminable by J. □

Proof: Consider any element $M_0 \in \mathcal{M}$, and a signal $w = (v, e) \in \mathcal{B}(M_0)$, with $e = 0$, $v = (y, u)$, and u satisfying

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} u(t)u^T(t+\tau) = I_m\delta(\tau)$$

with I_m the $m \times m$-identity matrix, and $\delta(\tau)$ the delta-function: $\delta(\tau) = 0$, for $\tau \neq 0$, and $\delta(\tau) = 0$, for $\tau = 0$. Consequently for any other model $M_1 \in \mathcal{M}$ it follows that

$$M_1 \in J(v^+, \mathcal{M}) \Leftrightarrow \{\ \exists e_1 \in \mathcal{E}(v^+, M_1) \mid \ell(e_1) = 0\}.$$

In [10] it is shown that the latter expression is equivalent with $M_1 \overset{i}{\underset{io}{\sim}} M_0$. Under the condition as formulated in the theorem, it follows that $\{M_0\} = J(v^+, \mathcal{M})$. Since M_0 can be chosen any element of $\mathcal{M}$, this proves discriminability of $\mathcal{M}$ by J. □

The theorem shows that absence of distinct models that are i/o transfer-equivalent, is sufficient for guaranteeing discriminability of the model set by J. Apparently, in the context of this theorem, the problem of constructing discriminable model sets is very closely related to the problem of constructing identifiable parametrizations, as e.g. discussed in [5]. Models that are equivalent through the equivalence relation $\overset{i}{\underset{io}{\sim}}$, should be removed from the model set. Further consequences of this theorem are analysed in more detail in section 5.2.
A second set of sufficient conditions, being less restrictive than the condition of theorem 5.1, is formulated in the following theorem.

Theorem 5.2 *Let $\mathcal{M}$ be a set of equation error models, $\mathcal{M} \subset \hat{\Sigma}_{p,m}$.*
Then $\mathcal{M}$ is discriminable by J if for any two models M_1, $M_2 \in \mathcal{M}$ satisfying $M_1 \overset{i/o}{\sim} M_2$ and $H_{e_2e_1} := H_{ey}^{(2)}(H_{ey}^{(1)})^{-1}$ a stable rational matrix [1], it holds that

$$\|H_{e_2e_1}(z)\|_2 \geq p \text{ with equality if and only if } M_1 = M_2.$$

The sufficient condition stated in this theorem is less restrictive than the condition of theorem 5.1; whereas in theorem 5.1 distinct i/o-transfer-equivalent models are abandoned, in theorem 5.2 restrictions are formulated on distinct i/o-transfer-equivalent models. When discussing the consequences of these theorems, we will most often deal with the transfer functions $H_e(z)$ of the models, see (4). Because of the fact that we deal with equation error models this transfer function is polynomial and uniquely related to (the behaviour of) the model.
In the next two subsections we will analyse the consequences of the sufficient conditions as formulated in the above theorems separately.

5.2 Removing i/o-transfer-equivalent models

If it is not premitted to allow distinct i/o-transfer-equivalent models in the model set, it is straightforward to impose the condition that for all models $M \in \mathcal{M}$, M^{io} is required to be controllable.

Corollary 5.3 *Let $\mathcal{M}$ be a set of equation error models, $\mathcal{M} \subset \hat{\Sigma}_{p,m}$, such that for all models $M \in \mathcal{M}$, M^{io} is controllable.*
Then $\mathcal{M}$ is discriminable by J if for all $M_1, M_2 \in \mathcal{M}$ with corresponding transfer functions $H_e^{(1)}$, $H_e^{(2)} \in \mathbb{R}^{p\times(p+m)}[z,z^{-1}]$:

$$\{H_e^{(1)} = UH_e^{(2)} \text{ with } U \in \mathbb{R}^{p\times p}[z,z^{-1}],\ \textit{unimodular}\} \Rightarrow \{U = I\} \tag{10}$$

The corollary clearly reflects the type of problem that is concerned. In this setting the construction of a discriminable model set comes down to the construction of a set of canonical forms for polynomial matrices based on the equivalence relation of unimodular premultiplication (unimodularity here has to be considered with respect to the ring $\mathbb{R}^{p\times(p+m)}[z,z^{-1}]$). This directly refers to a parametrization problem, as discussed in [5] and [4] for polynomials in one indeterminate, and in [6] for the binomial case. Note however that the problem of constructing the model set as such, is *not* a parametrization problem, since it is not a matter of representation, but the identified models finally obtained, are essentially dependent on the model set that has been chosen. This means that, referring to this problem as a parametrization-like problem, when we choose for a different set of canonical forms we will end up with a different set of identified models $J(v^+,\mathcal{M})$, since the *model set* $\mathcal{M}$ essentially will have changed, and not only its representation in terms of parameters.
A further specification of the results in the corollary are given below.

Corollary 5.4 *Let $\mathcal{M}$ be a set of equation error models, $\mathcal{M} \subset \hat{\Sigma}_{p,m}$, such that for all models $M \in \mathcal{M}$, M^{io} is controllable.*
Then $\mathcal{M}$ is discriminable by J if there exist rational matrices $K \in \mathbb{R}^{(p+m)\times q_1}(z)$, $N \in \mathbb{R}^{(p+m)\times q_2}(z)$, and $L \in \mathbb{R}^{(p+m)\times q_3}(z)$, with $q_1, q_2, q_3 \geq p$, such that for all $M, M_1, M_2 \in \mathcal{M}$ with corresponding transfer functions H_e, $H_e^{(1)}$, $H_e^{(2)} \in \mathbb{R}^{p\times(p+m)}[z,z^{-1}]$:

(i) H_eK, H_eN and H_eL are polynomial matrices

(ii) there exist integers $m_1,..,m_{q_1}$, satisfying $m_j \geq \mu_j^{(u)}(H_e^{(i)}K)$, $i = 1,2$, $j = 1,..,q_1$, such that $\textit{rank}\ \Gamma_{c,(m_1,..,m_{q_1})}(H_e^{(1)}K) = \textit{rank}\ \Gamma_{c,(m_1,..,m_{q_1})}(H_e^{(2)}K) = p$;

(iii) there exist integers $n_1,..,n_{q_2}$, satisfying $n_j \leq \mu_j^{(l)}(H_e^{(i)}N)$, $i = 1,2$, $j = 1,..,q_2$, such that $\textit{rank}\ \Gamma_{c,(n_1,..,n_{q_2})}(H_e^{(1)}N) = \textit{rank}\ \Gamma_{c,(n_1,..,n_{q_2})}(H_e^{(2)}N) = p$;

(iv) $H_e^{(i)}L = \sum_{k=s_i}^{t_i} G_k^{(i)} z^k$, for some $s_i \leq t_i$, $i = 1,2$, and $G_0^{(1)} = G_0^{(2)}$, with $\textit{rank}\ G_0^{(1)} = \textit{rank}\ G_0^{(2)} = p$.

The corollary formulates conditions on column degree properties of the postmultiplied transfer functions H_e of the models in the model set. Note that in the situation $q_1 = p$, implying that $H_e^{(i)}K$ is a square matrix, the integers $m_1,..,m_p$ are the upper column degrees of $H_e^{(i)}K$, which have to be equal for $H_e^{(1)}K$ and $H_e^{(2)}K$. Similarly in the situation $q_2 = p$, $H_e^{(i)}N$ is square, and the integers $n_1,..,n_p$ are the lower column degrees of $H_e^{(i)}N$.
Specific choices for matrices K and N can be made, e.g. in order to select columns within $H_e^{(i)}K$, $H_e^{(i)}N$ on which the restrictions formulated in conditions (ii), (iii) should be applied to. Consider for instance $K = N = \begin{bmatrix} I_p \\ 0 \end{bmatrix}$, leading to the situation that the restrictions operate on the polynomial matrices H_{ey} of the models in the model set. Condition (iv) states that one of the coefficient matrices (Markov parameters) of $H_e^{(i)}L$ should be fixed to a full row rank matrix.
The consequences of the corollary are best illustrated in a simple example.

Example 5.5 *Consider the model set $\mathcal{M}$:*

$$\{M \in \hat{\Sigma}_{1,1} | M \textit{ induced by } T(z,z^{-1}) = [a_0 + a_1z^{-1} | -b_0 - b_1z^{-1} | -c],\ a_0, a_1, b_0, b_1 \in \mathbb{R}, ((a_0 + a_1z^{-1}), (b_0 + b_1z^{-1})) \textit{ coprime},\ a_0 \neq 0 \vee a_1 \neq 0; c \in \mathbb{R}\backslash\{0\}\}.$$

it follows that the transfer functions of the models in this set can be written as:

$$H_e = [\ \frac{a_0}{c} + \frac{a_1}{c}z^{-1} \mid \frac{-b_0}{c} + \frac{-b_1}{c}z^{-1}\ ]$$

Applying corollary 5.4 for $K = N = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ now shows the restrictions: $\frac{a_0}{c} \neq 0$, (ii), and $\frac{a_1}{c} \neq 0$, (iii), while condition (iv) of the corollary is satisfied if any of the coefficients $\frac{a_0}{c}$, $\frac{a_1}{c}$, $\frac{b_0}{c}$ or $\frac{b_1}{c}$ is unequal to 0 and fixed over the model set. These four possibilities respectively refer to the choices $L = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, $L = \begin{bmatrix} z & 0 \end{bmatrix}^T$, $L = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, and $L = \begin{bmatrix} 0 & z \end{bmatrix}^T$. □

In order to restrict the number of conditions that have to be imposed on the model set in order to achieve discriminability, the rank conditions (ii) and (iii) of the corollary above, can be chosen in line with the rank condition in (iv), and with the requirement that M^{io} is controllable. For instance if, in example 5.5, we choose $K = L = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, and $N = I$, then (ii) and (iv) are satisfied by requiring that $\frac{a_0}{c} = 1$, while additionally the coprimeness condition on $((a_0 + a_1z^{-1}), (b_0 + b_1z^{-1}))$ and (iii) are satisfied by rank $\begin{bmatrix} \frac{a_1}{c} & \frac{-b_1}{c} \end{bmatrix} = 1$.

Remark 5.6 *For the situation of a higher order SISO ARX model, as in (1). Corollary 5.4 states (by (iv)) that we can restrict any of the coefficients a_i, b_j to 1, but that neverthless restrictions have to be satisfied on a_0, b_0 on the one hand, and on a_{n_a}, b_{n_b} on the other hand, in order to satisfy the requirements of the corollary.*

[1] A rational matrix is called stable if it is analytic in $|\ z\ | \geq 1$, except possibly in $z = \infty$.

Remark 5.7 *Conditions (ii), (iii) of corollary 5.4 have been formulated in terms of column degree and column coefficient properties of polynomial matrices. Similar statements can also be made based on row degree and row coefficient properties, however only in the situation that all upper (c.q. lower) row degrees are equal.*

We have already mentioned that constructing discriminable model sets, based on theorem 5.1, is a problem which is very similar to the problem of constructing identifiable parametrizations for polynomial matrices, dealing with the equivalence relation of unimodular premultiplication. This implies that any polynomial parametrization characterized by a set of matrices $H_e \in \mathbb{R}^{p\times(p+m)}[z,z^{-1}]$ that satisfies the conditions of corollary 5.4 constitutes a set of canonical forms under the equivalence relation mentioned. Specific parametrizations that are dealt with in the literature are covered by this corollary, as e.g.

1. the monic ARMA form (Deistler, 1983).
   This concerns polynomial matrices $T = [P|Q] \in \mathbb{R}^{p\times(p+m)}[z^{-1}]$, with $\det P \neq 0$, (P,Q) left coprime with respect to $\mathbb{R}[z^{-1}]$ and additionally $\Gamma_{hc}(P) = I$, $\mu_i^{(u)}(P) = 0$, $\mu_j^{(l)}(T) \geq n_j$, and rank $G_{c,(n_1,..,n_{p+m})}(T) = p$.
2. the canonical and pseudo-canonical (overlapping) observability forms (Guidorzi, 1981; Gevers and Wertz, 1987).
   This concerns polynomial matrices $T = [P|Q] \in \mathbb{R}^{p\times(p+m)}[z]$, with $\det P \neq 0$, (P,Q) left coprime with respect to $\mathbb{R}[z]$ and additionally $\Gamma_{hc}(P) = I$, $\mu_i^{(u)}(P)$ prespecified, $\mu_j^{(l)}(T) \geq 0$, and rank $G_{c,(0,..,0)}(T) = p$.

In terms of corollary 5.4 both parametrizations reflect the choices $L = \left[\ diag\ \{z^{[-\mu_j^{(u)}(T)]}, j = 1,..,p\}\ \ 0\ \right]^T$, $K = \left[\ I\ \ 0\ \right]^T$ and $N = I_{p+m}$.

5.3 Restricting i/o-transfer-equivalent models

When we analyse the results of theorem 5.2, we come to a second set of sufficient conditions for obtaining discriminability.

Corollary 5.8 *Let $\mathcal{M}$ be a set of equation error models, $\mathcal{M} \subset \hat{\Sigma}_{p,m}$.*
Then $\mathcal{M}$ is discriminable by J if there exist rational matrices K, $N \in \mathbb{R}^{(p+m)\times q}(z)$, $q \geq p$, such that for all $M, M_1, M_2 \in \mathcal{M}$ with corresponding polynomial transfer functions H_e, $H_e^{(1)}$, $H_e^{(2)} \in \mathbb{R}^{p\times(p+m)}[z,z^{-1}]$, H_eK and H_eN are polynomial and either one of the following conditions (i) to (iv) is satisfied.

(i)
- *a.* $\Gamma_{hr}(H_e^{(1)}K) = \Gamma_{hr}(H_e^{(2)}K)$, *having rank p, and*
- *b.* $\nu_i^{(u)}(H_e^{(1)}K) = \nu_i^{(u)}(H_e^{(2)}K)$ *for* $i = 1,..,p$;

(ii)
- *a.* $\Gamma_{hr}(H_e^{(1)}K) = \Gamma_{hr}(H_e^{(2)}K)$, *having rank p, and*
- *b.* $\sum_{i=1}^p \nu_i^{(u)}(H_e^{(1)}K) = \sum_{i=1}^p \nu_i^{(u)}(H_e^{(2)}K)$; *and*
- *c.* $\nu_i^{(l)}(H_e^{(1)}N) = \nu_i^{(l)}(H_e^{(2)}N)$, *for* $i = 1,..,p$, *and*
- *d.* *rank* $\Gamma_{lr}(H_e^{(1)}N)$ = *rank* $\Gamma_{lr}(H_e^{(2)}N) = p$;

(iii) *there exist integers* $m_1,..,m_q$, *satisfying* $m_j \geq \mu_j^{(u)}(H_e^{(i)}K)$, $i = 1,2$, $j = 1,..,q$, *such that* $\Gamma_{c,(m_1,..,m_{q_1})}(H_e^{(1)}K) = \Gamma_{c,(m_1,..,m_{q_1})}(H_e^{(2)}K)$, *having rank p;*

(iv)
- *a.* $q = p$;
- *b.* $\Gamma_{hc}(H_e^{(1)}K) = \Gamma_{hc}(H_e^{(2)}K)$, *having rank p, and*
- *c.* $\sum_{i=1}^p \mu_i^{(u)}(H_e^{(1)}K) = \sum_{i=1}^p \mu_i^{(u)}(H_e^{(2)}K)$; *and*
- *d.* $\nu_i^{(l)}(H_e^{(1)}N) = \nu_i^{(l)}(H_e^{(2)}N)$, *for* $i = 1,..,p$, *and*
- *e.* *rank* $\Gamma_{lr}(H_e^{(1)}N)$ = *rank* $\Gamma_{lr}(H_e^{(2)}N) = p$; □

Especially if we take a closer look at situations (i) and (iii) of the corollary, the conditions that have to be imposed on the model set are less restrictive than the conditions formulated in the previous subsection. In situations (ii) and (iv) even the row (column) degrees do not have to be specified, only the sum of the row (column) degrees. Moreover note that, in contrast with the situation of the previous subsection, it is not required that M^{io} is controllable for all models in the model set.
One of the main differences between the conditions of corollaries 5.4 and 5.8, is that in corollary 5.4 a surjective matrix is fixed (part (iv)) that is one of the coefficient matrices of a polynomial matrix, whereas in corollary 5.8 the matrix to be fixed is specifically a surjective leading row or column coefficient matrix. This means that the situation now becomes different e.g. for the type of models as discussed in remark 5.6. In this general SISO ARX situation, if we want to fix any of the coefficients a_i or b_j to 1, but *not* a_0 or b_0, then the situation does not match anymore with the results of corollary 5.8. Apparently in that situation, controllability of M^{io} is required - as is assumed in corollary 5.4 - in order to achieve discriminability. This is briefly illustrated in the following example.

Example 5.9 *Consider two equation error models M_1, M_2 induced by $H_e^{(1)}(z)$, $H_e^{(2)}(z)$, with*

$$H_e^{(1)}(z) = [\ (z-a)(z-c)\ |\ k(z-b)(z-c)\ ] \quad \textit{and} \quad (11)$$
$$H_e^{(2)}(z) = [\ (z-a)(z-d)\ |\ k(z-b)(z-d)\ ] \quad (12)$$

with $a, b, c, d, k \in \mathbb{R}\backslash\{0\}$. Consequently

$$H_{yu}^{(1)}(z) = H_{yu}^{(2)}(z) = k\frac{z-b}{z-a}$$

and M_1^{io}, M_2^{io} are not controllable. It follows that $H_{e_2e_1}(z) = \frac{z-d}{z-c}$. Since the highest powers of z in $H_{ey}^{(1)}$, $H_{ey}^{(2)}$ have been prespecified to 1, it follows that $H_{e_2e_1}(z)$ is proper with $\lim_{z\to\infty} H_{e_2e_1} = 1$. Consequently $\|H_{e_2e_1}\|_2 \geq 1$. If, alternatively, the coefficient of any of the other powers of z had been fixed in H_{ey}, as e.g. the coefficient of z^0 being related to the lower column/row degree, then $\|H_{e_2e_1}\|_2 \geq 1$ would not be guaranteed, and lack of discriminability would be possible. □

We want specifically to pay some more attention to situations (ii) and (iv) of the corollary. To this end we isolate the following set of restrictions.

Set 1	**Set 2**
• $\Gamma_{hr}(H_e) = I$	• $\Gamma_{hc}(H_{ey}) = I$
• $\sum_{i=1}^p \nu_i^{(u)}(H_e) = n$, fixed	• $\sum_{i=1}^p \mu_i^{(u)}(H_{ey}) = n$, fixed
• $\nu_i^{(l)}(H_e) = 0$, fixed	• $\nu_i^{(l)}(H_e) = 0$, fixed
• rank $\Gamma_{lr}(H_e) = p$	• rank $\Gamma_{lr}(H_e) = p$

If the models in the two sets reflect causal input-output systems, i.e. if the transfer functions $H_{yu}(z)$ are proper rational functions, then the McMillan degree of the models is fixed to be n. A detailed analysis of McMillan degree and structure indices of these polynomial representations is presented in [14].
Now taking a closer look at set 2, presented above, then this set of conditions exactly appears to be the set of restrictions that is imposed on polynomial matrices to generate the so-called pseudo-canonical or overlapping parametrization of all input-output models having a prespecified McMillan degree n, see e.g. [1, 4]. In

our setting, set 2 does not describe overlapping sets of models, but one set of models that for a given value of n can be split up into several subsets related to a specific choice of the integer indices $\mu_1^{(u)},..,\mu_p^{(u)}$, while all the elements in the whole set can be discriminated from eachother. One of the consequences is that when, choosing for a specific set of integer indices $\mu_1^{(u)},..,\mu_p^{(u)}$, and performing an (approximate) identification with a corresponding model set, different dynamical models will be identified, dependent on the specific structure indices that have been chosen (but still summing up to n). This above observation was already pointed out in [12]. It is supported by results of [9], where properties are derived of approximately identified equation error models, being dependent on the specific sequence of integer indices chosen in an overlaping parametrization.

5.4 Parametrization of equation error models

Having determined sufficient conditions for equation error model sets to be discriminable by J we have to spend a few words on the problem of parametrizing these model sets. Actually this problem does not exist. Because of the fact that the restrictions on the models have been formulated in terms of restrictions on the polynomial matrices $H_e(z)$, any equation error model can simply be represented by the polynomial representation

$$T(z,z^{-1}) = [\ H_{ey} \mid H_{eu} \mid -I\ ]$$

in line with (2). This polynomial representation is unique because of the identity matrix, which prevents the polynomial to be transformed into an equivalent form by premultiplication with a unimodular matrix not equal to I.

Conclusions

Equation error models are known to inherently require the a priori choice for specific signal variables to be considered as regressands and/or regressors. The choice for a different regressand will influence the models that will be obtained by applying a least squares identification criterion in an aproximative sense. We have generalized this notion to the multivariable case, by quantifying the restrictions that can be laid upon the model set, in order to guarantee that all models in a model set can be distinguished by a least squares identification criterion. This property, denoted by "discriminability of the model set by the identification criterion" has been analyzed, and different sets of sufficient conditions have been formulated, employing system representations dealing with both forward and backward shift operators.

The first set reflects a problem of constructing sets of canonical forms for polynomial matrices given the equivalence relation of unimodular premultiplication. The result presented directly resembles and generalizes results that are obtained in a related problem of constructing identifiable parametrizations.

The second set of sufficient conditions is less restrictive, and does not directly refer to such a parametrization problem. Nevertheless the result has close connections to parametrization issues, and it is shown that the pseudo-canonical or overlapping parametrization of all dynamical systems with prespecified McMillan degree, actually constitutes a nonoverlapping set of models that is discriminable by a least squares identification criterion.

References

[1] Corrêa, G.O. and K. Glover (1984). Pseudo-canonical forms, identifiable parametrizations and simple parameter estimation for linear multivariable systems: input-output models. *Automatica, 20*, pp. 429-442.

[2] Deistler, M. (1983). The properties of the parametrization of ARMAX systems and their relevance for structural estimation. *Econometrica, 51*, pp. 1187-1207.

[3] De Moor, B., M. Gevers and G.C. Goodwin (1991). Estimation of transfer functions: overbiased, underbiased and unbiased identification schemes. *Proc. 9th IFAC/IFORS Symp. on Identif. and System Param. Estim.*, Budapest, Hungary, July 8-12, 1991.

[4] Gevers, M. and V. Wertz (1987). On identifiable parametrizations for multivariable linear systems". In: C.T. Leondes (Ed.), *Control and Dynamic Systems*, Vol. 26, pp.35-86. Ac. Press, New York.

[5] Guidorzi, R.P. (1981). Invariants and canonical forms for systems structural and parametric identification. *Automatica, 17*, 117-133.

[6] Heij, Ch. (1989). *Deterministic Identification of Dynamical Systems.* Lecture Notes in Control and Inform. Sciences, Vol. 127. Springer Verlag, Berlin.

[7] Ljung, L. (1987). *System Identification: Theory for the User.* Prentice-Hall, Englewood Cliffs, N.J.

[8] Mullis, C.T. and R.A. Roberts (1976). The use of second order information in the approximation of discrete time linear systems. *IEEE Trans. Acoust., Speech, Signal Processing, ASSP-24*, pp. 226-238.

[9] Van den Hof, P.M.J. and P.H.M. Janssen (1987). Some asymptotic properties of multivariable models identified by equation error techniques. *IEEE Trans. Automat. Contr., AC-32*, pp. 89-92.

[10] Van den Hof, P.M.J. (1989a). *On Residual-Based Parametrization and Identification of Multivariable Systems.* Dr. Dissertation, Dept. Electrical Engineering, Eindhoven University of Technology, The Netherlands, 1989.

[11] Van den Hof, P.M.J. (1989b). A deterministic approach to approximate modelling of input-output data. *Proc. 28th IEEE Conf. Decision and Control*, Tampa, FL, pp. 659-664.

[12] Van den Hof, P.M.J. (1989c). Criterion-based equivalence for equation error models. *IEEE Trans. Automat. Contr., vol. AC-34*, 191-193.

[13] Van den Hof, P.M.J. (1991a). Discriminability of linear models under closed loop observations. To be published in *Proc. 1991 International Symposium MTNS*, Kobe, Japan.

[14] Van den Hof, P.M.J. (1991b). On system order and structure indices of linear systems in polynomial form. To be published in *Int. J. Control.*

[15] Van den Hof, P.M.J. (1991c). Model sets and parametrizations for identification of multivariable equation error models. *Submitted for publication.*

[16] Wahlberg, B. and L. Ljung (1986). Design variables for bias distribution in transfer function estimation. *IEEE Trans. Automat. Contr., AC-31*, pp. 134-144.

[17] Willems, J.C. (1986). From time series to linear system — Part I: Finite dimensional linear time-invariant systems, *Automatica, 22*, pp. 561-580.

[18] Willems, J.C. (1988). Models for dynamics, In: Kirchgraber, U. and H.O. Walther (Eds.), *Dynamics Reported*, Vol. 2, pp. 171-269. Wiley and Teubner, 1988.