A Necessary Condition for Network Identifiability With Partial Excitation and Measurement

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Abstract—**This article considers dynamic networks where vertices and edges represent manifest signals and causal dependencies among the signals, respectively. We address the problem of how to determine if the dynamics of a network can be identified when only partial vertices are measured and excited. A necessary condition for network identifiability is presented, where the analysis is performed based on identifying the dependency of a set of rational functions from excited vertices to measured ones. This condition is further characterized by using an edge-removal procedure on the associated bipartite graph. Moreover, on the basis of necessity analysis, we provide a necessary and sufficient condition for identifiability in circular networks.**

Index Terms—**Bipartite graph, data-driven modeling, directed graphs, graph theory, network systems, system identification.**

I. Introduction

The study of complex dynamic networks has flourished recently in the systems and control community, along with a wide range of applications in, e.g., robotic coordination, biochemical reactions, and smart power grids. Developing mathematical models for these interconnected systems is a fundamental step in understanding their behavior and eventually devising efficient techniques for prediction and control.

In this article, we consider identification of a class of dynamic networks consisting of vertex signals that are interconnected by causal rational transfer functions (modules) and possibly driven by external excitation signals [1]. A central concept here is *identifiability*, which essentially reflects if a unique model can be distinguished on the basis of measurement data. Different from identifiability defined in the classical system identification literature for fixed open-loop and closed-loop configurations [2], identifiability analysis in a network setting largely relies on the interconnection structure of networks [3], [4], [5], [6],

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[7], [8], [9], [10]. Based on the structural information, a set of models for a dynamic network is obtained, and then the core problem of this article is to *explore the conditions under which the network model set is identifiable*.

The identifiability problem can be formulated in the scale of a full network, see, e.g., [8], [9], [10], [11], [12], [13], and it can also focus on a single module or a subset of modules in a network [14], [15], [16], [17]. In this article, the latter is of particular interest. The majority of existing studies on the full network identifiability mainly consider two settings. For example in [8], [9], [10], all vertices are assumed to be excited by external excitation signals, and only partial vertex signals are measured, while the works, [12], [13], [18] follow a dual setting, where all internal variables are measured, and only a subset of vertices are driven by measured external excitation signals or unmeasured noises. Within these two settings, necessary and sufficient conditions for network identifiability have been derived. In [12], identifiability was interpreted as the full rank property of certain transfer matrix from external signals to measured internal signals. In contrast, a generic notion of identifiability was proposed in [8] and [9], which leads to attractive graph-theoretical conditions based on *vertex-disjoint paths* for checking network identifiability. These works have inspired [10], which extends the path-based condition to address global identifiability. Furthermore, [13] reformulates a new graphical characterisation for generic identifiability by means of *disjoint pseudotree covering*, which further leads to a scalable graph-based algorithm for allocating actuators.

While the abovementioned works require that all vertices are either measured or excited by sufficiently rich external signals, [11] provided identifiability results in the scenario where not all vertices are measured and not all vertices are excited. Sufficient conditions are discussed for the generic identifiability of a subset of modules. Yet these conditions require certain prior knowledge on network dynamics, and they are not entirely graph-based except for special networks with cycle and tree topologies. A generic local notion of network identifiability was considered in [19] that is weaker than generic identifiability. A necessary and sufficient condition for local identifiability could be obtained in terms of transfer matrix ranks, but a graph-theoretical analysis for local identifiability remained an open question. In contrast, we presented in [17] a sufficient condition for generic identifiability of a single module in a network.

In line with the network setting of [11], [17], this article studies identifiability of full dynamic networks, where only partial excitation and measurement signals are available. We develop a new necessary condition for the identifiability of general networks by recasting the identifiability problem into determining the dependency of a system of rational functions with the parametrized modules as indeterminate variables, where the functions represent individual mappings from the excitation signals to the measured vertices. The condition can be reformulated in terms of an edge-removal process for a bipartite graph, thus it only relies on the topology of networks. Furthermore, the necessity

analysis is then applied to circular networks, leading to a necessary and sufficient condition for identifiability that goes beyond the results in [11], where only a sufficient condition is given. Meanwhile, a parallel development in [24] presented alternative conditions for the identifiability of circular networks that do not rely on vertex disjoint paths.

The rest of this article is organized as follows. In Section II, we recap some basic notations used in graph theory and introduce the dynamic network model. Existing necessary conditions for network identifiability are introduced in Section II, and Section IV presents a necessary condition for the identifiability of dynamic networks and further zooms into the analysis of circular networks in Section V, where a necessary and sufficient condition for identifiability is provided. Finally, Section VI concludes the article.

Notation: Denote $\mathbb{R}$ as the set of real numbers, and $\mathbb{R}(q)$ is the rational function field over $\mathbb{R}$ with the variable q. The cardinality of a set $\mathcal{V}$ is represented by $|\mathcal{V}|$. A_{ij} denotes the (i,j)th entry of a matrix A, and more generally, $A_{\mathcal{U},\mathcal{V}}$ denotes the submatrix of A that consists of the rows and columns of A indexed by two positive integer sets $\mathcal{U}$ and $\mathcal{V}$, respectively. The normal rank of a transfer matrix $A(q)$ is denoted by $\text{Rank}(A(q))$, and $\text{Rank}(A(q)) = r$ if the rank of $A(q)$ is equal to r for almost all values of q.

II. PRELIMINARIES AND PROBLEM SETTING

A. Graph Theory

A graph $\mathcal{G}$ consists of a finite and nonempty vertex set $\mathcal{V} := \{1, 2, \ldots, L\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. A directed graph is such that each element in $\mathcal{E}$ is an ordered pair of elements of $\mathcal{V}$. If $(i,j) \in \mathcal{E}$, we say that the vertex i is an *in-neighbour* of j, and j is an *out-neighbour* of i. We use $\mathcal{N}_j^-$ and $\mathcal{N}_j^+$ to denote the sets that collect all the in-neighbours and out-neighbours of vertex j, respectively.

A graph $\mathcal{G}$ is called *simple*, if $\mathcal{G}$ does not contain self-loops (i.e., $\mathcal{E}$ does not contain any edge of the form (i,i), $\forall i \in \mathcal{V}$), and there exists only one directed edge from one vertex to each of its out-neighbours. In a simple graph, a directed *path* connecting vertices i_0 and i_n is a sequence of edges of the form (i_{k-1}, i_k), $k = 1, \ldots, n$, and every vertex appears at most once on the path. Particularly, a single vertex can also be regarded as a special path of length 0. Two directed paths are *vertex-disjoint* if they do not share any common vertex, including the start and the end vertices.

B. Dynamic Network Model

Consider a dynamic network whose topology is captured by a simple directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, 2, \ldots, L\}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Following the basic setup in [1], [11], each vertex is described by an internal variable $w_j(t) \in \mathbb{R}$, and a compact form of the overall network dynamics is

$$w(t) = G(q)w(t) + Rr(t) + v_e(t),$$
$$y(t) = Cw(t) + v_m(t) \tag{1}$$

where q^{-1} is the delay operator, and $w(t) := [w_1(t) \ w_2(t) \ \cdots \ w_L(t)]^\top$ collects all the internal signals. $G(q)$ is a hollow transfer matrix, in which the (i,j)th entry, denoted by $G_{ij}(q) \in \mathbb{R}(q)$, indicates the transfer operator from vertex j to vertex i.

Let $\mathcal{R} \subseteq \mathcal{V}$ and $\mathcal{C} \subseteq \mathcal{V}$ be the vertices that are excited and measured, respectively, and $K = |\mathcal{R}|$ and $N = |\mathcal{C}|$. The signals $r(t) \in \mathbb{R}^K$ and $y(t) \in \mathbb{R}^N$ are the external excitation and measurement signals with $R \in \mathbb{R}^{L \times K}$, $C \in \mathbb{R}^{N \times L}$ binary matrices indicating which vertices are excited or measured. Specifically, R and C consist of the columns and

rows of the identity matrix indexed by the set $\mathcal{R}$ and $\mathcal{C}$, respectively. The excitation and measurement noises are represented by $v_e(t) \in \mathbb{R}^L$, $v_m(t) \in \mathbb{R}^L$, respectively.

Assumption 1: Throughout the article, we consider the following standard assumptions for dynamic networks (see also [12], [15]).
1) The network (1) is *well-posed* and stable, i.e., $(I - G(q))^{-1}$ is proper and stable.
2) The function $G_{ji}(q)$ is nonzero if and only if $(i,j) \in \mathcal{E}$.
3) All the entries of $G(q)$ are proper and stable transfer operators.

In the context of network identification, we consider $M = (G, R, C)$ to be a network model of (1), where all the nonzero entries in G are parametrized independently. Therefore, we obtain a network model set

$$\mathcal{M} := \{M(q, \theta) = (G(q, \theta), R, C), \theta \in \Theta\}. \tag{2}$$

Denote the transfer matrix

$$T(q, \theta) := (I - G(q, \theta))^{-1} \tag{3}$$

and $T_{\mathcal{C}, \mathcal{R}}$ is the submatrix of T containing the rows and columns of T indexed by $\mathcal{C}$ and $\mathcal{R}$, respectively. The network identifiability is thereby defined as follows.

Definition 1 (Network identifiability): The network model set $\mathcal{M}$ in (2) is identifiable from the submatrix $T_{\mathcal{C}, \mathcal{R}}$ at $M_0 := M(\theta_0)$ with $\theta_0 \in \Theta$ if the implication

$$CT(q, \theta_1)R = CT(q, \theta_0)R \Rightarrow M(q, \theta_1) = M(q, \theta_0) \tag{4}$$

holds for all $\theta_1 \in \Theta$. Furthermore, the network model set $\mathcal{M}$ is identifiable from $T_{\mathcal{C}, \mathcal{R}}$ if (4) holds for all $\theta_0 \in \Theta$.

As a relevant concept, generic identifiability of the network model set $\mathcal{M}$ is defined when the implication (4) holds for almost all $\theta_0 \in \Theta$,[1] see more details in [8], [9], [15].

Remark 1: It is worth noting that while this article considers only excitation input r in (1), disturbances can also be taken into account as in [13]. Under some mild assumptions, disturbance inputs play a similar role as the excitation inputs, and thereby the results in this article can be directly generalized to the disturbance case.

In an identification setting, we can consistently identify the transfer matrix $T_{\mathcal{C}, \mathcal{R}}$ from the measured signals $y(t)$ and $r(t)$, provided that the network is fed with sufficiently exciting inputs $r(t)$. Therefore, identifiability reflects the ability to distinguish between models in the set $\mathcal{M}$ from measurement data, or more precisely, from the transfer matrix $T_{\mathcal{C}, \mathcal{R}}$ as described in Definition 1. In this sense, network identifiability essentially depends on the presence and location of external excitation signals r and the selection of measured vertex signals y.

III. EXISTING NECESSARY CONDITIONS

A preliminary necessary condition has been given in [11] as follows.
Lemma 1: Any network model set $\mathcal{M}$ is identifiable *only if* $|\mathcal{R}| \geq 1$, $|\mathcal{C}| \geq 1$, and $\mathcal{V} = \mathcal{R} \cup \mathcal{C}$.[11]

This condition means that to identify all the parametrized entries in G, each vertex in $\mathcal{G}$ must be either excited or measured.

In other studies including [9], [12], [20], necessary conditions for network identifiability have been provided in the setting of either full excitation or full measurement. Combining these works in the two settings, we immediately obtain a necessary condition for identifiability in the case of partial excitation and measurement.

Proposition 1: Consider the network model set $\mathcal{M}$ in (2) with $\mathcal{R} \subseteq \mathcal{V}$ and $\mathcal{C} \subseteq \mathcal{V}$ the excited and measured vertices. If $\mathcal{M}$ is identifiable,

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[1]"Almost all" refers to the exclusion of parameters that are in a subset of Θ with Lebesgue measure zero.

Fig. 1. Dynamic network that satisfies the necessary conditions in Lemma 1 and Proposition 1 but is not identifiable.
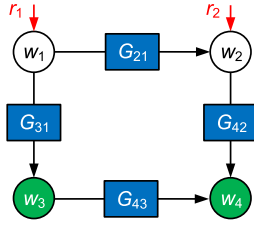

Fig. 2. Network that satisfies the necessary conditions in Lemma 1, Proposition 1 and (6) but is not identifiable.

then

$$\text{Rank}\left(T_{\mathcal{N}_i^-,\mathcal{R}}(q,\theta)\right) = |\mathcal{N}_i^-|, \tag{5a}$$

$$\text{and Rank}\left(T_{\mathcal{C},\mathcal{N}_i^+}(q,\theta)\right) = |\mathcal{N}_i^+| \tag{5b}$$

hold for each $i \in \mathcal{V}$ and for all $\theta \in \Theta$.

Proof: The necessity of (5a) can be proved following a reasoning similar to [12, Th. 2] for the full measurement case. Then, the necessity of (5b) is also validated, following from a dual analysis. ∎

Generally, the study of the necessary condition for network identifiability in the partial excitation and measurement setting is rarely addressed. The available necessary conditions can be rather loose in determining identifiability of general networks. For instance, these conditions are not sufficient for the identifiability of the four-vertex dynamic network in Fig. 1.

Example 1: In the network shown in Fig. 1, $\mathcal{R} = \{1,2\}$ and $\mathcal{C} = \{3,4\}$. The matrix T defined in (3) is computed as

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ G_{21} & 1 & 0 & 0 \\ G_{31} & 0 & 1 & 0 \\ G_{21}G_{42} + G_{31}G_{43} & G_{42} & G_{43} & 1 \end{bmatrix}.$$

It is not hard to verify that all the necessary conditions in Lemma 1 and Proposition 1 are fulfilled. However, the model set of this network $\mathcal{M}$ is not identifiable, which can simply be seen from the submatrix

$$T_{\mathcal{C},\mathcal{R}} = \begin{bmatrix} G_{31} & 0 \\ G_{21}G_{42} + G_{31}G_{43} & G_{42} \end{bmatrix}.$$

Note that identifiability of $\mathcal{M}$ essentially requires to obtain a unique solution of the four unknown modules G_{12}, G_{24}, G_{31}, and G_{43} from the entries of $T_{\mathcal{C},\mathcal{R}}$. However, $T_{\mathcal{C},\mathcal{R}}$ has a zero entry, thus it is impossible to identify four modules from it.

IV. MAIN RESULTS

In this section, we present a novel necessary condition for network identifiability and further develop a graph-based condition to check the necessary condition.

A. Necessary Condition for Network Identifiability

In this section, we derive a necessary condition for network identifiability, motivated by Example 1, that is based on the number of the nonzero transfer functions as the entries in $T_{\mathcal{C},\mathcal{R}}$. In the transfer matrix T, we have three categories of entries, namely, "0," "1," and *nonconstant elements* that are represented as functions of modules G_{ji},
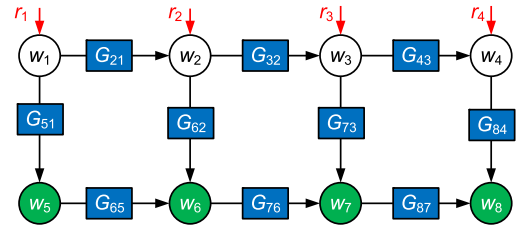
e.g., the element T_{41} in Example 1 is a function of the modules as $T_{41} = G_{21}G_{42} + G_{31}G_{43}$. For any $i \neq j$, the entry T_{ji} is nonzero if there is at least a directed path from i to j, and $T_{ji} = 0$ otherwise. Furthermore, a diagonal entry of T, $T_{ii} \neq 1$ if there exists a directed cycle that starts and ends at vertex i, and $T_{ii} = 1$ otherwise.

Identifiability essentially reflects whether we can uniquely solve all the modules in G from the entries of $T_{\mathcal{C},\mathcal{R}}$. Note that the 0 or 1 entries in T do not contain any information of the modules, and thus are not useful for identifiability. Let ξ be the number of nonconstant elements in $T_{\mathcal{C},\mathcal{R}}$. An immediate necessary condition for identifiability of the network model is

$$\xi \geq |\mathcal{E}| \tag{6}$$

where $\mathcal{E}$ is the edge set of the network. This simple condition shows why the network in Fig. 1 is not identifiable. It is found that $\xi = 3$, while there are four edges, and therefore this does not conform with (6). However, the condition in (6) is obviously not sufficient to verify the identifiability of a dynamic network in a more general setting. For instance, it is incapable to handle the network in Fig. 2. This network example will also be used as a lead-in to our new necessary condition to be developed later in this section.

Example 2: We consider a simple network in Fig. 2, where $\mathcal{R} = \{1,2,3,4\}$ and $\mathcal{C} = \{5,6,7,8\}$. To the best of our knowledge, there is currently no available condition in the literature that can determine identifiability of the model set. It is clear that this network satisfies all the available necessary conditions in Lemma 1 and Proposition 1. Furthermore, there are ten nonconstant elements in $T_{\mathcal{C},\mathcal{R}}$ that is equal to the number of unknown modules, i.e., $|\mathcal{E}| = 10$. Thus, (6) is also satisfied.

Here, we need a further analysis to determine whether this network model set is identifiable or not. Note that we can identify the submatrix of $T := (I - G)^{-1}$ from measurement data (r, y) as

$$T_{\mathcal{C},\mathcal{R}} = \begin{bmatrix} T_{51} & 0 & 0 & 0 \\ T_{61} & T_{62} & 0 & 0 \\ T_{71} & T_{72} & T_{73} & 0 \\ T_{81} & T_{82} & T_{83} & T_{84} \end{bmatrix}$$

where $T_{51} = G_{51}$, $T_{62} = G_{62}$, $T_{73} = G_{73}$, $T_{84} = G_{84}$, $T_{61} = G_{62}G_{21} + G_{65}G_{51}$, $T_{72} = G_{73}G_{32} + G_{76}G_{62}$, $T_{83} = G_{84}G_{43} + G_{87}G_{73}$, $T_{71} = G_{73}G_{32}G_{21} + G_{76}G_{62}G_{21} + G_{76}G_{65}G_{51}$, $T_{82} = G_{84}G_{43}G_{32} + G_{87}G_{73}G_{32} + G_{87}G_{76}G_{62}$, $T_{81} = G_{84}G_{43}G_{32}G_{21} + G_{87}G_{73}G_{32}G_{21} + G_{87}G_{76}G_{62}G_{21} + G_{87}G_{76}G_{65}G_{51}$.

It can be verified that

$$T_{83} = T_{73}(T_{61}T_{82} - T_{62}T_{81})(T_{61}T_{72} - T_{62}T_{71})^{-1} \tag{7}$$

which implies that the information of T_{83} is redundant in identifying the unknown modules in the network, since it can be represented by the other nonzero elements in $T_{\mathcal{C},\mathcal{R}}$. Thus we have ten equations that

are not independent, from which it is impossible to solve ten unknown modules. As a result, the network model set is not identifiable.

Motivated by Example 2, we aim to find a tighter necessary condition for network identifiability, of which the necessity is featured by the independence of a set of rational functions. To this end, we define two sets associated with a dynamic network (1). The unknown modules G_{ij} in the network are viewed as indeterminate variables, leading to a set of unknown modules

$$\mathscr{X} := \{G_{ji}(q) \mid (i,j) \in \mathcal{E}\}$$

with $\mathcal{E}$ the edge set of the network. Then, the nonconstant elements of $T_{\mathcal{C},\mathcal{R}}$ form a set of rational functions on $\mathscr{X}$, denoted by

$$\mathscr{F} = \{T_{\ell k} \mid T_{\ell k} \neq 0 \text{ and } T_{\ell k} \neq 1, \ell \in \mathcal{C}, k \in \mathcal{R}\} \qquad (8)$$

where $|\mathscr{F}| = \xi$ with ξ in (6). The rational functions in $\mathscr{F}$ are *dependent*, if there is a function in $\mathscr{F}$ that can be represented by the other functions in $\mathscr{F}$ with elementary arithmetic operations (i.e., addition, subtraction, multiplication, and division).

For instance, the ten functions in Example 2 are dependent due to (7). In order to analyse identifiability of the network, we can first remove dependent functions in $\mathscr{F}$, e.g., T_{83} in Example 2, to obtain a reduced set $\hat{\mathscr{F}}$, and then compare the cardinality of $\hat{\mathscr{F}}$ with the number of edges in the network. A key question therefore arises, i.e., how to *identify the dependency of the functions in $\mathscr{F}$*?

To address this question, we adopt the concept of *structural rank* from [21] that considers the nonzero pattern of T.

Definition 2 (Structural rank): The structural rank of T, denoted by $\mathrm{Sprank}(T)$, is the highest rank of all matrices with the same nonzero pattern as T.

To verify the dependency of the functions in $\mathscr{F}$ defined in (8), we consider $T_{\mathcal{C},\mathcal{R}}$ and check each submatrix of $T_{\mathcal{C},\mathcal{R}}$ that has full structural rank.

Lemma 2: Consider any transfer matrix T and the function set $\mathscr{F}$ in (8). The nonconstant rational functions in $\mathscr{F}$ are dependent, if there are two subsets $\bar{\mathcal{C}} \subseteq \mathcal{C}$ and $\bar{\mathcal{R}} \subseteq \mathcal{R}$ with $|\bar{\mathcal{C}}| = |\bar{\mathcal{R}}|$ such that

$$\mathrm{Rank}(T_{\bar{\mathcal{C}},\bar{\mathcal{R}}}) < \mathrm{Sprank}(T_{\bar{\mathcal{C}},\bar{\mathcal{R}}}) = |\bar{\mathcal{C}}|. \qquad (9)$$

Proof: The two equalities in (9) mean that $T_{\bar{\mathcal{C}},\bar{\mathcal{R}}}$ is a square matrix that has full structural rank. Therefore, the determinant $\det(T_{\bar{\mathcal{C}},\bar{\mathcal{R}}})$ is a function that can be written as combination of the functions in the set $\mathscr{F}$. While the inequality holds, i.e., $T_{\bar{\mathcal{C}},\bar{\mathcal{R}}}$ is not full rank, the determinant function $\det(T_{\bar{\mathcal{C}},\bar{\mathcal{R}}}) = 0$, from which a dependency of the functions as components of $\det(T_{\bar{\mathcal{C}},\bar{\mathcal{R}}})$ is obtained. ∎

We illustrate this lemma by using Example 2. Consider a 3×3 matrix $T_{\bar{\mathcal{C}},\bar{\mathcal{R}}}$ as the submatrix of $T_{\mathcal{C},\mathcal{R}}$ with $\bar{\mathcal{R}} = \{1,2,3\}$ and $\bar{\mathcal{C}} = \{6,7,8\}$. From the nonzero pattern of $T_{\bar{\mathcal{C}},\bar{\mathcal{R}}}$, we see that $\mathrm{Sprank}(T_{\bar{\mathcal{C}},\bar{\mathcal{R}}}) = 3$. However, it follows from (7) that $\det(T_{\bar{\mathcal{C}},\bar{\mathcal{R}}}) = T_{83}(T_{61}T_{72} - T_{62}T_{71}) - T_{73}(T_{61}T_{82} - T_{62}T_{81}) = 0$, i.e., $\mathrm{Rank}(T_{\bar{\mathcal{C}},\bar{\mathcal{R}}}) < 3$. Therefore, the nonconstant elements in $T_{\bar{\mathcal{C}},\bar{\mathcal{R}}}$ are dependent.

Remark 2: To identify dependent functions in $\mathscr{F}$, we need to check the rank of each square submatrix in $T_{\mathcal{C},\mathcal{R}}$, since the full rank of $T_{\mathcal{C},\mathcal{R}}$ does not guarantee the full rank of its submatrices. One example is $T_{\mathcal{C},\mathcal{R}}$ in Example 2 that has full rank, while the submatrix $T_{\bar{\mathcal{C}},\bar{\mathcal{R}}}$ is rank deficient.

To derive a tighter necessary condition for network identifiability than (6), we proceed to an iterative elimination of the entries in $T_{\mathcal{C},\mathcal{R}}$, when dependency of a nonconstant element of $T_{\mathcal{C},\mathcal{R}}$ is found. A detailed scheme is described as follows.

We use Example 2 to show how Algorithm 1 works. Initially, $\mathscr{F}$ contains 10 elements. In the first iteration, we can select $\bar{\mathcal{R}} = \mathcal{R}$ and $\bar{\mathcal{C}} = \mathcal{C}$ with $|\bar{\mathcal{C}}| = |\bar{\mathcal{R}}|$. Due to (7), $T_{\bar{\mathcal{C}},\bar{\mathcal{R}}}$ is not full rank, i.e.,

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Algorithm 1: Iterative Elimination of Dependent Functions.

1: **initialize** $\hat{T} = T_{\mathcal{C},\mathcal{R}}$, and the set $\mathscr{F}$ as defined in (8).
2: **repeat**
3:     Find $\bar{\mathcal{C}} \subseteq \mathcal{C}$ and $\bar{\mathcal{R}} \subseteq \mathcal{R}$ with $|\bar{\mathcal{C}}| = |\bar{\mathcal{R}}|$ such that

$$\mathrm{Rank}(\hat{T}_{\bar{\mathcal{C}},\bar{\mathcal{R}}}) < \mathrm{Sprank}(\hat{T}_{\bar{\mathcal{C}},\bar{\mathcal{R}}}) = |\bar{\mathcal{C}}|. \qquad (10)$$

4:     Select any pair of $i \in \bar{\mathcal{R}}$, $j \in \bar{\mathcal{C}}$ and remove the element T_{ji}
    in $\mathscr{F}$, and let the element in $\hat{T}$ corresponding to T_{ji} be zero.
5: **until** There are no $\bar{\mathcal{C}} \subseteq \mathcal{C}$ and $\bar{\mathcal{R}} \subseteq \mathcal{R}$ satisfying (10).
6: **return** The reduced set $\hat{\mathscr{F}}$ associated with $\hat{T}$.

$\mathrm{Rank}(T_{\bar{\mathcal{C}},\bar{\mathcal{R}}}) < 4 = \mathrm{Sprank}(T_{\bar{\mathcal{C}},\bar{\mathcal{R}}})$. Hence, we can remove an arbitrary element, e.g., T_{84}, yielding a reduced set $\hat{\mathscr{F}} = \mathscr{F} \setminus T_{84}$. Meanwhile, we obtain a new transfer matrix

$$\hat{T} = \begin{bmatrix} T_{51} & 0 & 0 & 0 \\ T_{61} & T_{62} & 0 & 0 \\ T_{71} & T_{72} & T_{73} & 0 \\ T_{81} & T_{82} & T_{83} & 0 \end{bmatrix}.$$

As we cannot find any more subsets $\bar{\mathcal{C}}$ and $\bar{\mathcal{R}}$ such that the resulting submatrix $\hat{T}_{\bar{\mathcal{C}},\bar{\mathcal{R}}}$ satisfies (10), then the algorithm terminates.

With the element elimination procedure in Algorithm 1, $T_{\mathcal{C},\mathcal{R}}$ is sparsified as $\hat{T}$, in which nonconstant elements form a reduced set $\hat{\mathscr{F}}$ after removing the dependent rational functions in $\mathscr{F}$. Thereby, it then yields a new necessary condition for network identifiability as follows.

Theorem 1: Consider the network model set $\mathcal{M}$ with $\mathcal{R}$ and $\mathcal{C}$ the sets of excited and measured vertices, respectively. Let $\hat{\mathscr{F}}$ be the set generated by Algorithm 1. $\mathcal{M}$ is identifiable only if $|\hat{\mathscr{F}}| \geq |\mathcal{E}|$, where $\mathcal{E}$ is the edge set of the network.

Proof: If $\mathcal{M}$ is identifiable, then it is clear that the number of independent functions in $\mathscr{F}$ should be greater than or equal to $|\mathcal{E}|$. Otherwise, there will be a smaller number of equations than the number of unknown modules as indeterminate variables, such that the system of equations becomes underdetermined and cannot yield a unique solution.

From Algorithm 1, the returned function set $\hat{\mathscr{F}}$ is generated by removing a subset of elements in $\mathscr{F}$ defined as in (8), and these removed elements are nonconstant rational functions that are dependent on the rest of functions in $\hat{\mathscr{F}}$. Then, it is necessary to have $|\hat{\mathscr{F}}| \geq |\mathcal{E}|$ if $\mathcal{M}$ is identifiable. ∎

A special case is discussed, where there is a single excited vertex or only one measured vertex. In this case, we do not need to implement the elimination procedure in Algorithm 1.

Corollary 1: Consider the network model set $\mathcal{M}$ with $\mathcal{R}$ and $\mathcal{C}$ the sets of excited and measured vertices, respectively. Let $|\mathcal{R}| = 1$ or $|\mathcal{C}| = 1$. If $\mathcal{M}$ is identifiable, then $|\mathscr{F}| \geq |\mathcal{E}|$, where $\mathscr{F}$ is the set of nonconstant elements in $T_{\mathcal{C},\mathcal{R}}$.

Proof: If $|\mathcal{R}| = 1$, $T_{\mathcal{C},\mathcal{R}}$ is a matrix with only one column, and thus $\mathrm{Rank}(T_{\bar{\mathcal{C}},\mathcal{R}}) = \mathrm{Sprank}(T_{\bar{\mathcal{C}},\mathcal{R}}) = 1$, for any $\bar{\mathcal{C}} \subseteq \mathcal{C}$. Consequently, no dependent rational functions can be found in $\mathscr{F}$, and the result is immediate from Theorem 1. For the case that $|\mathcal{C}| = 1$, we can prove the statement in a similar way. ∎

Example 3: Consider a circular network in Fig. 3 with six vertices, where $\mathcal{R} = \{1,2,3\}$ and $\mathcal{C} = \{4,5,6\}$. It was shown in [11] that such circular network is identifiable if one node is both excited and measured, which however is not the case in this example. Here, we illustrate how to use Algorithm 1 and Theorem 1 to check network identifiability.
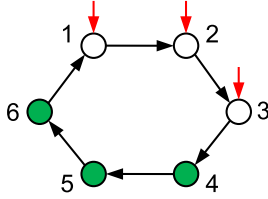
Fig. 3. Six-vertex dynamic network with $\mathcal{R} = \{1,2,3\}$ and $\mathcal{C} = \{4,5,6\}$. The network model set is not identifiable by applying Theorem 1.

Initially, we have

$$\hat{T} = T_{\mathcal{C},\mathcal{R}} = \begin{bmatrix} T_{41} & T_{42} & T_{43} \\ T_{51} & T_{52} & T_{53} \\ T_{61} & T_{62} & T_{63} \end{bmatrix}$$

which gives $\mathscr{F} = \{T_{41}, T_{42}, T_{43}, T_{51}, T_{52}, T_{53}, T_{61}, T_{62}, T_{63}\}$. Note that

$$T_{51} = (1 - \phi_c)^{-1} G_{54} G_{43} G_{32} G_{21},$$
$$T_{41} = (1 - \phi_c)^{-1} G_{43} G_{32} G_{21},$$
$$T_{52} = (1 - \phi_c)^{-1} G_{54} G_{43} G_{32},$$
$$T_{42} = (1 - \phi_c)^{-1} G_{43} G_{32}$$

where $\phi_c := G_{16} G_{65} G_{54} G_{43} G_{23} G_{21}$. The four equations satisfy $T_{51} T_{41}^{-1} = T_{52} T_{42}^{-1} = G_{54}$, meaning that $T_{\{4,5\},\{1,2\}}$ is rank deficient while $T_{\{4,5\},\{1,2\}}$ has full structural rank and thus fulfils (10). We select T_{41} and remove this element from $\mathscr{F}$ and let

$$\hat{T} = \begin{bmatrix} 0 & T_{42} & T_{43} \\ T_{51} & T_{52} & T_{53} \\ T_{61} & T_{62} & T_{63} \end{bmatrix}. \tag{11}$$

Further, we observe that $T_{52} T_{42}^{-1} = T_{53} T_{43}^{-1} = G_{54}$, such that $T_{\{4,5\},\{2,3\}}$ satisfies (10). We can further eliminate an element T_{42} in $\mathscr{F}$ and let $T_{42} = 0$. We can repeat this process and eventually obtain a reduced set $\hat{\mathscr{F}} = \{T_{43}, T_{53}, T_{61}, T_{62}, T_{63}\}$ associated with the matrix

$$\hat{T} = \begin{bmatrix} 0 & 0 & T_{43} \\ 0 & 0 & T_{53} \\ T_{61} & T_{62} & T_{63} \end{bmatrix}.$$

Since there are only five elements, which is less than the unknown modules in the original network in Fig. 3, it follows from Theorem 1 that the model set is not identifiable.

Although Algorithm 1 and Theorem 1 provide a necessary condition for network identifiability in the partial excitation and partial measurement setting, this condition is limited since it is difficult to check the relevant ranks in (10) when encountering a large-scale network. Therefore, one of the main contributions of this article will be to provide a graphical characterization for dependent functions in $\mathcal{F}$, and a graph-based condition will be given in the next section.

B. Graph-Based Condition

In this section, we provide a graph-based version for the condition in Theorem 1. The basic idea is to use graphical alternatives to characterize the rank and the structural rank appearing in (10).

First, a graph-based characterisation of a transfer matrix is revisited. Consider the transfer matrix $T_{\mathcal{C},\mathcal{R}}$ in which each nonconstant element
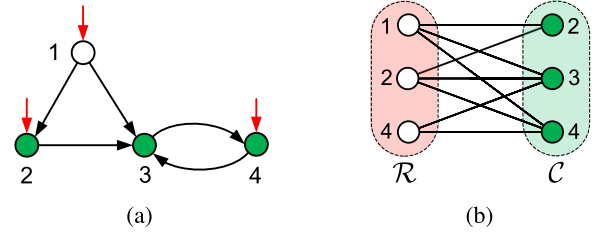

Fig. 4. (a) Four-vertex dynamic network with $\mathcal{R} = \{1,2,4\}$ and $\mathcal{C} = \{2,3,4\}$. (b) The associated bipartite graph $\mathcal{B}$ with a maximum matching $\{(1,2),(2,3),(4,4)\}$.

is a function of $\theta \in \Theta$. It then follows from [22] that

$$\max_{\theta \in \Theta} \operatorname{Rank}(T_{\mathcal{C},\mathcal{R}}(q,\theta)) = b_{\mathcal{R} \to \mathcal{C}} \tag{12}$$

where $b_{\mathcal{R} \to \mathcal{C}}$ denotes the maximal number of vertex-disjoint paths from $\mathcal{R}$ to $\mathcal{C}$. The expression on the left-hand side of (12) is referred to as the *generic rank* of the matrix $T_{\mathcal{C},\mathcal{R}}$. From (12), we further have

$$\operatorname{Rank}(T_{\mathcal{C},\mathcal{R}}(q,\theta)) \leq b_{\mathcal{R} \to \mathcal{C}} \tag{13}$$

in which the *equality generically holds*, i.e., holds for almost all $\theta \in \Theta$.

Next, the structural rank of $T_{\mathcal{C},\mathcal{R}}$ is interpreted graphically. The nonzero pattern of $T_{\mathcal{C},\mathcal{R}}$ is associated with a so-called *bipartite graph* defined by a triplet $\mathcal{B} := (\mathcal{R}, \mathcal{C}, \mathcal{E}_b)$, where $\mathcal{E}_b$ is the edge set of $\mathcal{B}$, and every edge $(i,j) \in \mathcal{E}_b$ links a vertex $i \in \mathcal{R}$ and $j \in \mathcal{C}$, if $T_{ji} \neq 0$ (see [23]). Note that we allow a vertex i to be excited and measured simultaneously, thus an edge $(i,i) \in \mathcal{E}_b$ may exist.

Definition 3: In a bipartite graph $\mathcal{B} = (\mathcal{R}, \mathcal{C}, \mathcal{E}_b)$, a **matching** between two sets $\bar{\mathcal{R}} \subseteq \mathcal{R}$ and $\bar{\mathcal{C}} \subseteq \mathcal{C}$ is a set of edges between $\bar{\mathcal{R}}$ and $\bar{\mathcal{C}}$ that do not share any common vertices. Furthermore, a **maximum(-cardinality) matching** between $\bar{\mathcal{R}}$ and $\bar{\mathcal{C}}$, denoted by $\mathcal{M}(\bar{\mathcal{R}}, \bar{\mathcal{C}})$, is a matching between $\bar{\mathcal{R}}$ and $\bar{\mathcal{C}}$ with the largest possible number of edges.

This concept then leads to the following result.

Lemma 3: Consider the matrix T in (3), and any $\bar{\mathcal{R}} \subseteq \mathcal{R}$, $\bar{\mathcal{C}} \subseteq \mathcal{C}$. It holds that $\operatorname{Sprank}(T_{\bar{\mathcal{C}}, \bar{\mathcal{R}}}) = |\mathcal{M}(\bar{\mathcal{R}}, \bar{\mathcal{C}})|$, where $\mathcal{M}(\bar{\mathcal{R}}, \bar{\mathcal{C}})$ is any maximum matching between $\bar{\mathcal{R}}, \bar{\mathcal{C}}$ in the associated bipartite graph.[23]

The following example is used to demonstrate how to determine the structural rank of a matrix through the maximum matching in its associated bipartite graph.

Example 4: A four-vertex dynamic network is shown in Fig. 4(a), in which $\mathcal{R} = \{1,2,4\}$, and $\mathcal{C} = \{2,3,4\}$. The associated bipartite graph $\mathcal{B}$ of $T_{\mathcal{C},\mathcal{R}}$ is constructed as in Fig. 4(b). A maximum matching of this bipartite graph is given as $\{(1,2),(2,3),(4,4)\}$, which has cardinality 3. Thus, $\operatorname{Sprank}(T_{\mathcal{C},\mathcal{R}}) = 3$. Note that the maximum matchings of a bipartite graph may not be unique. An alternative in this case can be $\{(1,2),(2,4),(4,3)\}$.

By means of Lemma 3 and the relation (13), the result in Lemma 2 can be reformulated on the basis of graphs.

Corollary 2: Consider the network model in (1) with the underlying graph $\mathcal{G}$. Let $\mathcal{R}$ and $\mathcal{C}$ be the sets of vertices that are excited and measured, respectively. The nonconstant rational functions in the set $\mathscr{F}$ defined in (8) are dependent, if there exist two subsets $\bar{\mathcal{C}} \subseteq \mathcal{C}$ and $\bar{\mathcal{R}} \subseteq \mathcal{R}$ with $|\bar{\mathcal{C}}| = |\bar{\mathcal{R}}|$ such that

$$b_{\bar{\mathcal{R}} \to \bar{\mathcal{C}}} < |\mathcal{M}(\bar{\mathcal{R}}, \bar{\mathcal{C}})| \tag{14}$$

where $\mathcal{M}(\bar{\mathcal{R}}, \bar{\mathcal{C}})$ is a maximum matching between $\bar{\mathcal{R}}$ and $\bar{\mathcal{C}}$ in the bipartite graph $\mathcal{B} := \{\mathcal{R}, \mathcal{C}, \mathcal{E}_b\}$, and $b_{\bar{\mathcal{R}} \to \bar{\mathcal{C}}}$ is the maximum number of vertex-disjoint paths from $\bar{\mathcal{R}}$ to $\bar{\mathcal{C}}$ in $\mathcal{G}$.

Algorithm 2: Edge-Removal in Bipartite Graph.

1: **initialize** $\hat{\mathcal{E}}_t = \emptyset$
2: **for** $k = 2 : \min\{|\mathcal{R}|, |\mathcal{C}|\}$ **do**
3:      For all $\bar{\mathcal{C}} \subseteq \mathcal{C}$ and $\bar{\mathcal{R}} \subseteq \mathcal{R}$ with $|\bar{\mathcal{C}}| = |\bar{\mathcal{R}}| = k$,
4:      **if** $b_{\bar{\mathcal{R}} \to \bar{\mathcal{C}}} < |\mathscr{M}(\bar{\mathcal{R}}, \bar{\mathcal{C}})| = k$ and $\mathcal{E}_b(\bar{\mathcal{R}}, \bar{\mathcal{C}}) \nsubseteq \hat{\mathcal{E}}_t$ **then**
5:          Remove an arbitrary edge (i, j) in the bipartite graph
             with $i \in \bar{\mathcal{R}}, j \in \bar{\mathcal{C}}$;
6:          $\hat{\mathcal{E}}_t \leftarrow \mathcal{E}_b(\bar{\mathcal{R}}, \bar{\mathcal{C}}) \cup \hat{\mathcal{E}}_t$.
7:      **end if**
        if $\hat{\mathcal{E}}_t = \mathcal{E}_b$ **then** Stop the iteration. **end if**
8: **end for**
9: **return** A simplified bipartite graph $\hat{\mathcal{B}}$.

Corollary 2 is an immediate result of Lemmas 2 and 3, and (13), and hence, its proof is omitted here. We show how to apply Corollary 2 to check if the nonconstant elements in $\mathscr{F}$ are dependent in Example 4. Consider two subsets $\bar{\mathcal{R}} = \{1, 2\}$ and $\bar{\mathcal{C}} = \{3, 4\}$. Observe that $\{(1, 3), (2, 4)\}$ is a maximum matching between the two subsets with $|\mathscr{M}(\bar{\mathcal{R}}, \bar{\mathcal{C}})| = 2$, while the maximum number of vertex disjoint paths from $\bar{\mathcal{R}}$ to $\bar{\mathcal{C}}$ is only 1. Therefore, the elements in $\mathscr{F}$ are not independent.

Now we derive a necessary condition for network identifiability based on a more comprehensive procedure that iteratively removes edges in the bipartite graph $\mathcal{B}$ of $T_{\mathcal{C}, \mathcal{R}}$. Note that each edge in $\mathcal{B}$ is associated with an nonconstant entry in $T_{\mathcal{C}, \mathcal{R}}$, or an element in $\mathscr{F}$. This simplification process corresponds to the element removal steps in Algorithm 1. Consider a network model as in (1) with $\mathcal{R}$ and $\mathcal{C}$ the excited and measured vertices, where $|\mathcal{R}| \geq 2$ and $|\mathcal{C}| \geq 2$. In the case where $|\mathcal{R}| = 1$ or $|\mathcal{C}| = 1$, we can simply apply Corollary 1. Let $\mathcal{B} := (\mathcal{R}, \mathcal{C}, \mathcal{E}_b)$ be the bipartite graph associated with $T_{\mathcal{C}, \mathcal{R}}$. A graph simplification process is performed on $\mathcal{B}$, see Algorithm 2, in which the set $\mathcal{E}_b(\bar{\mathcal{R}}, \bar{\mathcal{C}}) \subseteq \mathcal{E}_b(\mathcal{R}, \mathcal{C})$ collects all the edges between $\bar{\mathcal{R}} \subseteq \mathcal{R}$ and $\bar{\mathcal{C}} \subseteq \mathcal{C}$.

The path-based characterisation (13) for the rank of the transfer matrix $T_{\mathcal{C}, \mathcal{R}}$ does not hold for a matrix $\hat{T}$ in Algorithm 1 that is a sparsification of $T_{\mathcal{C}, \mathcal{R}}$ with certain entries in $T_{\mathcal{C}, \mathcal{R}}$ assigned to zero. Therefore, in Algorithm 2, we identify the independent functions in $\mathscr{F}$ based on the original $T_{\mathcal{C}, \mathcal{R}}$ matrix in order to utilize the path-based characterisation (13). To this end, we define an edge set $\hat{\mathcal{E}}_t$, which collects all the edges corresponding to the rational functions whose dependency have been detected. If the subsets $\bar{\mathcal{C}} \subseteq \mathcal{C}$ and $\bar{\mathcal{R}} \subseteq \mathcal{R}$ have been found to satisfy $b_{\bar{\mathcal{R}} \to \bar{\mathcal{C}}} < |\mathscr{M}(\bar{\mathcal{R}}, \bar{\mathcal{C}})|$, meaning that the matrix $T_{\bar{\mathcal{C}}, \bar{\mathcal{R}}}$ contains dependent elements, we then include all the edges between $\bar{\mathcal{R}}$ and $\bar{\mathcal{C}}$ in $\mathcal{B}$ into the set $\hat{\mathcal{E}}_t$. For the latter iteration, it is required that $\mathcal{E}_b(\bar{\mathcal{R}}, \bar{\mathcal{C}}) \nsubseteq \hat{\mathcal{E}}_t$, namely, we avoid checking subsets $\bar{\mathcal{C}}$ and $\bar{\mathcal{R}}$ if all the edges between the two subsets are in $\hat{\mathcal{E}}_t$, as the dependency of the relevant functions associated with $\hat{\mathcal{E}}_t$ has been examined. We will illustrate the procedure of Algorithm 2 in Example 5.

Algorithm 2 starts with inspecting 2×2 submatrices $T_{\bar{\mathcal{C}}, \bar{\mathcal{R}}}$ in $T_{\mathcal{C}, \mathcal{R}}$, which are the smallest submatrices for detecting dependent elements in $\mathscr{F}$. Note that we do not start with the higher-dimensional submatrices in $T_{\mathcal{C}, \mathcal{R}}$, since it may not be able to find dependent elements in some lower-dimensional submatrices. Take the network in Fig. 3 as an example, if we first check the submatrix $T_{\bar{\mathcal{C}}, \bar{\mathcal{R}}}$ with $\bar{\mathcal{R}} = \{1, 2, 3\}$ and $\bar{\mathcal{R}} = \{4, 5, 6\}$, then all the edges of the associated bipartite graph will be included in $\hat{\mathcal{E}}_t$. As a result, Algorithm 2 will stop checking dependent elements in 2×2 submatrices, which leads to a more conservative result.
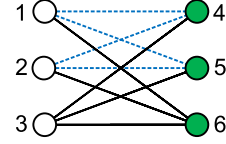

Fig. 5.   Bipartite graph associated with the network in Fig. 3. The dashed lines represent the edges can be removed.

With the simplified bipartite graph $\hat{\mathcal{B}}$ generated by Algorithm 2, we obtain a graph-based necessary condition for network identifiability as follows.

Corollary 3: Consider the network model set $\mathcal{M}$ with $\mathcal{R}$ and $\mathcal{C}$ the sets of excited and measured vertices, respectively. Let $\hat{\mathcal{E}}_b$ be the edge set of the simplified bipartite graph $\hat{\mathcal{B}}$ generated by Algorithm 2. The model set $\mathcal{M}$ is identifiable only if $|\hat{\mathcal{E}}_b| \geq |\mathcal{E}|$.

Example 5: We implement Corollary 3 to check identifiability of the model set of the network in Fig. 3. The associated bipartite graph $\mathcal{B}$ is shown in Fig. 5 . Consider the subsets $\bar{\mathcal{R}} = \{1, 2\}, \bar{\mathcal{C}} = \{4, 5\}$ with $|\mathscr{M}(\bar{\mathcal{R}}, \bar{\mathcal{C}})| = 2$. We find that the maximal number of disjoint paths from $\bar{\mathcal{R}}$ to $\bar{\mathcal{C}}$ in Fig. 3 is less than 2, implying that the four entries in $T_{\bar{\mathcal{C}}, \bar{\mathcal{R}}}$ are dependent. Therefore, we remove an edge (1,4) in $\mathcal{B}$ and meanwhile obtain the edge set $\hat{\mathcal{E}}_t = \{(1, 4), (2, 4), (1, 5), (2, 5)\}$.

We proceed to check the subsets $\bar{\mathcal{R}} = \{1, 2\}, \bar{\mathcal{C}} = \{5, 6\}$, which allows to remove an edge $(1, 5)$ in $\mathcal{B}$, and the edge set $\hat{\mathcal{E}}_t$ is enlarged as $\hat{\mathcal{E}}_t = \{(1, 4), (2, 4), (1, 5), (2, 5), (1, 6), (2, 6)\}$. This process can be continued by considering $\bar{\mathcal{R}} = \{2, 3\}, \bar{\mathcal{C}} = \{4, 5\}$, and then $\bar{\mathcal{R}} = \{2, 3\}, \bar{\mathcal{C}} = \{5, 6\}$. Accordingly, we remove two more edges (2,4) and (2,5) in $\mathcal{B}$, yielding a reduced bipartite graph with five edges, see the edges in Fig. 5 indicated by solid lines. Note that although there are more pairs satisfying (14), e.g., $\bar{\mathcal{R}} = \{1, 3\}, \bar{\mathcal{C}} = \{4, 6\}$, and $\bar{\mathcal{R}} = \{1, 2, 3\}$, $\bar{\mathcal{C}} = \{4, 5, 6\}$, we no longer impose an edge removal step, since $\hat{\mathcal{E}}_t$ now has covered all the edges in $\mathcal{B}$. Eventually, it follows from Corollary 3 that the model set of the network in Fig. 3 is not identifiable as $|\hat{\mathcal{E}}_b| = 5 < |\mathcal{E}| = 6$.

V. IDENTIFIABILITY OF CIRCULAR NETWORKS

In this section, we zoom in network identifiability of directed circular networks, which also includes isolated cycles in a larger network, namely, the ones that do not share any common vertices with other cycles. This article shows a necessary and sufficient condition for identifiability of circular graphs based on vertex disjoint paths. Alternative conditions on the identifiability of circular graphs can also be found in [24].

In the existing work [11], only sufficient conditions for identifiability of circular networks are presented. They are as follows.

1) A directed circular network is identifiable if $\mathcal{R} \cup \mathcal{C} = \mathcal{V}$ and $\mathcal{R} \cap \mathcal{C} \neq \emptyset$.

2) In the special case that the vertex number L is even and larger than 3, a circular network is identifiable if its nodes are alternately measured and excited.

However, the above two conditions can be conservative in some circumstances, see Example 6, in which the network does not satisfy the two conditions. However this network is actually identifiable. We will show this by exploring a new necessary and sufficient condition for identifiability in circular networks.

Theorem 2: A directed circular network is identifiable if and only if $\mathcal{R} \cup \mathcal{C} = \mathcal{V}$ and one of the following conditions holds.
1) $\mathcal{R} \cap \mathcal{C} \neq \emptyset$.
2) $|\mathcal{R}| \geq 2$, $|\mathcal{C}| \geq 2$, and there are at least two vertex disjoint paths from $\mathcal{R}$ to $\mathcal{C}$ in the cycle.

Proof: Consider a circular network with L vertices, which also has L unknown modules.

Necessity: It has been shown in [11] that $\mathcal{R} \cup \mathcal{C} = \mathcal{V}$ is a necessary condition for identifiability of any directed network. In the following, we prove the two conditions by contradiction. Suppose the two conditions do not hold, i.e., there exist two vertex disjoint paths from $\mathcal{R}$ to $\mathcal{C}$ on a directed cycle when $|\mathcal{R}| = K \geq 2$, $|\mathcal{C}| = N \geq 2$, and $\mathcal{R} \cap \mathcal{C} \neq \emptyset$. Let $\mathcal{R} = \{j_1, \ldots, j_K\}$ and $\mathcal{C} = \{i_1, \ldots, i_N\}$. In this case, $\hat{T} = T_{\mathcal{C},\mathcal{R}}$ is an $N \times K$ transfer matrix, where each entry is nonconstant (see Example 3). Due to $\mathcal{C} \cap \mathcal{R} = \emptyset$, it holds $N + K = L$. Furthermore, each 2×2 submatrix of $\hat{T}$, e.g.,

$$T_{\{i_1,i_2\},\{j_1,j_2\}} = \begin{bmatrix} T_{i_1,j_1} & T_{i_1,j_2} \\ T_{i_2,j_1} & T_{i_2,j_2} \end{bmatrix}$$

is not full rank. Following the element removal process in Algorithm 1, we can assign $T_{i_1,j_1} = 0$ in T. Repeating the operation for each 2×2 submatrix of T, it leads to

$$\hat{T} = \begin{bmatrix} 0 & \cdots & 0 & T_{i_1,j_K} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & T_{i_{N-1},j_K} \\ T_{i_N,j_1} & \cdots & T_{i_N,j_{K-1}} & T_{i_N,j_K} \end{bmatrix}$$

whose bipartite graph contains only $N + K - 1$ edges. It follows from Theorem 1 that the circular network is not identifiable. Therefore, the two conditions are necessary.

Sufficiency: It directly follows from [11] that the condition 1) is sufficient for network identifiability of the circular network. Now we focus on the case that there are more than one excited and measured vertices and assume there exist at least two vertex disjoint paths from $\mathcal{R}$ to $\mathcal{C}$. Without loss of generality, let vertices 1 and i be excited and vertices j, k be measured, with $1 \leq k < i \leq j \leq L$, and the two paths from vertices 1 to k and from vertices i to j are vertex disjoint. Denote $\phi_c = G_{21}G_{32} \cdots G_{L,L-1}G_{1L}$ as the transfer function for the cycle that starts and ends at vertex 1. Note that

$$T_{k1} = (1 - \phi_c)^{-1}G_{21} \cdots G_{k,k-1},$$

$$T_{j1} = (1 - \phi_c)^{-1}G_{21} \cdots G_{k,k-1}G_{k+1,k} \cdots G_{j,j-1},$$

$$T_{ki} = (1 - \phi_c)^{-1}G_{i+1,i} \cdots G_{j,j-1}G_{j+1,i} \cdots G_{k,k-1},$$

$$T_{ji} = (1 - \phi_c)^{-1}G_{i+1,i} \cdots G_{j,j-1}.$$

Therefore, we obtain

$$T_{j1}T_{k1}^{-1}T_{ki}T_{ji}^{-1} = G_{k+1,k} \cdots G_{j,j-1}G_{j+1,j} \cdots G_{k,k-1} \qquad (15)$$

which is equivalent to the cycle transfer function ϕ_c.

For any vertex u on the cycle, identifiability of $G_{u+1,u}$ can be analysed as follows. If both vertices u and $u + 1$ are measured, then we can find an excited vertex s such that $G_{u+1,u} = T_{u+1,s}T_{u,s}^{-1}$; If vertex u is excited while $u + 1$ is measured, then we have $G_{u+1,u} = (1 - \phi_c)T_{u+1,u}$, where ϕ_c is identified from (15). If $u, u + 1 \in \mathcal{R}$ or $u \in \mathcal{C}, u + 1 \in \mathcal{R}$, we can prove identifiability of $G_{u+1,u}$ in a similar way. As the vertex u can be chosen arbitrarily in the cycle, we thus can identify all the modules on the cycle. ∎

The first two conditions in Theorem 2 can be interpreted as that a vertex has to be measured (excited) if it is the only excited (measured) vertex in the cycle. The minimal number of signals, i.e., $\gamma := |\mathcal{R}| + |\mathcal{C}|$, required for identifiability of a circular network is now discussed. When the number of vertices $L \leq 3$ in the cycle, we have $\gamma = L + 1$, since at least one vertex has to be excited and measured simultaneously. When $L > 3$, then identifiability is guaranteed if $\gamma = L$ with two vertex
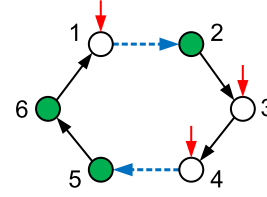

Fig. 6. Circular network with two vertex disjoint paths indicated by the dashed edges. The model set is identifiable.

disjoint paths from the excited vertices to the measured ones. Note that it is not necessary to impose the vertices to be alternately measured and excited as in [11]. In the following example, identifiability of the circular network model set cannot be determined by the conditions in [11], while it can be checked by using Theorem 2.

Example 6: Recall the circular network Fig. 3, where $\mathcal{R} \cap \mathcal{C} = \emptyset$, i.e., none of the vertices are excited and measured simultaneously. Furthermore, there are no vertex disjoint paths from $\mathcal{R}$ to $\mathcal{C}$. Therefore, this network model set is not identifiable, according to Theorem 2. Now we apply a different excitation and measurement scheme to the circular network as in Fig. 6 with $\mathcal{R} = \{1, 3, 4\}$ and $\mathcal{C} = \{2, 5, 6\}$. Observe that this network does not satisfy the identifiability conditions proposed in [11]. However, there are two vertex disjoint paths from $\mathcal{R}$ to $\mathcal{C}$, highlighted by the dashed edges. Therefore, the second condition in Theorem 2 is fulfilled, showing identifiability of the model set of this circular network.

VI. CONCLUSION

In this article, we have analyzed identifiability of a dynamic network where only partial excitation and measurement signals are available. We presented a necessary condition for identifiability of general networks, where identifiability is determined by the dependency of a set of rational functions with the parametrized modules as indeterminate variables. The merit of this result is that the necessary condition can be reinterpreted as a graph-theoretical condition only dependent on network topology. Moreover, we obtain a necessary and sufficient identifiability condition for circular networks.

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