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BOUNDS ON THE SIMULATION POWER OF EQUATION ERROR ESTIMATES

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Summary Several results concerning the properties of least squares models for linear, time-invariant, discrete-time SISO systems are derived. The model set used is the set of all autoregressive moving-average (ARMA) models of order (μ,ν) or less. This paper is concerned with the properties of least squares (equation error) models of systems which are not in the model set. In particular, we characterize the difference between the actual and model impulse responses. It is shown that although asymptotically the first $\mu+1$ impulse response elements are matched exactly (where μ is the numerator order of the model transfer function) the rest of the fit can be quite poor. Several theorems are presented which describe the properties of this fit.

1. Setting of the problem

We study the discrete time description of single input/single output, linear, time-invariant, causal systems, as outlined and defined in Fig. 1.

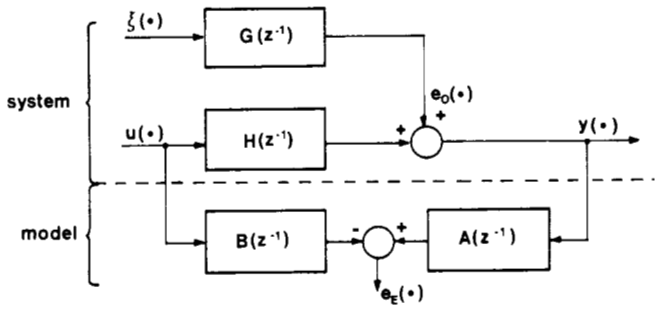

Fig. 1 Outline of the system and equation error model. A,B,G,H are polynomials in the delay operator z^{-1} of the respective order ν,μ,∞,∞. $u(.)$ and $\xi(.)$ are independent white noise sequences of variances σ_i^2 and σ_o^2.

The input signal will be assumed very rich so that it can be conceived of as white noise. Furthermore we only consider large data sets, i.e. the number of samples tends to infinity. An equation error model is obtained by minimizing the sum of squared equation errors $e_E(k)$. For a large number of samples this is asymptotically equivalent to minimizing the expectation of the squared equation error:

$$\min_{\underline{\alpha},\underline{\beta}} V_E = \min_{\underline{\alpha},\underline{\beta}} E\{e_E^2(k)\} = \delta_E \qquad (1.1)$$

$\underline{\alpha} = (\alpha_1,\alpha_2,\alpha_3 \ldots \alpha_\nu)^T$; $\underline{\beta} = (\beta_0,\beta_1,\beta_2,\ldots \beta_\mu)^T$ are the coefficients in the polynomials $A(z^{-1})$ and $B(z^{-1})$ while $\alpha_0 = 1$.

From now on $\underline{\alpha}$ and $\underline{\beta}$ represent the values of the parameters for which V_E takes the minimum δ_E. Of course $\delta_E = 0$ if and only if the system is in the model set (see Mullis and Roberts[3]) and $\sigma_0^2 = 0$. Conversely, the $\underline{\alpha}$ and $\underline{\beta}$ parameters define the equation error model with impulse response $h(.)$ and covariance

$$\hat{\psi}(k) = \sum_{i=0}^{\infty} \hat{h}(i)\,\hat{h}(i-k) \qquad (1.2)$$

Correspondingly, for the output covariance sequence we get: $\psi(k) = \sum_{i=0}^{\infty} h(i)\,h(i-k) + \sigma_0^2 \sum_{i=0}^{\infty} g(i)g(i-k)$ (1.3)

where $h(.)$ and $g(.)$ are the inverse z-transforms of H and G and where we have put $\sigma_i^2 = 1$ without loss of generality.

2. Some theorems

The proof of the following theorems can be found in Damen and Moses[1]. The first theorem is already known (see Mullis and Roberts[3]) though in our opinion it is not yet sufficiently emphasized in the literature:

Theorem 1: Given a stable linear system represented by its impulse response $\{h(i)\}_{i=0,\ldots,\infty}$. Let the input be white noise. A (μ,ν) order equation error model fits the first $\mu+1$ samples of this impulse response asymptotically exactly, i.e. $\hat{h}(i)=h(i)$ for $i=0,1\ldots\mu$.

Remark: Deeper analysis of the proof reveals that for accomplishing this, all degrees of freedom in the $\underline{\beta}$ parameters are used for this purpose so that only the freedom in the $\underline{\alpha}$-parameters is left for the fit of the tail of the impulse response.

For the next theorems we need the following lemma and corollary:

Lemma 1

- Let $\underline{\alpha}$ define a matrix Q as in (2.2) and the 'stable' polynomial $z^\nu + \alpha_1 z^{\nu-1} + \ldots + \alpha_\nu$ (roots inside unit disc).
- Let the corresponding reflection coefficients (see e.g. Söderström and Stoica[4]) be given by $\underline{k} = [k_1\ k_2\ \ldots\ k_\nu]^T$.
- Let $-\underline{k}$ (i.e. the negative refl. coef.) define the 'stable' polynomial $z^\nu + \gamma_1 z^{\nu-1} + \ldots + \gamma_\nu$ and a corresponding vector $\underline{\gamma} = [\gamma_1\ \gamma_2\ \ldots\ \gamma_\nu]^T$.

Then: $\underline{\gamma}$ is the unique solution of $Q^T \begin{bmatrix}1\\ \underline{\gamma}\end{bmatrix} = \begin{bmatrix}p_\nu\\ \underline{0}\end{bmatrix}$ (2.1)

where $p_\nu = \prod_{i=1}^{\nu}(1-k_i^2)$ and consequently $0 < p_\nu \leq 1$.

$$Q = \begin{bmatrix} 1 & & & \emptyset \\ \alpha_1 & 1 & & \\ \alpha_2 & \alpha_1 & 1 & \\ & & & \\ \alpha_{\nu-1} & & \alpha_1 & 1 \\ \alpha_\nu & \alpha_{\nu-1} & \alpha_2 & \alpha_1 \quad 1 \end{bmatrix} + \begin{bmatrix} 0 & \alpha_1 & \alpha_2 & \cdot & \alpha_{\nu-1} & \alpha_\nu \\ 0 & \alpha_2 & & & \alpha_\nu & \\ 0 & \cdot & & \cdot & & \\ \cdot & \alpha_{\nu-1}\alpha_\nu & & & \emptyset & \\ 0 & \alpha_\nu & & & & \\ 0 & 0 & & 0\ 0 & & 0 \end{bmatrix} \qquad (2.2)$$

Corollary 1: $p_\nu\psi(0) = \delta_H + \delta_E$ (2.3)

$$p_\nu\hat{\psi}(0) = \delta_H \qquad (2.4)$$

where $\delta_H = (1\ \underline{\gamma}^T)H^T H \begin{bmatrix}1\\ \underline{\alpha}\end{bmatrix}$ (2.5)

$$H = \begin{bmatrix} h(0) & & & \\ h(1) & h(0) & & \emptyset \\ h(2) & h(1) & h(0) & \\ \cdot & \cdot & \cdot & \\ h(\mu) & h(\mu-1) & \cdot & h(\mu-\nu) \end{bmatrix} \quad \text{a } [(\mu+1)\times(\nu+1)] \text{ matrix} \qquad (2.6)$$

Subtraction of the equations in corollary 1 immediately yields:

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Theorem 2: $$\psi(0) - \hat{\psi}(0) = \frac{\delta_E}{p_\nu} \qquad (2.7)$$

or in words: The power in the output equals the power of the simulated output plus the least squares error divided by p_ν.

Remarks: Since both $\delta_E \geq 0$ and $p_\nu > 0$ it is clear that $\psi(0) \geq \hat{\psi}(0)$. If the system is in the model set, this was to be expected because the real output also contains the noise influence. If on the other hand there is no noise at all ($\sigma_0^2 = 0$) and the system is not in the model set, the relation still holds, which is quite remarkable. It says that even in the absence of noise the equation error model produces a simulated output which is less in power than the actual output. This difference in power δ_E/p_ν can be quite substantial because δ_E (squared one-step-ahead prediction error) may be very small but so can $p_\nu = \prod_{i=1}^{\nu} (1-k_i^2)$. If the estimated poles are close to the unit circle, the reflection coefficients will be close to 1 and consequently p_ν will become very small too.

Similar remarks can be made when we consider the ratio of the powers obtained by division of (2.4) by (2.3):

Theorem 3: $$\frac{\hat{\psi}(0)}{\psi(0)} = \frac{\delta_H}{\delta_H + \delta_E} \leq 1 \qquad (2.8)$$

The inequality is now caused by the fact that $\delta_E > 0$ and $\delta_H \geq 0$ (follows from (2.4). The ratio of powers will be very much less than 1 as soon as $\delta_H < \delta_E$. Further study of this δ_H will be done in the next section.

3. Bounds

The quantity $\delta_H = (1\ \underline{\alpha}^T)\ H^T H \begin{bmatrix} 1 \\ \underline{\alpha} \end{bmatrix}$ depends on the first $\mu+1$ impulse response samples $h(\cdot)$ and the estimated auto-regressive parameters $\underline{\alpha}$. It has been shown (see Damen and Moses[1]) that for all $\underline{\alpha}$ corresponding to stable models this quantity δ_H is less than a value depending on $\underline{h} = (h(0),h(1),\ldots h(\mu))^T$. The more knowledge available of $\underline{h}$ (and possibly $\underline{\alpha}$) the tighter this bound will be. If we have some a priori knowledge about $\underline{h}$ such as the (maximum) energy, we can use the following bound:

$$\max_{\text{stable } \underline{\alpha}} \delta_H = \|\underline{h}\|_2^2 \min(\mu+1, \nu+1) = \delta_{H\ max} \qquad (3.1)$$

(see Damen and Moses[1]) where the norm is the standard norm on ℓ_2. Combination of (2.8) and (3.1) puts an upper limit on the ratio of output powers:

$$\frac{\hat{\psi}(0)}{\psi(0)} \leq \frac{\delta_{H\ max}}{\delta_{H\ max} + \delta_E} \qquad (3.2)$$

The interpretation of (3.2) becomes particularly interesting if we apply it for the ideal situation when there is no noise (i.e. $\sigma_0^2 = 0$) but all deviations are due to the fact that the system is not representable by a (μ,ν) ARMA model. Under that condition $\psi(0)$ represents the power of the noise-free system output. We are interested in δ_o, the expected power of the output error if we use the equation error model for a simulation, so $\delta_o = \sum_{i=0}^{\infty} (h(i)-\hat{h}(i))^2$. We can use Schwartz inequality in ℓ^2 infinite sequence Hilbert space:

$$\sqrt{\frac{\delta_0}{\psi(0)}} \geq 1 - \sqrt{\frac{\hat{\psi}(0)}{\psi(0)}} \geq 1 - \sqrt{\frac{\delta_{H\ max}/\psi(0)}{\delta_{H\ max}/\psi(0) + \delta_E/\psi(0)}} \qquad (3.3)$$

Clearly δ_E is the expected power of the one step ahead prediction error. $\delta_{H\ max}$ is the maximal energy in the first $\mu+1$ impulse response samples times $\min(\mu+1,\nu+1)$. Finally, $\psi(0)$ is the energy in the actual impulse response or the expected power of the output.

As expected, we immediately observe that the worse the relative prediction is (greater $\delta_E/\psi(0)$) the worse the relative simulation (greater $\delta_0/\psi(0)$) will be. But we also note that if the energy in the first $\mu+1$ impulse response samples become small with respect to the total impulse response energy (i.e. small $\delta_{H\ max}/\psi(0)$) a good simulation performance is impossible. Of course, in practice noise will disturb this ideal picture - but only slightly. If we allow $\sigma_0^2 \neq 0$ and mark the resulting powers of output and equation error with an asterisk ($\psi^*(0)$ and δ_E^*) and preserve $\psi(0)$ for the noise-free output power, reconsideration of (3.2) and (3.3) yields:

$$\sqrt{\frac{\delta_0}{\psi^*(0)}} \geq \sqrt{\frac{\psi(0)}{\psi^*(0)}} - \sqrt{\frac{\delta_{H\ max}/\psi^*(0)}{\delta_{H\ max}/\psi^*(0) + \delta_E^*/\psi^*(0)}} \qquad (3.4)$$

Note that $\delta_{H\ max}$ exclusively depends on the maximum energy of the first $\mu+1$ impulse response samples. The smaller this energy the worse the simulation will be.

4. Example

As a system we used the impulse response of an experiment which concerned a retina phenomenon and which is shown in Fig. 2. No output noise has been applied. A 5th order model ($\mu = \nu = 5$) has been used in an equation error estimation and we indeed observe that the first $\mu+1$ estimates of the impulse response fit perfectly. The tail of the impulse response $h(k)$ for $k > \mu+1$ is much too small (remember $\hat{\psi}(0) < \psi(0)$). We have observed this phenomenon again and again in many simulations and industrial data. One should consider that continuous processes always need some time previous to response so that a sufficiently high sampling rate will always cause $\|\underline{h}\|_2$ to be small compared to $\psi(0)$. It is certainly not true that the modelset is too tight because results of an output error estimation are also given in Fig. 2 and these give a satisfactory fit. More examples can be found in Damen, Tomita, Van den Hof[2].

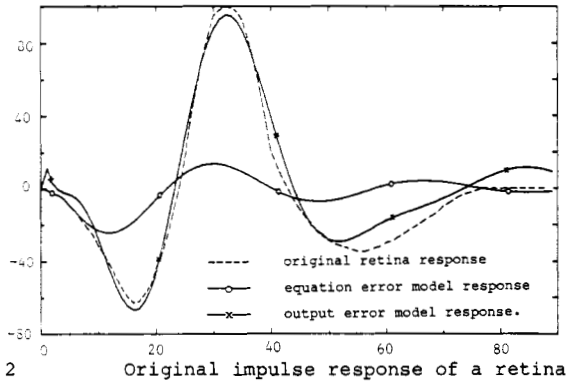

Fig. 2 Original impulse response of a retina phenomenon and the impulse responses of its 5-th order equation error and output error models

5. Conclusions

We summarise the main results:

- An equation error model gives a perfect fit (asymptotically) for the first $\mu+1$ impulse response samples if the input is sufficiently rich.
- The later impulse response samples are generally (far) too small.
- This phenomenon is increasingly pronounced when the energy of the first $\mu+1$ impulse response samples of the actual system is small compared to the energy of the complete response.
- The simulation performance of an equation error model is always worse than the prediction error.

6. References

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