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On partial realization and interpolation of models from orthogonal basis function expansions*

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Abstract

In this paper we address the problem of computing a minimal state-space realization from partial knowledge of an expansion in terms of generalized rational orthogonal basis functions. The basis functions considered are generated by stable all-pass filters. It is shown how a minimal state-space realization can be found on the basis of complete knowledge of the expansion coefficients. Subsequently an algorithm is given that solves the partial realization problem, meaning that it computes a minimal realization on the basis of a finite number of expansion coefficients. The analysis also results in compact expressions for computing the Hambo transform underlying this basis expansion as well as its inverse. Finally it is shown how these realization problems are related to the interpolation problem of finding a rational model of minimal degree that interpolates to the derivatives of a transfer function in a given set of points outside the unit disc.

Keywords: orthogonal basis functions; partial realization; Hambo transform; interpolation

1 Introduction

The description of linear time-invariant systems in terms of basis function expansions has a long history in the field of system modeling and system identification. The most familiar example are the „standard" basis functions z^{-k}, with $k \in \mathbb{N}$, that form a basis for the space H_2 of strictly proper, stable discrete-time systems. The associated expansion coefficients, the so-called Markov parameters, play a fundamental role in system theory, e.g. in the context of the realization of minimal state-space models, or in FIR (finite impulse response) modeling in system identification and filter design. A number of other basis function families consisting of real-rational transfer functions have been applied for the purpose of system approximation and system analysis, such as the Laguerre and Kautz bases [1]. The unifying property of these constructions is that the basis functions are generated by repeated multiplication of a first (standard, Laguerre) or second (Kautz) order function with an all-pass function that has the same pole(s).

During the past decade the use of this type of basis functions has received a considerable amount of attention, specifically in the context of system identification [2, 3]. Several generalizations of the Laguerre/Kautz construction have been proposed. Heuberger [4] has considered a generalized construction in which a complete orthonormal basis of H_2 is constructed by repeated multiplication with a single all-pass transfer function of arbitrary order. Ninness has focused attention on the construction in which the basis functions are generated by multiplication with first and second order all-pass transfer functions that have different poles at each multiplication step [5].

The principal motivation for using alternative bases lies with the differences in efficiency of different bases to compactly represent a certain class of signals and/or systems. Representation of signals/systems in alternative coordinates can significantly improve the efficiency of the computations involved. The constructions of Heuberger and Ninness can be seen in this light as they were introduced in order to increase the flexibility of the previous constructions by allowing a more diverse selection of poles in the rational basis functions. If a proper choice of poles is made, approaching the 'real' poles of the underlying system, then the number of parameters to be estimated can be kept small. In this way negative effects of the bias/variance tradeoff can, to a certain extent, be limited [6].

In this paper we will restrict attention to the case of scalar transfer functions. Also it is assumed throughout that the basis functions are real rational implying that the associated coefficients are real valued. We will consider the use of the Heuberger basis construction and discuss the problem of how we can, once a finite number of expansion coefficients are known, realize a minimal state-space model of which the corresponding expansion coefficients match the given ones. Apart from being an interesting system theoretic problem in its own right, a solution will be of practical relevance when one has estimated a finite number of expan-

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sion coefficients from input/output data and would like to find a compact state-space model that is in accordance with this information.

The Heuberger construction begins with building a stable all-pass transfer function $G_b(z)$ with minimal, balanced state-space realization (A_b, B_b, C_b, D_b), where A_b is of dimension $n_b \times n_b$. As a result of balancing the state trajectories ϕ_i of this all-pass function to a unit pulse input will form a set of n_b mutually orthonormal elements of the signal space $\ell_2[1,\infty)$. This implies that the functions $\phi_i(z) = e_i^T(zI - A_b)^{-1}B_b$,with e_i the i-th Euclidean basis vector of $\mathbb{E}^{n_b}$, form an orthonormal set in the H_2. The other basis elements are obtained from this initial set by repeated multiplication of the ϕ_i by the all-pass filter $G_b(z)$. Orthogonality of this construction can most easily be verified by residue calculus. It follows that

$$\frac{1}{2\pi i}\oint \phi_i(z)G_b^k(z)\phi_j(1/z)G_b^l(1/z)\frac{dz}{z} = \delta_{i,j}\delta_{k,l}$$

for $i,j = 1..n_b$ and $k,l \in \mathbb{N}$. Completeness of the basis is guaranteed when $G_b(z)$ is asymptotically stable as was shown e.g. in [4].

In our notation we will often group the basis functions $\phi_i(z)G_b^{k-1}(z)$ in batches of n_b functions by stacking them in the vectors $V_k(z)$ indexed according to

$$V_k(z) = \left(\phi_1(z)\dots\phi_{n_b}(z)\right)^T G_b^{k-1}(z).$$

This grouping will facilitate our later analysis as it takes full benefit of the structure which is present in the basis construction: the basis elements in each group share the property that they have been obtained by the same number of multiplications with G_b. The basis function expansion of an arbitrary strictly-proper, stable transfer function $G(z)$ in terms of this basis is written as

$$G(z) = \sum_{k=1}^{\infty}\sum_{i=1}^{n_b} l_k^i\phi_i(z)G_b^{k-1}(z) = \sum_{k=1}^{\infty} L_k^T V_k(z)$$

where the l_k^i are the basis function coefficients which are grouped into coefficient vectors L_k and the $V_k(z)$ are the associated basis function vectors.

The problem considered in this paper is the following. Given the expansion coefficient vectors L_k for $k = 1..N$, compute a minimal state-space realization of the underlying system such that its expansion coefficient vectors match the given ones. Minimal in this context means that it should have the lowest possible McMillan degree. This problem is closely related to the classical problem of partial realization from Markov parameters which was already solved by Tether in 1970 [7]. In the classical setting the minimal realization is obtained by applying the Ho-Kalman algorithm to the finite Hankel matrix associated with the Hankel operator representation of the system. In the approach presented here, which is an extension of the approach suggested by Szabó and Bokor [8], we make use of the fact that we can reduce our realization problem to a classical realization problem by exploiting properties of the basis transformation involved.

As is well-known, the classical partial realization problem is equivalent to an interpolation problem in which a rational-transfer function of minimal degree order is computed that interpolates to $\frac{d^kG}{dz^k}(z)$ for $k = 1..N$ evaluated at infinity. As we will show the generalized algorithm presented here can be interpreted as providing a way to compute a rational transfer function of finite order interpolating to $\frac{d^kG}{dz^k}(z)$ in a set of arbitrarily assigned, but distinct points outside the unit disk.

In the next sections we start with explaining the basic ideas behind the approach by which the generalized partial realization problem can be solved. This analysis results in an algorithm for the solution of the partial realization problem which is presented in section 3. In section 4 we will discuss how the partial realization problem is connected to an equivalent interpolation problem. We will finish with some conclusions.

2 Realization from Hankel operators

The solution of the classical realization problem is based upon the representation of the underlying system in Hankel operator form. In this form a linear time-invariant system is viewed as a mapping from past input signals $u \in \ell_2(-\infty,0]$, to the resulting future outputs $y \in \ell_2[1,\infty)$. The signals u and y are represented in terms of their expansion coefficients with respect to the standard basis for the $\ell_2(-\infty,\infty)$ space. The Hankel operator in that case takes on the form of an infinite Hankel matrix $\mathbf{H}$ containing as its elements the Markov parameters g_k, $k \in \mathbb{N}$ as given by $G(z) = \sum_{k=1}^{\infty} g_k z^{-k}$. In abstract notation we can write this as $\mathbf{y} = \mathbf{Hu}$, in which $\mathbf{y}$ and $\mathbf{u}$ are infinite sequences.

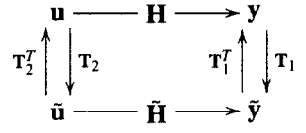
The realization of a state-space representation via the Ho-Kalman algorithm [9] makes use of a factorization of the infinite Hankel matrix into a full rank decomposition $\mathbf{H} = \Gamma\Delta$ which reflects the division of the input/output map into a map from input to states via the infinitely extended controllability matrix Δ and a map from states to output via the extended observability matrix Γ. The B and C matrices of the minimal realization are directly obtained as the first column, respectively row of Δ and Γ. To obtain the associated A matrix one makes use of the fact that the Hankel operator $\bar{\mathbf{H}}$ pertaining to the shifted system $zG(z)$ is related to $\mathbf{H}$ according to $\bar{\mathbf{H}} = \Gamma A\Delta$. The A matrix is then obtained as $A = \Gamma^+\bar{\mathbf{H}}\Delta^+$, where $^+$ denotes pseudo-inverse.

The complete generalized realization problem, i.e. concerning the situation in which one has knowledge of all expansion coefficients L_k in the generalized basis up to infinity, was solved by Szabó and Bokor in [8]. The approach is

based on the representation of the Hankel operator as a matrix $\tilde{\mathbf{H}}$ that operates on signals that are represented as coefficient expansions in a generalized basis. Use is made of the fact that the representations of signals in different orthonormal bases are related through unitary transformations. Denoting the infinite series of expansion coefficients of y and u in the generalized basis by $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{u}}$ we can write this unitary relation as

$$\tilde{\mathbf{y}} = \mathbf{T}_1\mathbf{y},\ \tilde{\mathbf{u}} = \mathbf{T}_2\mathbf{u}.$$

Since $\mathbf{y}$ and $\mathbf{u}$ are infinite sequences we can think of the transformations $\mathbf{T}_1$ and $\mathbf{T}_2$ as unitary matrices of infinite dimension. It follows that there exists a simple relation between the two Hankel operators as is illustrated by the commutative diagram below.



Specifically we have that $\mathbf{H} = \mathbf{T}_1^T\tilde{\mathbf{H}}\mathbf{T}_2$. Using this relation one can easily compute a full rank decomposition $\mathbf{H} = \Gamma\Delta$ from a full rank decomposition of $\tilde{\mathbf{H}} = \tilde{\Gamma}\tilde{\Delta}$, and apply the Ho-Kalman algorithm as before. We can write

$$\Gamma = \mathbf{T}_1^T\tilde{\Gamma},\ \Delta = \tilde{\Delta}\mathbf{T}_2,\ A = \tilde{\Gamma}^{+}\tilde{\tilde{\mathbf{H}}}\tilde{\Delta}^{+},$$

with $\tilde{\tilde{\mathbf{H}}}$ the infinite Hankel matrix pertaining to the shifted system $zG(z)$, represented with respect to the generalized basis.

The above argument illustrates how the generalized realization problem can be solved. The next question deals with how the necessary ingredients of the realization scheme can be computed. Computation of the transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ is straightforward. If we group the coefficients into coefficient vectors, as described in the introduction, we find that the k-th block row of $\mathbf{T}_1$ is given by the impulse response of $(zI - A_b)^{-1}B_bG_b^{k-1}(z)$, with $k \in \mathbb{N}$. Similarly the block-rows of $\mathbf{T}_2$ are given by the impulse responses of $(zI - A_b^T)^{-1}C_b^TG_b^{k-1}(z)$, with $k \in \mathbb{N}$.

We will now analyze the composition of the matrix $\tilde{\mathbf{H}}$. Suppose that we have an output signal $y \in \ell_2[1,\infty)$ and a signal $u \in \ell_2(-\infty,0]$ that are related according to

$$y(t) = P_{\ell_2[1,\infty)}G(q)u(t),$$

with $P_{\ell_2[1,\infty)}$ the orthogonal projector onto $\ell_2[1,\infty)$. We can expand the signals y and u in terms of the generalized basis functions to obtain the sequences $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{u}}$. Grouping the coefficients according to the number of multiplications with $G_b(z)$ gives us a means to efficiently compute the elements of $\tilde{\mathbf{H}}$ by making use of the inherent structure in the underlying basis construction. Beware however that by grouping the basis functions in vectors the basic elements of the Hankel operator will be matrices of dimension $n_b \times n_b$. If we denote the k-th coefficient vector of $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{u}}$ respectively by $\mathcal{Y}(k)$ and $\mathcal{U}(k)$, then it is not difficult to show that there exists a relation

$$\mathcal{Y}(t) = \sum_{k=0}^{\infty} M_k\,\mathcal{U}(t-k),$$

with $M_k \in \mathbb{R}^{n_b \times n_b}$ given by

$$M_k = \frac{1}{2\pi i}\oint V_1(1/z)G_b^k(1/z)G(z)V_1^T(z)\frac{dz}{z}. \qquad (1)$$

We see that the coefficients $\{\mathcal{Y}(t)\}$ are obtained from $\{\mathcal{U}(t)\}$ by means of a convolution operation that is determined by the Markov parameters M_k and hence the associated Hankel operator takes on the form of a block Hankel matrix of infinite dimension given by:

$$\tilde{\mathbf{H}} = \begin{pmatrix} M_1 & M_2 & \cdots \\ M_2 & M_3 & \\ \vdots & & \ddots \end{pmatrix},$$

In fact this convolution operator which maps from $\ell_2^{n_b}$ to $\ell_2^{n_b}$ is the so-called *Hambo* transform of $G(z)$. This Hambo transform was discussed extensively in [10, 6]. We will not go into detail about it here, suffice it to say that it can be obtained from $G(z)$ directly by means of a variable transformation, and that the McMillan degree of the underlying system is invariant under Hambo transformation [10].

From (1), one can easily calculate the Markov parameters M_k. In the following we will give an outline of the method by which they can be obtained. Detailed proofs of most of the steps taken can be found in [11]. We make use of the fact that we can write

$$G(z) = \sum_{k=1}^{\infty}\sum_{i=1}^{n_b} l_k^i\phi_k^i(z)$$

and that there exist matrices P_i, Q_i such that

$$V_1(z)\phi_i(z) = P_iV_1(z) + Q_iV_1G_b(z), \qquad (2)$$

with $P_i, Q_i \in \mathbb{R}^{n_b \times n_b}$. Substituting this in (1) reveals that the Markov parameters M_k are directly related to the expansion coefficients of the transfer function $G(z)$ as follows:

$$M_k = \sum_{i=1}^{n_b} l_{k+1}^iP_i^T + l_k^iQ_i^T. \qquad (3)$$

A similar relation is found for the Markov parameters of the shifted system of $zG(z)$ which are denoted by $\bar{M}_k$. The $\bar{M}_k$ are also needed for the computation of the realization since they form the Hankel matrix $\bar{\tilde{\mathbf{H}}}$. From (2) it can be deduced that the matrices P_i and Q_i are the solutions of the set of coupled Sylvester equations

$$AP_iA^T + Be_i^TA^T = P_i$$
$$AQ_iA^T + Be_i^TC^TB^T + AP_iC^TB^T = Q_i.$$

Equation (3) implies that any Markov parameter M_k depends solely on the coefficient vectors L_{k+1} and L_k. In order to calculate the first $N-1$ Markov parameters it therefore suffices to have available the first N coefficient vectors.

Although we now, in principle, have the necessary tools for constructing the matrices $\tilde{\mathbf{H}}, \tilde{\tilde{\mathbf{H}}}, \mathbf{T}_1$ and $\mathbf{T}_2$, we need to take some additional steps before we can solve the partial realization problem.

3 Partial realization

The main difficulty with applying the Hankel operator and transformation concepts directly, lies with fact that the Hankel matrices involved are of infinite dimension, while in the partial realization problem setting we are only given a finite number of Markov parameters. However, we can resolve the problem by observing that the minimal partial realization problem of the Hambo transform of $G(z)$ can be solved under the same rank conditions as in the classical realization problem setting [7]. Once a realization $(\tilde{A}, \tilde{B}, \tilde{C})$ of the Hambo transforms of $G(z)$ has been derived we have available the necessary information to construct complete full rank decomposition of $\tilde{\mathbf{H}}$. Further, once a state-space realization of the Hambo transform of $G(z)$ is known, a realization of the strictly proper part of the Hambo transform of $zG(z)$, and hence a full rank decomposition of $\tilde{\tilde{\mathbf{H}}}$, can be found according to the following proposition.

Proposition 1 *Suppose that the Hambo transform of $G(z)$ with respect to the basis generated by the all-pass function $G_b(z)$ has a realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$. Then a realization of the strictly proper part of the Hambo transform of $zG(z)$ is $(\tilde{A}, \tilde{B}, A_b^T \tilde{C} + C_b^T X_C \tilde{A})$, with X_C the solution to the Sylvester equation*

$$D_b^T X_C \tilde{A} + B_b^T \tilde{C} = X_C.$$

Proof: We make use of the fact that the Hambo transform of $zG(z)$ is equal to $N^T(z^{-1})$ times the Hambo transform of $G(z)$, with $N(z) = A_b + B_b(zI - D_b)^{-1} C_b$, for an explanation of this see e.g. [6]. Hence it is equal to the product of infinite series

$$\left(A_b^T + C_b^T B_b^T z + C_b^T D_b^T B_b^T z^2 + \ldots\right)\left(\tilde{D} + \tilde{C}\tilde{B} z^{-1} + \ldots\right). \quad (4)$$

Evaluating this expression term by term and discarding the terms with non-negative powers of z leads to an expression that has the given realization[1]. ∎

Denoting the first block-column (of dimension $\infty \times n_b$) of $\mathbf{T}_1$ and $\mathbf{T}_2$ by T_1 and T_2 respectively we get that our minimal realization is given by

$$A = \tilde{\Gamma}^{+} \tilde{\tilde{\Gamma}} \tilde{\tilde{\Delta}} \tilde{\Delta}^{+}, B = \tilde{\Delta} T_2, C = T_1^T \tilde{\Gamma}.$$

[1] Another possible realization is found by interchanging the factors in expression (4). This leads to an alternative (dual) formulation of the realization algorithm.

Substituting $\tilde{\Delta}^{+} = \tilde{\Delta}^T \tilde{X}_c^{-1}$ and $\tilde{\Gamma}^{+} = \tilde{X}_o^{-1} \tilde{\Gamma}^T$ with $\tilde{X}_c = \tilde{\Delta}\tilde{\Delta}^T$ and $\tilde{X}_o = \tilde{\Gamma}^T \tilde{\Gamma}$ (the controllability and observability Gramians of $(\tilde{A}, \tilde{B}, \tilde{C})$) we see that we are left with computing

$$\tilde{\Gamma}^T \tilde{\tilde{\Gamma}}, \tilde{\tilde{\Delta}} \tilde{\Delta}^T, \tilde{\Delta} T_2, T_1^T \tilde{\Gamma}.$$

These four entities are, as can easily be shown, the solutions to a set of four separate discrete-time Sylvester equations, involving only known quantities. In particular from proposition 1 it follows that $\tilde{\tilde{\Delta}}\tilde{\Delta}^T$ is in fact equal to $\tilde{X}_c$ which simplifies the computation of A considerably, as it cancels against $\tilde{X}_c^{-1}$.

The procedure is summarized in the following partial realization algorithm.

Algorithm 1 *Let M_{k-1} for $k = 1..N$ be the first N Markov parameters of the Hambo transform of the system $G(z)$. Construct a block Hankel matrix $\tilde{\mathbf{H}}_{N-1,N-1}$, where the index pair $(N-1, N-1)$ denotes the block-row, respectively block-column dimension of the matrix. A minimal realization of the system $G(z)$ is obtained following these steps:*

1. Check whether there exists a $j < (N-1)$ for which the rank condition

$$\operatorname{rank}(\tilde{\mathbf{H}}_{j,j}) = \operatorname{rank}(\tilde{\mathbf{H}}_{j+1,i}) = \operatorname{rank}(\tilde{\mathbf{H}}_{j,j+1}) = n$$

holds. If not then the algorithm fails. If the rank condition holds then proceed with the next steps.

2. Compute from $\tilde{\mathbf{H}}_{N-1,N-1}$ a minimal realization (McMillan degree n) $(\tilde{A}, \tilde{B}, \tilde{C})$. This can be done by applying the Ho-Kalman algorithm.

3. Compute the observability Gramian $\tilde{X}_o$ of the pair $(\tilde{A}, \tilde{C})$. Also compute the solutions to the following set of Sylvester equations:

$$\begin{aligned} &\tilde{A} X_B D_b^T + \tilde{B} C_b^T = X_B \\ &D_b^T X_C \tilde{A} + B_b^T \tilde{C} = X_C \\ &\tilde{A}^T X_A \tilde{A} + \tilde{C}^T \left(A_b^T \tilde{C} + C_b^T X_C \tilde{A}\right) = X_A \end{aligned} \quad (5)$$

A minimal state-space realization (A,B,C) of McMillan degree n of $G(z)$ is given by

$$A = \tilde{X}_o^{-1} X_A,\ B = X_B,\ C = X_C. \quad (6)$$

We can view equations (5) and (6) as a means to compute the Hambo inverse transform, i.e. to obtain a minimal (A,B,C) from $(\tilde{A}, \tilde{B}, \tilde{C})$, under the condition that the latter indeed constitute a valid Hambo transform. Following a reverse route,finding a minimal $(\tilde{A}, \tilde{B}, \tilde{C})$ from (A,B,C) by the same realization approach provides us with a dual result: a straightforward way to compute the Hambo transform.

Corollary 1 ***(Hambo transform)*** *Suppose that $G(z)$ has a minimal realization (A,B,C) and we are given a generalized basis generated by the all-pass function with minimal,*

balanced realization (A_b, B_b, C_b). *A minimal realization of the strictly proper part of the Hambo system transform of* $G(z)$ *is given by*

$$\tilde{A} = X_o^{-1} X_{\tilde{A}},\ \tilde{B} = X_{\tilde{B}},\ \tilde{C} = X_{\tilde{C}},$$

with X_o *the observability Gramian of the pair* (A,C) *and* X_Γ, $X_{\tilde{B}}$, $X_{\tilde{C}}$ *given by*

$$\begin{aligned} AX_{\tilde{B}}A_b + BC_b &= X_{\tilde{B}} \\ A_b X_{\tilde{C}} A + B_b C &= X_{\tilde{C}} \\ A^T X_{\tilde{A}} A + C^T (D_b C + C_b X_{\tilde{C}} A) &= X_{\tilde{A}}. \end{aligned}$$

Note that a Hambo transform of a strictly proper system is in general only proper. When necessary the direct feedthrough term $\tilde{D} = M_0$ can be computed from the first expansion coefficient vector L_1 using relation (1).

4 The underlying interpolation problem

It is well-known that the classical problem of minimal partial realization from the first N Markov parameters is equivalent to the problem of constructing a stable strictly-proper real-rational transfer function of minimal degree that interpolates to the first $N-1$ derivatives of $G(z)$ evaluated at infinity [12]. Also it is easy to see that the least-squares approximation of a stable transfer function $G(z)$ in terms of a finite set of rational basis functions interpolates to the function $G(z)$ and/or its derivatives in the points $1/\lambda_i$, with λ_i the poles of the basis functions involved [13].

In the basis construction considered in this paper the error function of an N-th order approximation $\hat{G}(z)$ takes on the form

$$E(z) = \sum_{k=N+1}^{\infty} L_k^T V_k(z) = G_b^N(z) \sum_{k=1}^{\infty} L_{N+k}^T V_1(z) G_b(z)^{k-1}.$$

The basis function vectors $V_k(z)$ are obtained by repeated multiplication of the first vector $V_1(z)$ with $G_b(z)$. This transfer function is all-pass and of McMillan degree n_b and henceforth it can be written in the form (modulo multiplication by -1)

$$G_b(z) = \prod_{i=1}^{n_b} \frac{1 - z\lambda_i^*}{z - \lambda_i}.$$

This function has poles λ_i and zeros $1/\lambda_i$. Due to the repetition of the all-pass function $G_b(z)$ in V_k, the error $E(z)$ will have as a factor the function $G_b^N(z)$. This means that $E(z)$ has zeros of order N at each of the points $1/\lambda_i$ and subsequently $\hat{G}(z)$ interpolates to $\frac{d^{k-1}G}{dz^{k-1}}$ in $z = 1/\lambda_i$ for $k = 1..N$ and $i = 1..n_b$. This interpolation property in fact holds true for any model of which the first N expansion coefficient vectors match those of the system, in particular for a model found by solving the partial realization problem.

In view of the interpolating property of the basis function expansion it is not surprising that there exists a one-to-one correspondence between the expansion coefficient vector sequence $\{L_k\}_{k=1..N}$ and the interpolation data $\{\frac{d^{k-1}G}{dz^{k-1}}(1/\lambda_i)\}_{k=1..N}$. An explicit expression for this relation can be derived by exploiting the linear transformation that links the set of basis function vectors V_k and the set of vectors that consists of single-pole transfer functions as given by

$$S_k^T(z) = \left(\frac{1}{(z-\lambda_i)^k}, \quad \cdots, \quad \frac{1}{(z-\lambda_{n_b})^k} \right)^T,$$

with λ_i the poles of the basis generating function $G_b(z)$. If we assume the poles λ_i to be distinct we can write

$$\begin{pmatrix} V_1(z) \\ V_2(z) \\ \vdots \end{pmatrix} = \begin{pmatrix} T_{11} & 0 & \cdots \\ T_{21} & T_{22} & \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} S_1(z) \\ S_2(z) \\ \vdots \end{pmatrix},$$

with $T_{kl} \in \mathbb{R}^{n_b \times n_b}$. The coefficient vectors L_k are obtained as the solutions of

$$L_k = \frac{1}{2\pi i} \oint V_k(1/z) G(z) \frac{dz}{z}.$$

When substituting $V_k(1/z) = \sum_{l=1}^{k} T_{kl} S_l(1/z)$ and applying Cauchy's integral formula we find that

$$\begin{pmatrix} L_1 \\ L_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} T_{11} & 0 & \cdots \\ T_{21} & T_{22} & \\ \vdots & & \ddots \end{pmatrix} \Pi \begin{pmatrix} F_1 \\ F_2 \\ \vdots \end{pmatrix},$$

with

$$F_k^T = \left(\frac{1}{\lambda_1^k} \frac{d^{k-1}G}{dz^{k-1}}(1/\lambda_1), \quad \cdots, \quad \frac{1}{\lambda_{n_b}^k} \frac{d^{k-1}G}{dz^{k-1}}(1/\lambda_{n_b}) \right)^T,$$

and Π a matrix that is given by

$$\Pi = \begin{pmatrix} \Lambda^{-1} & 0 & \cdots \\ 0 & -\Lambda^{-2} & \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} \binom{0}{0} I & 0 & \cdots \\ \binom{1}{0} I & \binom{1}{1} I & \\ \vdots & & \ddots \end{pmatrix}, \qquad (7)$$

with $\Lambda^{-k} = \mathrm{diag}(1/\lambda_1^k, \cdots, 1/\lambda_{n_b}^k)$. Since $1/\lambda_i$ exists only for $\lambda_i \neq 0$, we must make the additional assumption that $\lambda_i \neq 0$, $\forall i$. Equation (7) shows that there exists a direct correspondence between the first N coefficient vectors L_k and the first N vectors F_k that contain the data $\frac{d^{k-1}G}{dz^{k-1}}$ for $k = 1..N$ evaluated at the points λ_i.

A similar relation can be derived that shows the correspondence between the generalized Markov parameter sequence $\{M_{k-1}\}_{k=1..N}$ and the interpolation data $\{\frac{d^{k-1}G}{dz^{k-1}}\}_{k=1..N}$, starting from equation (1).

Now that we are able to compute the parameters $\{L_k\}_{k=1..N}$ and $\{M_{k-1}\}_{k=1..N}$ from $\{\frac{d^{k-1}G}{dz^{k-1}}(1/\lambda_i)\}_{k=1..N}$ we can solve

the following interpolation problem, by means of the minimal partial realization algorithm given in the previous section. Given the interpolation conditions

$$\frac{d^{k-1}G}{dz^{k-1}}(1/\lambda_i) = p_{i,k},\ p_{i,k} \in \mathbb{C}$$

for $i = 1..n_b$ and $k = 1..N$, with $\lambda_i \neq 0$ distinct points inside the unit disc, find the rational transfer function of minimal degree that interpolates these points. The problem is solved by constructing a balanced all-pass function (A_b, B_b, C_b, D_b) such that the eigenvalues of A_b are λ_i. From this all-pass function we can then obtain all the parameters that are necessary to compute the set of Markov parameters M_k that correspond to the interpolation data. Realization of a state-space model from these Markov parameters gives us the desired transfer function. Note that apart from the requirement that the λ_i should be distinct and unequal to zero they must satisfy the restriction that if complex they should come in conjugate pairs. This latter requirement ensures that the resulting transfer function has only real valued parameters.

The relation between the Markov parameters M_k and the derivatives of $G(z)$ evaluated at $1/\lambda_i$ has been treated in a similar context in the work of Audley and Rugh [14] on the representation of systems in so-called H-matrix form. The H-matrix is not to be mistaken for the Hankel operator discussed earlier but it is closely connected to it. It takes on a Toeplitz instead of a Hankel matrix form but the basic elements of the H-matrix for the basis considered in this paper are still the Markov parameters M_k. Audley and Rugh provided an algorithm to realize a transfer function of minimal degree from a finite dimensional H-matrix representation, by directly solving the underlying interpolation problem.

5 Conclusions

In this paper we have shown how to realize a minimal state-space model on the basis of partial knowledge of the expansion of a transfer function $G(z)$ in terms of a set of rational orthogonal basis functions. The problem was solved by computing the Markov parameters that correspond to the representation of the underlying system as a convolution operator that operates on expansion coefficient sequences. A minimal state-space realization of this operator, also known as the Hambo transform of $G(z)$, can be found under the same conditions that apply in the classical partial realization problem setting. After this state-space form has been obtained, one simply has to transform it back to the original domain to obtain the minimal realization of $G(z)$. It was shown that the analysis related to the realization problem provides us with compact expressions for computing the Hambo inverse transform as well as the Hambo transform itself.

References

[1] Y. Lee, *Statistical Theory of Communication.* New York: Wiley, 1960.

[2] B. Wahlberg, "System identification using Laguerre models," *IEEE Transactions on Automatic Control*, vol. 36, pp. 551–562, May 1991.

[3] B. Wahlberg, "System identification using Kautz models," *IEEE Transactions on Automatic Control*, vol. 39, pp. 1276–1282, June 1994.

[4] P. Heuberger, P. Van den Hof, and O. Bosgra, "A generalized orthonormal basis for linear dynamical systems," *IEEE Transactions on Automatic Control*, vol. 40, pp. 451–465, Mar. 1995.

[5] B. Ninness and F. Gustafsson, "A unifying construction of orthonormal bases for system identification," *IEEE Transactions on Automatic Control*, vol. 42, pp. 515–521, April 1997.

[6] P. Van den Hof, P. Heuberger, and J. Bokor, "System identification with generalized orthonormal basis functions," *Automatica*, vol. 31, no. 12, pp. 1821–1834, 1995.

[7] A. Tether, "Construction of minimal linear state-variable models from finite input-output data," *IEEE Transactions on Automatic Control*, vol. 15, pp. 427–436, August 1970.

[8] Z. Szabó and J. Bokor, "Minimal state space realization for transfer functions represented by coefficients using generalized orthonormal basis," in *Proceedings of the 36th Conference on Decision and Control*, (San Diego, CA), pp. 169–174, December 1997.

[9] B. Ho and R. Kalman, "Effective construction of linear state-variable models from input/output functions," *Regelungstechnik*, vol. 14, no. 12, pp. 545–592, 1966.

[10] P. Heuberger and P. Van den Hof, "The Hambo transform: A signal and system transform induced by generalized orthonormal basis functions," in *Proc. 13th IFAC World Congress*, (San Francisco), pp. 357–362, 1996.

[11] P. Heuberger, Z. Szabó, T. de Hoog, P. Van den Hof, and J. Bokor, "Realization algorithms for expansions in generalized orthonormal basis functions," in *Volume H of the Proceedings of the 14th IFAC World Congress*, (Beijing), pp. 385–390, July 1999.

[12] B. Anderson and A. Antoulas, "Rational interpolation and state-variable realizations," *Linear Algebra and its Applications*, vol. 137/138, pp. 479–509, 1990.

[13] J. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain.* Rhode Island: American Mathematical Society, 2nd ed., 1956.

[14] D. Audley and W. Rugh, "On the H-matrix system representation," *IEEE Transactions on Automatic Control*, vol. 18, pp. 235–242, June 1973.