Robust Optimization of Oil Reservoir Flooding

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***Abstract*— Over the recent years a variety of new developments have been introduced within the field of oil recovery, with the aim to maximize production of oil and gas from petroleum reservoirs. One of these new developments is the introduction of so-called "smart wells", which are equipped with control valves to actively control the oil production. The optimal operational strategy of these control valves can be found using a dynamic optimization procedure. However, due to geological uncertainty inherent to reservoir modelling, the mismatch between the reservoir model and the real reservoir may become considerable. As a result, a model-based optimal solution may seize to be the optimal, but will yield sub-optimal or even worse results. Within this work a robust optimization approach is presented that takes the possibly large impact of the mismatch between model and reservoir into account using a set of multiple reservoir realizations.**

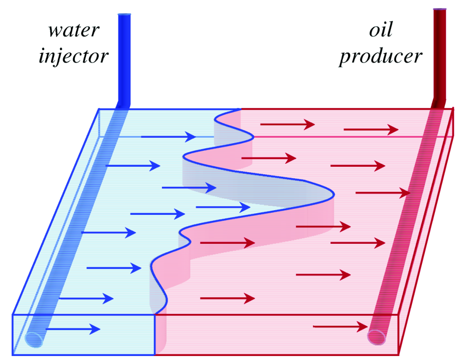

Fig. 1. Process of water flooding using a (horizontal) injection and production well. The irregular-shaped oil-water front is a result of the heterogeneous nature of the reservoir (after [2]).

I. INTRODUCTION

The field of petroleum engineering is concerned with the search for ways to extract more oil and gas from the earth's subsurface. In a world in which an increase in production of tenths of a percentage may result into a growth in profit of millions of dollars, no stone is left unturned.

A common technique in oil recovery, known as "water flooding", makes use of two types of wells: injection and production wells. The production wells are used to transport liquid and gas from the reservoir to the subsurface. The injection wells inject water into the oil reservoir with the aim to push the oil towards the production wells and keep up the pressure difference. The oil-water front progresses toward the production wells until water breaks through into the production stream. An increasing amount of water is produced, while the oil production rate diminishes, until at some time the recovery is no longer profitable and production is brought to an end.

Due to the strongly heterogeneous nature of oil reservoirs, the oil-water front does not travel uniformly towards the production wells, but is usually irregularly shaped, as depicted in Figure 1. As a result, large amounts of oil may be still trapped within the reservoir as water breakthrough occurs and production is brought to an end. Using water flooding, up to about 35% of the oil can be recovered economically.

The introduction of so-called "Smart" or "Intelligent" wells is one of the most promising developments in this field over the recent years. These types of wells allow for advanced downhole measurement and control devices, which expand the possibilities to manipulate and control fluid flow paths through the oil reservoir. The ability to manipulate (to some degree) the progression of the oil-water front provides the possibility to search for a control strategy that will result in maximization of oil recovery (in an economic sense). A straightforward approach to find such a control strategy is to use a dynamic optimization technique, based on a predictive reservoir model.

Brouwer & Jansen [2] have investigated the optimization of the oil recovery using water flooding. The resulting control strategy applied to the reservoir model at hand shows that optimal control can potentially increase recovery by several percentages by delaying water breakthrough and increasing sweep. The main drawback on the way to implementation however, remains the uncertain nature of the reservoir modelling process. The mismatch between physical oil reservoir and model may be that profound that a model-based control strategy may turn out to be sub-optimal or even worse.

Dealing with uncertainty is a topic encountered in many fields related to modelling and control. It can essentially be divided into two different strategies, which are not mutually exclusive: reducing the uncertainty itself using measurements and reducing the sensitivity to the uncertainty. Within reservoir modelling, uncertainty reduction is known as *history matching* and is a promising, but complicated research field [4], [5]. The measurement information is usually very limited and the number of uncertain parameters is considerable. This work however focuses on reducing the sensitivity to the uncertainty and no measurement information is assumed available.

The range of techniques available to reduce the effects of uncertain model parameters is quite broad [8], however the characteristics of this particular control problem limit to a large extent their applicability. Any closed-loop solution is ruled out, as no measurement information is assumed available.

Within the field of reservoir engineering, describing subsurface uncertainty is referred to as *geostatistics*. A common approach resulting from geostatistics to determine the effect of modelling uncertainty *a posteriori*, is to create a set of possible realizations of the reservoir model [3], [11]. The finite set of realizations is created in such a way that it gives a discretized approximation of the uncertainty space associated with the modelling process of the real oil reservoir. The impact of uncertainty is determined by applying the same strategy to each of the realizations and evaluating the outcomes.

A suggested approach from within chemical process engineering, to optimization problems which suffer from uncertainty and limited measurement information, is the use of a *robust optimization* technique [9], [13], [7]. In robust optimization, the optimization procedure is carried out over a set of realizations, in this way *actively* accounting for the influence of the uncertainty.

The goal of this paper is to present a robust optimization procedure based on a set of 100 realizations of a 3-dimensional petroleum reservoir, which leads to a control strategy that explicitly accounts for geological uncertainty. A gradient-based optimization procedure is used to obtain a (possibly local) optimal solution in which the gradient information is computed using a forward integration of the system equations and a backward integration of a system of adjoint equations. The 100 realizations are created using a geological training image, which reflects the range of possible geological structures which honor the statistics of the modelling uncertainty. Strebelle [10] reports on designing such a training image, however within this study a different approach is used to obtain realizations from it. The optimization objective adopted within this work is a max-mean objective, which determines the optimal control strategy over the set of realizations in an average sense.

This paper proceeds as follows. In Section II the properties and characteristics of the reservoir model are described. Section III presents the gradient-based optimization procedure adopted within this work. Section IV deals with the way the set of geological scenarios is implemented within the robust optimization routine. The results of this procedure applied to the reservoir model at hand are presented in section V.

II. RESERVOIR MODELLING

Oil reservoirs consist of porous rock in which the interconnected pores contain the oil (and possibly gas). Usually, a number of injection and production wells ($10^1 - 10^2$) are drilled. Reservoir rock has a strongly heterogeneous structure, due to for instance varying pore sizes, faults and channels, which to a great extent affects the flow paths through the reservoir. As a result, the oil-water front does not move uniformly through the reservoir, which is the main reason for early water breakthrough resulting in (prematurely) closing a production well (shut in).

Reservoir simulators use conservation of mass and momentum equations to describe the flow of oil, water or gas through the reservoir rock. Although oil consist of a large number of chemical components with varying properties, in many reservoir modelling cases the Black Oil Model is adopted for simplicity reasons [1]. This model distinguishes between three phases: water (w), oil (o) and gas (g). For further simplification, in the oil reservoirs models used within this work no gas is assumed present, hence reducing the number of phases to two.

The mass balance is expressed as follows:

$$\nabla(\rho_i u_i) + \frac{\partial}{\partial t}(\phi \rho_i S_i) = 0, \quad i = o, w, \tag{1}$$

where t is time, ∇ the spatial derivative operator, ϕ is the porosity (volume fraction of void space), ρ_i is the density of the phase i, u_i the superficial velocity and S_i the saturation, defined as the proportion of the pore space occupied by phase i.

Conservation of momentum is governed by the Navier-Stokes equations, but is normally simplified for low velocity flow through porous materials, to be described by the semi-empirical Darcy's equation as follows (discarding gravity):

$$u_i = -k \frac{k_{ri}}{\mu_i} \nabla p_i, \quad i = o, w, \tag{2}$$

where p_i is the pressure of phase i, k is the absolute permeability, k_{ri} is the relative permeability and μ_i is the viscosity of phase i. The permeability k is a measure of the resistance a fluid (or gas) experiences flowing through the porous medium. The relative permeability k_{ri} relates to the additional resistance phase i experiences when other phases are present, due to differences in viscosity. As a result, it is a strong non-linear function of the saturation S_i.

Substituting (2) into (1) results into 2 flow equations with 4 unknowns, p_o, p_w, S_o and S_w. Two additional equations are required to complete the system description. The first is the closure equation requiring that the sum of phase saturations must equal 1:

$$S_o + S_w = 1. \tag{3}$$

Second, the relation between the individual phase pressures is given by the capillary pressure equation:

$$p_{cow} = p_o - p_w = f_{cow}(S_w). \tag{4}$$

Common practice in reservoir simulation is to substitute (3) and (4) into the flow equations, by taking the oil pressure p_o and water saturation S_w as primary state variables:

$$\nabla(\lambda_o \nabla p_o) = \frac{\partial}{\partial t}(\phi \rho_o \cdot [1 - S_w]), \tag{5}$$

$$\nabla\left(\lambda_w \nabla p_o - \lambda_w \frac{\partial p_{cow}}{\partial S_w} \nabla S_w\right) = \frac{\partial}{\partial t}(\phi \rho_w S_w), \tag{6}$$

where $\lambda_o = \frac{k \cdot k_{ro} \cdot \rho_o}{\mu_o}$ and $\lambda_w = \frac{k \cdot k_{rw} \cdot \rho_w}{\mu_w}$ are the oil and water mobilities. Flow equations (5) and (6) are defined over the entire volume of the reservoir. It is assumed that there is no flow across the boundaries of the reservoir geometry over which (5)-(6) is defined (Neumann boundary conditions).

Due to the complex nature of oil reservoirs, (5)-(6) generally cannot be solved analytically, hence they are evaluated numerically. To this purpose the equations are discretized in space and time, resulting in the following state space form:

$$V(x_k) \cdot x_{k+1} = T(x_k) \cdot x_k, \tag{7}$$

where k is the time index and x is the state vector containing the oil pressures (p_o) and water saturations (S_w) in all grid blocks.

The discretization in space leads to a system built up of a finite number of blocks, referred to as "grid blocks". The geological properties inside each grid block are assumed constant. The strongly heterogeneous nature of the reservoir can be characterized by assigning different property values to each of the grid blocks.

Usually a very large number of grid-blocks is required ($10^2 - 10^6$) to adequately describe the fluid dynamics of a real petroleum reservoir. As each grid-block relates to two state variables (or three, depending on whether gas is present), the models used in reservoir simulation are usually very large, which results in very long simulation times.

Almost all dynamic behavior in a petroleum reservoir originates from the wells activities. A well can be introduced to the discretized reservoir model description (7) by adding a source term to each grid block at which a well is located. The model description is extended by adding a source vector q comprising of zeros and source terms to (7):

$$V(x_k) \cdot x_{k+1} = T(x_k) \cdot x_k + q_k, \quad x(0) = x_0, \tag{8}$$

where x_o is a vector containing the initial conditions, which are assumed to be known.

The source terms are usually represented by a so-called well model, which relates the source term to the pressure difference between the well and grid block pressure:

$$q_k^j = \alpha_k^j \cdot w^j \cdot (p_{bhp,k}^j - p_k^j), \tag{9}$$

where $p_{bhp,k}$ is the well's bottom hole pressure, j the index of the grid block containing the well and p_k^j the grid block pressure in which the well is located. The term w is a well constant which contains the well's geometric factors and the rock and fluid properties of the reservoir directly around the well. The term α_k represents the control valve setting at time index k and is simply a multiplication factor ranging from 0 to 1.

The most realistic way to control the flow paths trough the reservoir is to manipulate the control valve settings α_k. However, the adjoint-based optimization procedure used within this work does not support the use of α_k as input variables. At this point it only allows for a direct manipulation of the source terms in q_k and therefore these are chosen as the manipulated control variables.

To summarize, reservoir models exhibit the following characteristics:

- The reservoir models consist of a large number of state variables, which result in long simulation times.
- Reservoirs contain a number of injection and production wells, which (possibly using a well model) serve as multiple inputs to the reservoir model. Also, in many cases additional measurements are performed within each well, in which case the reservoir models are MIMO.
- The models are strongly non-linear, mainly due to the relative permeabilities k_{ri}.

The reservoir simulations used within this study are performed using the reservoir simulation software package MoReS, which has been developed by Shell.

III. OPTIMIZATION ALGORITHM

As mentioned in Section II, the source terms q_k in (8) serve as input variables to the reservoir model, by which the fluid flow paths can be manipulated. The first step towards finding the optimal injection and production flow rates is determining a quantitative performance measure over a fixed time horizon. Within this work, a performance measure (J) of the following kind is used [2]:

$$J = \sum_{k=1}^{N} \Delta t_k \left[r_o \cdot q_{o,k} - r_w \cdot q_{w,k} - r_i \cdot q_{i,k} \right], \tag{10}$$

where r_o is the oil revenue $[\frac{\$}{m^3}]$, r_w the water production costs $[\frac{\$}{m^3}]$ and r_i the water injection costs $[\frac{\$}{m^3}]$, which are all assumed constant. The term N represents the total number of time steps k of a fixed time span and Δt_k the time interval of time step k in $[day]$. The terms $q_{o,k}$ represents the total flow rate of produced oil $[\frac{m^3}{day}]$, $q_{w,k}$ the total flow rate of produced water $[\frac{m^3}{day}]$ and $q_{i,k}$ the total flow rate of injected water $[\frac{m^3}{day}]$, at time step k. They are defined as follows:

$$q_{o,k} = \sum_{j \in \mathcal{N}_p} \left(1 - f_{w,k}^j\right) \cdot q_k^j, \tag{11}$$

$$q_{w,k} = \sum_{j \in \mathcal{N}_p} f_{w,k}^j \cdot q_k^j, \tag{12}$$

$$q_{i,k} = \sum_{j \in \mathcal{N}_i} q_k^j, \tag{13}$$

where $f_{w,k}^j$ is the water fraction at time step k and is defined as $\frac{\lambda_w}{\lambda_w + \lambda_o}$. $\mathcal{N}_p$ represents the set of grid block indices in which a production well is located and $\mathcal{N}_i$ the set of grid block indices in which a injection well is located.

The optimization problem involves finding the optimal injection and production flow rates q that maximize the performance measure (10), while honoring the dynamic model description (7). The injection and production rates are subject to inequality constraints as they are bounded by a minimum and maximum rate. Secondly, the reservoir pressure should remain constant, as an increasing reservoir pressure may lead to undesirable ruptures in the reservoir and a decrease of the reservoir pressure below the so-called bubble-point pressure leads to the formation of gas within the oil, which complicates production. To accomplish this, an additional equality constraint is implemented, stating that the total injection rate must equal the total production rate at each time step. This results in the following mathematical formulation:

$$\max_q J(q) = \max_q \sum_{k=1}^{N} L(x_k, q_k), \tag{14}$$

$$s.t. \quad x_{k+1} = F(x_k, q_k), \quad x(0) = x_0, \tag{15}$$

$$q_{min} \leq q_k \leq q_{max}, \tag{16}$$

$$q_{o,k} + q_{w,k} = q_{i,k}, \tag{17}$$

where F represents the system equations as described in (8), which are implemented in MoReS. L represents the integral part of the performance measure J and x_0 is a vector containing the initial conditions.

Various approaches to dynamic optimization problems exist, from which four have been considered to be applied to the optimization of the oil recovery: a gradient-based method, a shooting method, a simultaneous method and dynamic programming. The latter two are not able to deal with large-scale systems and are therefore not suitable to handle complex reservoir models. Both a shooting method as a gradient-based method can handle large-scale models. However, a shooting method may experience stability problems in solving the required system of adjoint equations. For this reason, a gradient-based optimization procedure is implemented within this work.

The gradients of the performance measure J towards the rates q, to be used within the optimization procedure, are obtained using a system of adjoint equations λ. In order to use the adjoint variables to obtain the gradients, the Hamiltonian function H needs to be determined. The Hamiltonian function at time step k, in which the equality (17) and inequality constraint (16) are discarded, is defined as follows:

$$H_k = L(x_k, q_k) + \lambda_k^T \cdot F(x_k, q_k), \tag{18}$$

where λ is a vector containing the adjoint variables and are obtained by integrating the adjoint equations backward in time after integrating the system equations forward in time:

$$\lambda_{k+1} = -\left.\frac{\partial L}{\partial x}\right|_k - \left.\frac{\partial F}{\partial x}\right|_k^T \cdot \lambda_k, \quad \lambda_N = 0. \tag{19}$$

The state and adjoint variables are subsequently used to obtain the gradients of H towards the injection and production rates q

$$\left.\frac{\partial H}{\partial q}\right|_k = \left.\frac{\partial L}{\partial q}\right|_k + \lambda_k^T \left.\frac{\partial F}{\partial q}\right|_k. \tag{20}$$

Subsequently, the gradients $\left.\frac{\partial H}{\partial q}\right|_k$ can easily be reformulated into $\left.\frac{\partial J}{\partial q}\right|_k$:

$$\left.\frac{\partial J}{\partial q}\right|_k = \left.\frac{\partial H}{\partial q}\right|_k \cdot \Delta t_k. \tag{21}$$

The gradients $\left.\frac{\partial J}{\partial q}\right|_k$ are used in a Steepest Descent algorithm to iteratively converge to the optimal input trajectory:

$$q_k^{n+1} = q_k^n + \tau \cdot \left.\frac{\partial J}{\partial q}\right|_k, \tag{22}$$

where τ is the step size of the algorithm. Within this work, a *fixed* step size is used. A line search to find the direction of the greatest descent will speed up convergence, however as this study is not aimed at improving on convergence speed, a fixed τ is used for simplicity reasons.

Using (22), a situation may occur in which the new flow rates q_k^{n+1} do not obey (17) and (16), as they were discarded in the Hamiltonian function (18). In order to ensure that they comply with the constraints, a feasible search direction d_k of the gradient vector $\left.\frac{\partial J}{\partial q}\right|_k$ needs to be determined. As both the equality and inequality constraints are linear, we can simply apply the *gradient projection method* as described in Luenberger (1984), to determine d_k.

Using d_k, the Steepest Descent algorithm thus becomes:

$$q_k^{n+1} = q_k^n + \tau \cdot d_k \tag{23}$$

Determining the feasible search direction d_k however does not guarantee that q_k^{n+1} is feasible, given the fixed step size τ. It merely states that a certain $\tau > 0$ *exists* for which it is feasible. For this reason, after q_k^{n+1} is determined, it is subsequently checked for its feasibility. If not the case, τ is scaled down until a feasible q_k^{n+1} is reached.

IV. ROBUST OPTIMIZATION

The uncertainty in reservoir modelling results from the limited information on the usually complex geological structure of the real petroleum reservoir. The limited information, obtained from seismic measurements and bottom-hole core samples, allows for a broad range of possible geological structures. In many cases, it is left up to geologists to lift out the structures which most likely resemble the true reservoir, usually based on their experience with similar cases.

The resulting set of most likely geological structures is referred to as geological realizations. The set of realizations is used to perform *a posteriori* estimations on the expected oil recovery. Robust optimization offers a way to actively incorporate the set of realizations within the optimization procedure.

The field of robust optimization covers various ways in which a set of realizations may be used to account for the impact of uncertainty within the optimization procedure [9], [13], [7]. These different approaches are represented by so-called robust optimization objectives. The adopted robust optimization objective within this work is a max-mean objective. It is selected because reservoir engineers in practice often take decisions based on the average value.

The idea behind the max-mean objective is to maximize the average outcomes of the performance measure (10) related to each of the realizations. Basically, this comes down to finding an optimal control input q^*, which maximizes a new (robust) performance measure.

$$\max_q \bar{J}(q) = \max_q \left(\frac{1}{N_r} \cdot \sum_{r=1}^{N_r} J_r(q) \right), \tag{24}$$

where N_r is the total number of realizations within the set.

Calculating the "average" performance measure $\bar{J}$ involves a linear operation, hence the gradients of (24) are calculated using the gradients of each realization:

$$\left.\frac{\partial \bar{J}}{\partial q}\right|_k = \frac{1}{N_r} \sum_{r=1}^{N_r} \left.\frac{\partial J_r}{\partial q}\right|_k . \tag{25}$$

Equation (25) shows that $\left.\frac{\partial \bar{J}}{\partial q}\right|_k$ is obtained by calculating the gradients of each realization in a sequential manner. It has the advantage that the dimensions of the individual dynamic models remain the same, which is usually bounded by the computational capacity of the computer. However, it does lead to an extended simulation time by a factor N_r.

V. EXAMPLE

The robust optimization procedure is implemented on a 3-dimensional oil reservoir model over a time period of 10 years. The reservoir model consists of 18553 grid blocks, as depicted in Figure 2. and has dimensions of $480 \times 480 \times 28$ meter. The reservoir model contains eight injection wells and four production wells, placed in such a way that the distance from each injection well to its closest production well remains approximately the same. The minimum rate for each well is chosen equal to 0.02 $[\frac{m^3}{day}]$, as setting the rate equal to 0 $[\frac{m^3}{day}]$ presents numerical problems when solving the system of adjoint equation. The maximum rate for each well is fixed at a rate of 160 $[\frac{m^3}{day}]$.

The geological structure, shown in Figure 2., involves a network of meandering channels in which the fluids flows experience less resistance, in other words higher permeability. It is assumed that, based on seismic measurements, the main direction of the channels is known. However, exactly *how* the channels meander through the reservoir is assumed unknown and the main contributor to the geological uncertainty.

Based on the available information a set of 100 realizations is created, representing this type geologic uncertainty.

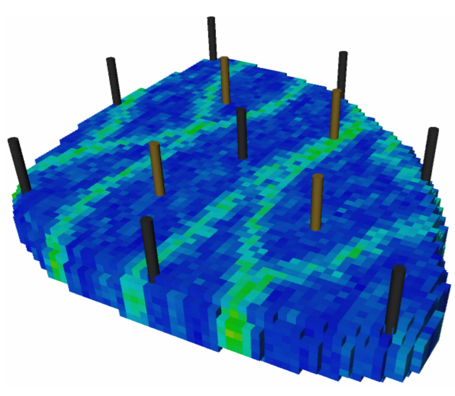

Fig. 2. Reservoir model containing 8 injection and 4 production wells, showing a meandering network of channels.

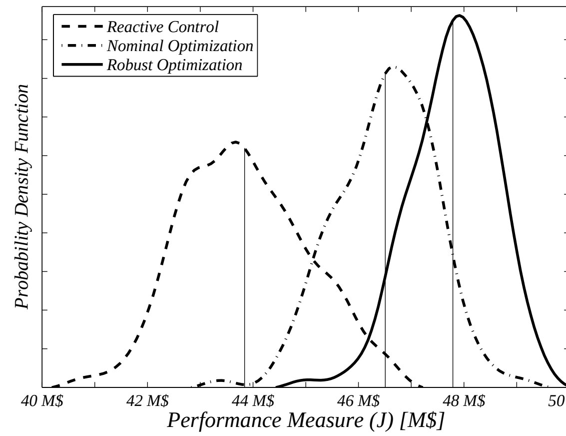

Fig. 3. Estimated pdf's resulting from alternative control strategies for oil recovery

Each realization has the same size, spatial characteristics and well locations, but displays an alternative channeling configuration. It is assumed that the realizations have an equal probability of being equal (or close) to the actual reservoir.

The robust optimization procedure is performed using (10), with $r_o = 126\,[\frac{\$}{m^3}]$, $r_w = 19\,[\frac{\$}{m^3}]$ and $r_i = 6\,[\frac{\$}{m^3}]$. The performance of the robust optimization procedure is determined by applying the acquired control strategy to each of the realizations and determining the resulting J, using (10). The 100 J's are subsequently used to determine $\bar{J}$ and are also used to estimate a probability density function (pdf), using a normal kernel smoothing function with a bandwidth of 3×10^5[12].

In order to evaluate the performance of the robust optimization procedure, its results are compared to two alternative approaches: a *nominal* optimization approach and a *reactive control* approach. The nominal optimization is based on a single realization, in this case realization number 1. Using the reactive approach, each production well is simply closed (shut in) if production is no longer profitable. With $r_o = 126\,[\frac{\$}{m^3}]$ and $r_w = 19\,[\frac{\$}{m^3}]$, this profitability threshold corresponds to a maximum water cut of 87%. The injection flow rates and production flow rates are fixed at 24 and 48 $\frac{m^3}{day}$ respectively. However, when a production well is shut in, the injection rate of each injection well is proportionally scaled down in order to meet (17). The two alternative control strategies, applied to the set of realizations, lead to 100 J's each, form which two pdf's are estimated.

The three estimated pdf's from the reactive, nominal optimization and robust optimization approach are depicted in Figure 3. The following table shows the minimum, maximum and mean J of each of the control strategies.

	reactive	*nominal*	*max-mean*
minimal J	40,7 M$	43,4 M$	45,0 M$
maximal J	46,5 M$	49,1 M$	49,4 M$
average J ($\bar{J}$)	43,8 M$	46,5 M$	47,8 M$

Table 1. Minimum, maximum and mean J of each control strategy.

Figure 3. shows that the robust performance of the nominal optimization approach, based on the first realization, is only slightly better than the reactive control strategy. Although it leads to a major improvement of J of the first realization, it's results on the remaining 99 show no real overall improvement.

Secondly, Figure 3. and Table 1. show that the robust optimization method results in a higher $\bar{J}$ compared to the other two methods. As the objective to the robust optimization procedure is the maximization of $\bar{J}$, this is not unexpected and indicates that the procedure is successful.

Finally, although no claims were made on the range of possible J's, the estimated pdf of the robust optimization approach has a smaller distribution compared to the other two methods. This variance reduction is obviously an advantageous quality, as it provides more certainty within the decision process to whether or not exploit a particular oil reservoir.

VI. CONCLUSION

A robust optimization technique is an attractive approach to oil recovery optimization problems, as it creates a bridge between two research fields within reservoir engineering: dealing with geological uncertainty (geostatistics) and maximizing oil recovery. The results following from the motivating example point out that a robust optimization procedure is able to improve the average (expected) oil recovery revenues significantly, on which decisions within reservoir engineering are often based.

Within this work, injection and production flow rates are used to manipulate the progression of the oil-water front in the reservoir. In reality however it is unlikely that the wells can be completely operated on flow rates. Besides this, the equality constraint resulting from the use of flow rates limits the search space of the optimization algorithm. Incorporating a well model into the system of adjoint equations allows for the optimization to be carried out over the control valve settings. Using these valve settings, the use of an additional equality constraint is no longer necessary and is closer to reality.

No measurement information is assumed available, within this study. Additional measurement information can however be used to reduce the geological uncertainty associated within the reservoir model. In future research on robust optimization of reservoir flooding, a more integrated approach is advised, in which measurements can be used to narrow the set of realizations or estimate the probability of each realization.

NOMENCLATURE

p	pressure	[Pa]
S	saturation	[-]
ϕ	porosity	[-]
k	permeability	[m^2]
k_r	relative permeability	[-]
λ	mobility	[-]
μ	viscosity	[Pa s]
ρ	density	[kg/m^3]
q	flow rate	[m^3/day]
α	valve setting	[-]
w	well constant	[m/Pa s]
p_{bhp}	bottom hole pressure	[Pa]
∇	spatial derivative operator	[1/m]
$_o$	oil	[-]
$_w$	water	[-]
$_i$	injection	[-]
$_p$	production	[-]

J	performance measure
$\bar{J}$	average performance measure
q	control vector (flow rates)
$q_{\min}, q_{\max}$	minimum/maximum flow rate
x	state vector
x_0	initial condition
H	Hamiltonian
λ	adjoint vector
$_k$	time index
N	number of time steps
N_r	number realizations
τ	step size SD algorithm
d	feasible gradient vector

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