# Control Relevant Identification for $H_\infty$-norm based Performance Specifications

Raymond A. de Callafon[‡] Paul M.J. Van den Hof

Mechanical Engineering Systems and Control Group
Delft University of Technology
Mekelweg 2, 2628 CD Delft, The Netherlands
Tel. +31-15-784703; Fax: +31-15-784717;
E-mail: callafon@tudw03.tudelft.nl

**Abstract**

For consecutive model-based control design, approximate identification of linear models should be performed on the basis of a feedback-relevant criterion, compatible with the control design. For an $H_\infty$-norm based control design, a procedure is presented to estimate a possibly unstable and feedback controlled plant by using an $H_\infty$-norm based feedback-relevant identification criterion. It is shown that the formulated identification problem can be handled by taking (noisy) closed loop frequency domain measurements of the plant and fitting a model of a prespecified McMillan degree, which is parametrized in a stable factorization, using a certain non-linear constrained minimization. The procedure is illustrated by an example.

## 1 Introduction

To tackle the problem of designing an enhanced and robust control system for a plant with unknown dynamics, a simultaneous (off-line) optimization of identification and model-based control design criteria would be required, as formulated in [1]. On the other hand, it has been widely motivated to separate the two stages of identification and control design and to use an iterative scheme of identification and model-based control design [19]. In such an iterative scheme, closed loop experimental conditions are indispensable to obtain data from the (possibly unstable) plant [9, 12, 14].

Compatible criteria in both the identification and control design are a prerequisite in order to ensure performance improvement of the plant to be controlled during the subsequent iterations of identification and model-based control design [1]. For 'classical' (weighted) $H_2$-norm based identification criteria, control performance specifications are implicitly restricted to $H_2$/LQG-type control design criteria, see e.g. [9], [14], [24] or [7] for a nice overview. For $H_\infty$-norm based control performance specifications this unleashes the need for an $H_\infty$-norm based identification procedure. Furthermore, compatible (weighted) $H_\infty$-norm based criteria in both the identification and the control design of an iterative scheme, opens the possibility to incorporate (performance) robustness considerations [5] to ensure performance improvement of the plant to be controlled [1].

In this paper a procedure is presented to estimate a linear multivariable discrete or continuous time model with a prespecified McMillan degree that fits (noisy) frequency response data of a (possibly unstable) plant operating under closed-loop conditions on the basis of an $H_\infty$-criterion. The $H_\infty$-criterion will be approximated by a pointwise evaluation of frequency response data, which is the main motivation in this paper to consider frequency domain data of the plant. Furthermore, convergence aspects of an iterative scheme of identification and model-based control design employing an $H_\infty$-norm control performance are being discussed. A unified approach to handle both stable and unstable plants is obtained by estimating a model via a *stable* factorization similar as in [14] or [20]. Alternative approaches with an $H_\infty$-criterion can also be found in the area of identification in $H_\infty$, see e.g. [8] or [11]. In these approaches a stable transfer function having some worst-case optimality properties is being derived on the basis of (noisy) frequency response data. A drawback is the lack of ability to prespecify the McMillan degree of the model being estimated, which may result in relatively high order models [6].

As standard curve fit procedures do not guarantee stability of the resulting estimate of the factorization, a canonical parametrization given in [18] will be used in this paper to parametrize all stable, minimal and balanced state space systems with distinct Hankel singular values of a prespecified McMillan degree. An alternative curve fit procedure with conditions on stability of the estimate can also be found in [10]. The procedure in [10] is based on a maximum amplitude criterion, which has a close connection with an $H_\infty$-criterion only in the case of fitting a stable scalar transfer function. Using the parametrization results of [18], it is shown that the minimization of the feedback-relevant $H_\infty$-criterion can be handled by a non-linear constrained minimization, where the parameters lie in a convex set.

## 2 Preliminaries

Let $P$ be used to denote either the plant $P_o$ or the model $\hat{P}$, then the feedback configuration of $P$ and a controller $C$ is denoted with $\mathcal{T}(P,C)$ and defined as the connection structure depicted in Figure 1. If $P$ equals $P_o$ in Figure 1, then the signals $u$ and $y$ reflect respectively the inputs and outputs of the plant $P_o$, where $v$ is an additive noise on the output $y$ of the plant. It is presumed that the noise $v$ is uncorrelated with the external reference signals $r_1$, $r_2$ and can be modelled by a monic stable and stably invertible noise filter $H$ having a white noise input $e$ [15]. The signals $u$ and

‡The work of Raymond de Callafon is sponsored by the Dutch Systems and Control Theory Network

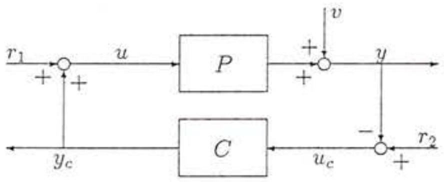

Fig. 1: feedback connection structure $\mathcal{T}(P, C)$

$y$ are being measured and $r_1$, $r_2$ (and consequently $u_c$, $y_c$) are possibly at our disposal.

It is assumed that the feedback connection structure is well posed, that is $\det[I + CP] \not\equiv 0$. In this way the mapping of $[r_2\ r_1]^T$ to $[y\ u]^T$ is given by the transfer function matrix $T(P, C)$ with

$$T(P,C) := \begin{bmatrix} P \\ I \end{bmatrix} [I + CP]^{-1} \begin{bmatrix} C & I \end{bmatrix}, \tag{1}$$

and the data coming from the closed loop system $\mathcal{T}(P_o, C)$ can be described by

$$\begin{bmatrix} y \\ u \end{bmatrix} = T(P_o, C) \begin{bmatrix} r_2 \\ r_1 \end{bmatrix} + \begin{bmatrix} I \\ -C \end{bmatrix} [I + P_o C]^{-1} v. \tag{2}$$

In case of an *internally* stable closed loop system $\mathcal{T}(P, C)$, all four transfer function matrices in $T(P, C)$ will be stable which implies $T(P, C) \in \mathbb{R}\mathcal{H}_\infty$ for a real rational $P$, where $\mathbb{R}\mathcal{H}_\infty$ denotes the set of all rational stable transfer functions.

Using the theory of fractional representations, $P$ will be expressed as a ratio of two stable mappings $N$ and $D$. Following [23], $P$ has a right coprime factorization (*rcf*) $(N, D)$ over $\mathbb{R}\mathcal{H}_\infty$ if there exists $X$, $Y$, $N$ and $D$ such that $P = ND^{-1}$ and $XN + YD = I$. In addition, a *rcf* $(N, D)$ is normalized if it satisfies $N^*N + D^*D = I$, where $^*$ denotes the complex conjugate transpose. Dual definitions apply for left coprime factorizations (*lcf*).

## 3 Motivation for $H_\infty$-criterion

### 3.1 Norm-based control design and identification

In the analysis of feedback relevant identification, the characterization of a closed loop (nominal) performance criterion plays a crucial role. If again the symbol $P$ is used to denote either the actual plant $P_o$ or the model $\hat{P}$ to be estimated, this criterion can be formalized as follows [21]. Let $\mathcal{X}$ denote a complete normed space, where $\|\cdot\|_\mathcal{X}$ is the norm defined on $\mathcal{X}$. Let $J(P, C)$ be any function with an image in $\mathcal{X}$, then an objective function can defined by $\|J(P, C)\|_\mathcal{X}$.

In this paper the normed space $\mathcal{X}$ is chosen to be the space $\mathbb{R}\mathcal{H}_\infty$ and consequently the norm function $\|J(P, C)\|_\mathcal{X}$ is the $H_\infty$-norm. The control objective function $J(P, C) \in \mathbb{R}\mathcal{H}_\infty$ to be minimized in a norm based control design is taken to be

$$\|J(P, C)\|_\infty := \|W_2 T(P, C) W_1\|_\infty \tag{3}$$

with $W_2$, $W_1 \in \mathbb{R}\mathcal{H}_\infty$, which is a weighted form of the closed loop dynamics described by the transfer function matrix $T(P, C) \in \mathbb{R}\mathcal{H}_\infty$ given in (1).

The actual plant $P_o$ under consideration is unknown and hence the minimization of (3) cannot be solved straightforwardly for $P = P_o$. Instead of minimizing $\|J(P_o, C)\|_\infty$ directly, generally $\|J(P_o, C)\|_\infty$ is minimized iteratively [7] using a sequence of controllers $C_i$, based on observations of the plant $P_o$, that satisfy at least

$$\|J(P_o, C_i)\|_\infty \leq \gamma_i \text{ and } \gamma_i \leq \gamma_{i-1}. \tag{4}$$

See e.g. [13] for an approach on direct tuning of controllers $C_i$ based on a 2-norm performance specification. If, on the other hand, a model $\hat{P}_i$ (obtained by identification techniques) is introduced and used to design the controller $C_i$ in the $i$th step of the iterative scheme, then generally *an upper bound* on $\|J(P_o, C)\|_\infty$ is minimized iteratively. The upper bound is inspired by the following triangular inequality [19].

$$\|J(P_o, C_i)\|_\infty \leq \|J(\hat{P}_i, C_i)\|_\infty + \|J(P_o, C_i) - J(\hat{P}_i, C_i)\|_\infty \tag{5}$$

The right hand side of (5) can be minimized by minimizing $\|J(\hat{P}_i, C)\|_\infty$ by a norm based control design,

$$C_i := \min_C \|J(\hat{P}_i, C)\|_\infty \tag{6}$$

while the minimization of the second term can be viewed as an identification problem [19]

$$\hat{P}_i := \min_P \|J(P_o, C_i) - J(P, C_i)\|_\infty \tag{7}$$

Unfortunately, the controller $C_i$ is based on the model $\hat{P}_i$ via (6), while $\hat{P}_i$ will depend on the controller $C_i$ via (7) and both are unknown (yet). Hence, the introduction of the model $\hat{P}_i$ in the right hand side of (5) does not help in minimizing the upper bound directly. Usually one tries to alternate on the minimization of (6) and (7) leading off with some initial controller, hoping that a sequence of controllers will be obtained that satisfy at least (4).

### 3.2 Enforcing performance enhancement

In order to have a sequence of controllers $C_i$ found by (6) that satisfy (4), conditions on both the models $\hat{P}_i$ and the controllers $C_i$ should be derived. These conditions can be found by evaluating the triangular inequality on the previous step $i - 1$ of the iteration as follows.

$$\begin{aligned} \|J(P_o, C_{i-1})\|_\infty \leq \|J(\hat{P}_i, C_{i-1})\|_\infty + \\ + \|J(P_o, C_{i-1}) - J(\hat{P}_i, C_{i-1})\|_\infty \end{aligned} \tag{8}$$

Now the minimization of the second term of the right hand side of (8), similar as in (7), can indeed be seen as an identification problem, since $C_{i-1}$ from the previous iteration is assumed to be known. The first term on the right hand side of (8) is now simply an evaluation of the control objective function, using the model being estimated. In this perspective the following iterative scheme of identification and control can be considered.

**Proposition 3.1** *Given a controller $C_{i-1}$ and a $\gamma_{i-1}$ such that $\|J(P_o, C_{i-1})\|_\infty \leq \gamma_{i-1}$. Consider the following iterative scheme.*

1. *Estimate a model $\hat{P}_i$ by the minimization*

$$\hat{P}_i := \min_P \|J(P_o, C_{i-1}) - J(P, C_{i-1})\|_\infty \tag{9}$$

*and accept the model $\hat{P}_i$ only if $\beta_i \leq \gamma_{i-1}$, with*

$$\beta_i := \underbrace{\|J(\hat{P}_i, C_{i-1})\|_\infty}_{(a)} + \underbrace{\|J(P_o, C_{i-1}) - J(\hat{P}_i, C_{i-1})\|_\infty}_{(b)}. \tag{10}$$

2. *Design a controller $C_i$ on the basis of $\hat{P}_i$ by the minimization*

$$C_i := \min_C \|J(\hat{P}_i, C)\|_\infty \quad (11)$$

*and accept the controller only if $\gamma_i \leq \beta_i$, with*

$$\gamma_i := \underbrace{\|J(\hat{P}_i, C_i)\|_\infty}_{(c)} + \underbrace{\|J(P_o, C_i) - J(\hat{P}_i, C_i)\|_\infty}_{(d)} \quad (12)$$

3. *$i := i + 1$, goto 1*

*Then the sequence of controllers $C_i$ satisfy* (4).

**Proof:** The inequality given in (10) is an upper bound for $\|J(P_o, C_{i-1})\|_\infty$, while (12) is an upper bound for $\|J(P_o, C_i)\|_\infty$. $\|J(P_o, C_i)\|_\infty \leq \gamma_i$ and $\|J(P_o, C_{i-1})\|_\infty \leq \beta_i$ with $\gamma_i \leq \beta_i \leq \gamma_{i-1}$. □

Note that proposition 3.1 is a rather general set-up of an iterative scheme to generate a sequence of controllers that satisfies (4). In this set-up (10) and (12) reflect respectively a model and a controller *validation test* in order to enforce (4). The numerical values of (b) and (c) are simply the minimizing values of respectively the minimization (9) and (11), while (a) is just an evaluation of the control objective function. The contribution of (d) can be overestimated by incorporating an estimation of the model mismatch between $\hat{P}_i$ and $P_o$, see also [1]. Moreover, this model mismatch can be used in the minimization (11) of step 2 by formulating a *robust* control design problem [5].

For notational convenience the *known* controller $C_{i-1}$ (coming from the previous iteration) and implemented on the actual feedback controlled plant $P_o$, will be simply denoted by $C$ in the sequel of this paper. With the choice of the objective function given in (3), the minimization (9) to estimate a model $\hat{P}_i$ can now be formulated as $\hat{P}_i := P(\hat{\theta}_i)$ with

$$\begin{aligned} \hat{\theta}_i &:= \arg\min_{\theta\in\Theta} \|\Delta T(P_o, P_i(\theta), C)\|_\infty \text{ and} \\ \Delta T(P_o, P_i(\theta), C) &:= W_2[T(P_o, C) - T(P_i(\theta), C)]W_1. \end{aligned} \quad (13)$$

For the minimization of (11) one is referred to standard $H_\infty$-control design problems present in the literature [2, 17]. In the remaining part of this paper, the minimization (13) will be considered. The minimization will be based on closed loop data coming from the plant and employing a stable factorization of the model to be estimated.

# 4 Estimation and parametrization of stable factorizations

## 4.1 Identification of coprime factors

A coprime factorization of a plant $P_o$ operating under closed loop conditions can be accessed by performing a filtering of the signals present in the closed loop system [22] and defined by

$$x := F[r_1 + Cr_2] = F[u + Cy]. \quad (14)$$

By considering the map from $x$ in (14) onto $col(y, u)$, a stable right factorization $(P_o S_{in} F^{-1}, S_{in} F^{-1})$ of $P_o$ is readily available from the data of the closed loop controlled plant. For exact details, one is referred to [22] or [3], but in order to evaluate the usefulness of the filtering (14), the following result can be given.

**Lemma 4.1** *Let the plant $P_o$ and a controller $C$ form an internally stable feedback system $\mathcal{T}(P_o, C)$ and denote $S_{in} = [I + CP_o]^{-1}$. Then the following statements are equivalent.*

(i) $(P_o S_{in} F^{-1}, S_{in} F^{-1})$ *is a* rcf.

(ii) *there exists a* rcf $(N_x, D_x)$ *of an auxiliary model $P_x$ with $T(P_x, C) \in \mathbb{R}\mathcal{H}_\infty$ such that*

$$F = [D_x + CN_x]^{-1} \quad (15)$$

*Both conditions imply $F \begin{bmatrix} C & I \end{bmatrix} \in \mathbb{R}\mathcal{H}_\infty$.*

**Proof:** See [22] or [3]. □

Using the notation

$$N_{o,F} := P_o S_{in} F^{-1}, \; D_{o,F} := S_{in} F^{-1}, \quad (16)$$

Lemma 4.1 characterizes the freedom in choosing the (non unique) right coprime factorization $(N_{o,F}, D_{o,F})$ of the plant $P_o$ by the choice of any stable right factorization of any auxiliary model $P_x$ that is internally stabilized by the controller $C$. Using the *rcf* $(N_{o,F}, D_{o,F})$ of the plant $P_o$, the error $\Delta T(P_o, P_i(\theta), C)$ in (13) can be expressed in the following way.

**Lemma 4.2** *Let $P_o$ and $C$ create an internally stable feedback system $\mathcal{T}(P_o, C)$ and let $(N_{o,F}, D_{o,F})$ be the* rcf *of $P_o$ given in* (16), *where $F$ is given in* (15). *Consider any $P_i(\theta)$, then*

(i) *for all $\theta \in \Theta$ there exists a* rcf $(N_i(\theta), D_i(\theta))$ *of $P_i(\theta)$ such that $D_i(\theta) + CN_i(\theta) = F^{-1}$.*

(ii) $\|\Delta T(P_o, P_i(\theta), C)\|_\infty$ *in* (13) *equals*

$$\left\| W_2 \left( \begin{bmatrix} N_{o,F} \\ D_{o,F} \end{bmatrix} - \begin{bmatrix} N_i(\theta) \\ D_i(\theta) \end{bmatrix} \right) F \begin{bmatrix} C & I \end{bmatrix} W_1 \right\|_\infty \quad (17)$$

*where $(N_i(\theta), D_i(\theta))$ is any* rcf *of $P_i(\theta)$ that satisfies* (i).

**Proof:** See [3] □

With the result mentioned in lemma 4.2 it can be seen that $\Delta T(P_o, P_i(\theta), C)$ in (13) is simply a *weighted* difference between the *rcf* $(N_{o,F}, D_{o,F})$ and $(N_i(\theta), D_i(\theta))$ respectively of the plant $P_o$ and the model $P_i(\theta)$. However, the weighting contains the filter $F$, which depends on the factorization $(N_i(\theta), D_i(\theta))$ of the model. In [3] an iterative procedure has been presented to update the filter $F$ on the basis of the factorization $(N_i(\hat{\theta}), D_i(\hat{\theta}))$ being estimated, to satisfy (i) in Lemma 4.2.

Since the *rcf* $(N_{o,F}, D_{o,F})$ can be accessed via the map from $x$ in (14) onto $col(u, y)$, complex (noisy) frequency domain data can be obtained. The data of the factorization $(N_{o,F}, D_{o,F})$ will be denoted by $(N(\xi_j), D(\xi_j))$, where $\xi_j$ for $j = 1, 2, \ldots, l$ denotes a (prespecified) frequency grid $\xi_j = e^{i\omega_j}$ for discrete time systems and $\xi_j = i\omega_j$ for continuous time systems. In this way the minimization of the $H_\infty$-criterion (13) will be approximated by performing a pointwise evaluation of (17). The actual minimization problem can now be formalized as follows.

$$\min_{\theta} \max_{j=1,2,\ldots,l} \bar{\sigma}\{W_2(\xi_j) \left( \begin{bmatrix} N(\xi_j) \\ D(\xi_j) \end{bmatrix} - \begin{bmatrix} N_i(\theta,\xi_j) \\ D_i(\theta,\xi_j) \end{bmatrix} \right) \cdot \\ \cdot F(\xi_j) \begin{bmatrix} C(\xi_j) & I \end{bmatrix} W_1(\xi_j)\} \quad (18)$$

where $\bar{\sigma}\{\cdot\}$ denotes the maximum singular value. To ensure stability of the estimate $(N_i(\hat{\theta}), D_i(\hat{\theta}))$, a special parametrization presented in [18] of minimal, stable and balanced state space representations for the factorization $(N_i(\theta), D_i(\theta))$ will be used. Combining the requirements for minimality and stability along with a balancing property of the state space representation, will lead to parameter constraints that can be dealt with relatively easily.

### 4.2 Parametrization of stable factorizations

The parametrization results on stable, minimal and balanced state space realization in [18] and further elaborated in [4], are based on continues time systems having multiple common Hankel singular values. For discrete time systems an indirect state space parametrization can be based on a Möbius transformation, since this transformation preserves both stability, minimality and the balanced property of the continues time state space realization. Furthermore, in this paper only the case of distinct Hankel singular values will be discussed, which can be considered to be the generic case [4].

**Lemma 4.3** *Let $G(s)$ be defined by*

$$G(s) := \begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$$

*where $(N(s), D(s))$ is a* rcf *of the $p \times m$ rational transfer function $P(s)$, then the following statements are equivalent*

1. *$G(s)$ is a $(p+m) \times m$ stable rational transfer function matrix of McMillan degree $n$, with $n$ distinct Hankel singular values.*

2. *$G(s)$ has a state space representation $[\bar{A}, \bar{B}, \bar{C}, \bar{E}]$ with*

$$\begin{aligned} \bar{B} &:= \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \text{ with } b_j = [b_{ij}] \in \mathbb{R}^{1\times m} \\ &\quad \text{and } b_{1j} > 0 \text{ for } 1 \le j \le n \qquad (19) \\ \bar{C} &:= \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \text{ with } c_j = u_j[b_j b_j^T]^{1/2} \in \mathbb{R}^{(p+m)\times 1}, \\ &\quad u_j \in \mathbb{R}^{(p+m)\times 1} \text{ and } u_j^T u_j = 1 \text{ for } 1 \le j \le n \quad (20) \\ \bar{A} &:= [a_{ij}] \in \mathbb{R}^{n\times n} \text{ with } a_{ij} = -\frac{b_j b_j^T}{2\sigma_j} \text{ for } i = j, \\ &\quad \text{and } a_{ij} = \frac{\sigma_j b_i b_j^T - \sigma_i c_i^T c_j}{\sigma_i^2 - \sigma_j^2} \text{ for } i \ne j, \text{ with} \\ &\quad \sigma_{j+1} > \sigma_j > 0 \text{ for } 1 \le j \le n-1 \qquad (21) \\ \bar{E} &:= [e_{ij}] \in \mathbb{R}^{(p+m)\times m} \end{aligned}$$

*which is a balanced state space representation having distinct Hankel singular values $\sigma_j$.*

**Proof:** Direct application of theorem 2.1 in [18] for a system with distinct Hankel singular values. □

Splitting up $\bar{C}$ and $\bar{E}$ respectively into $[\bar{C}_N^T\ \bar{C}_D^T]^T$ and $[\bar{E}_N^T\ \bar{E}_D^T]^T$, the operation $P = ND^{-1}$ leads to a state space realization $(A, B, C, E)$ of $P$ given by

$$\begin{aligned} A &= \bar{A} - \bar{B}\bar{E}_D^{-1}\bar{C}_D & B &= \bar{B}\bar{E}_D^{-1} \\ C &= \bar{C}_N - \bar{E}_N\bar{E}_D^{-1}\bar{C}_D & E &= \bar{E}_N\bar{E}_D^{-1} \end{aligned} \quad (22)$$

It should be noted that (19) and (21) reflect parameter constraints that can be dealt with relatively easy. In standard pseudo canonical (overlapping) parametrizations [15] more complicated parameter constraints should have been specified to enforce stability. Unfortunately, (20) reflect a non-linear equality constraint to parametrize the elements of a unitary vector.

To circumvent (20), in [4] a parametrization based on rotating actions has been proposed. This parametrization parametrizes *almost* all unitary vectors in $\mathbb{R}^{k\times 1}$ for $k > 1$, without a non-linear constraint. Since $p + m > 1$, this can be applied without objections and it will be shown here that *all* unitary vectors $\in \mathbb{R}^{(p+m)\times 1}$ can be parametrized.

**Lemma 4.4** *Let $\mathcal{U} := \{\bar{u} \in \mathbb{R}^{(p+m)\times 1} \mid \bar{u}^T\bar{u} = 1\}$, $\phi := [\varphi_1 \cdots \varphi_{p+m-1}] \in \mathbb{R}^{(p+m-1)\times 1}$ and the set $\Phi := \{\phi \in \mathbb{R}^{(p+m-1)\times 1} \mid \varphi_i \in (-\pi/2, \pi/2]$ for $1 \le i < p+m-1$, $\varphi_{p+m-1} \in (-\pi, \pi]\}$. Consider the map $f : \mathbb{R}^{(p+m-1)\times 1} \to \mathbb{R}^{(p+m)\times 1}$ given by*

$$u = \prod_{i=1}^{p+m-1} x_i, \text{ with } x_i := \begin{bmatrix} \cos(\varphi_i)x_{i-1} \\ \sin(\varphi_i) \end{bmatrix}, \ x_0 = 1 \qquad (23)$$

*then*

*(a) $f(\phi) \in \mathcal{U}$ for all $\phi \in \mathbb{R}^{(p+m-1)\times 1}$*

*(b) the map $f : \mathbb{R}^{(p+m-1)\times 1} \to \mathcal{U}$ is surjective*

*(c) the map $f : \Phi \to \mathcal{U}$ is bijective.*

**Proof:** (a) The fact that $f(\phi) \in \mathcal{U}\ \forall \phi \in \mathbb{R}^{(p+m-1)\times 1}$ can be found by induction, see also [4].
(b) Take any $u := [u_1 \cdots u_{p+m}]^T \in \mathcal{U}$ and define $k$ to denote the index of the *first* non-zero entry $u_k$ in $u$. Using the map (23) for verification, the elements $\varphi_i$ of $\phi$ can be given by $\varphi_i = \pi/2$ for $1 \le i \le k-1$,

$$\varphi_i = \tan^{-1}\left( \frac{u_{i+1}}{u_k} \prod_{j=k}^{i-1} \cos(\varphi_j) \right)$$

for $k \le i < p+m-1$ and finally $\varphi_{p+m-1} = \text{sign}(u_{p+m})[1 - \text{sign}(u_k)]\pi/2 + \text{sign}(u_k)\sin^{-1}(u_{p+m})$, where sign$(\cdot)$ denotes the sign function defined by $\text{sign}(x) := 1$ if $x \ge 0$ and $\text{sign}(x) := -1$ if $x < 0$. Clearly, $\exists\ \phi \in \mathbb{R}^{p+m-1}$, which proves (b).
(c) It can be verified that the elements $\varphi_i$ of $\phi_i$ defined above, restricts $\phi$ to be an element of $\Phi$ and therefore uniquely determines $\phi \in \Phi$. □

With the result mentioned in lemma 4.4, the parametrization of the unitary vectors $u_j \in \mathbb{R}^{p+m}$ in (20) can be replaced by the alternative parametrization in terms of $\phi \in \mathbb{R}^{p+m-1}$. Putting linear constraints on the elements $\varphi_i$ such that $\phi \in \Phi$ according to lemma 4.4, will yield a *unique* parametrization. However, due to the periodicity of the elements $\varphi_i$ of $\phi$, the parametrization will be locally identifiable. Therefore, the constraints on the elements $\varphi_i$ of $\phi$ can be omitted during the optimization.

Resuming, the minimization problem given in (18) can be formulated as follows. Let $G_i(\theta)$ be defined as

$$G_i(\theta) := \begin{bmatrix} N_i(\theta) \\ D_i(\theta) \end{bmatrix}$$

having $m$ inputs and $p+m$ outputs and parametrized according to the results mentioned in lemma 4.3 and lemma 4.4. Then $[\bar{A}, \bar{B}, \bar{C}]$ are given by the parameter

$$\theta = [\sigma_1 \ \cdots \ \sigma_n \ b_1 \ \cdots \ b_n \ \phi_1^T \ \cdots \ \phi_n^T] \in \Theta \subset \mathbb{R}^{1 \times n(2m+p)}$$

where $\Theta$ is determined by the additional constraints given in (19) and (21). These constraints can be rewritten into

$$\begin{array}{l} \sigma_n - \sigma_{n-1} > 0, \ \cdots, \ \sigma_2 - \sigma_1 > 0, \ \sigma_1 > 0 \\ b_{11} > 0, \ b_{21} > 0, \ \cdots, \ b_{n1} > 0 \end{array} \tag{24}$$

to ensure a minimal, stable and balanced continuous time state space realization of the stable factorization $(N_i(\theta), D_i(\theta))$. In this way the minimization given in (18) is in fact a non-linear constrained minimization, where the parameters lie in a convex set $\Theta$. This can be solved by standard constrained minimization routines, for example available in the optimization toolbox of the MatLab package [16].

Compared to the curve fit procedure presented in [10], an alternative parametrization along with linear parameter constraints is presented here, to enforce stability of the multivariable estimate having multiple outputs. In addition, the frequency grid is used here to evaluate a maximum singular value, instead of a maximum amplitude criterion over all possible transfer functions.

## 5 Example

In this section, only the results on the minimization of (18) will be illustrated for a fixed filter $F$ and unity weightings $W_1$ and $W_2$. Using lemma 4.3 and lemma 4.4 to parametrize the state space realization of $(N(\theta), D(\theta))$ and the convex parameter constraints given in (24), (18) will be solved by a non-linear constrained optimization.

Consider a 5th order SISO discrete time plant $P_o$ having a DC-gain of 5, zeros located at $0.52 \pm 0.44i$, $0.97 \pm 0.06i$, poles located at $0.76 \pm 0.40i$, $0.99 \pm 0.06i$ and 0.94. The plant $P_o$ is controlled by a discrete time controller $C$ given by

$$C(q) = \frac{2.04q^3 - 1.66q^2 - 1.14q + 1.24}{q^3 - 1.68q^2 + 1.03q - 0.35} \tag{25}$$

As mentioned before, the minimization of (18) will be illustrated and $W_1$ and $W_2$ are chosen to be identity. Furthermore, the filter $F$ in (15) is taken to be fixed and based on an auxiliary model $P_x$ that equals $P_o$, where the factorization $(N_x, D_x)$ is taken to be a discrete time normalized right coprime factorization. Noisy frequency response measurements were generated at 100 frequency points distributed between $10^{-3}$ and 1Hz and based on time domain observations of $x$ in (14) and $col(y, u)$.

The aim is to fit a 3rd order stable factorization on the frequency response data $(N(e^{i\omega_j}), D(e^{i\omega_j}))$ of the 5th order plant $P_o$, using the non-linear constrained minimization. In order to start up the non-linear minimization, an ordinary least-squares equation error fit is applied to have an initial estimate of the factorization. The initial estimate happens to be stable and therefore it can be balanced and converted to the parametrization of lemma 4.3 and lemma 4.4. After this, the non-linear constraint optimization is started up. The final result in transfer function representation is given by

$$\begin{bmatrix} \hat{N}(q) \\ \hat{D}(q) \end{bmatrix} = \frac{\begin{bmatrix} 0.0005q^3 + 0.1162q^2 - 0.1165q + 0.0594 \\ 0.8802q^3 - 2.1267q^2 + 1.8395q - 0.5871 \end{bmatrix}}{[q^3 - 2.1861q^2 + 1.7704q - 0.5244]}$$

which is of course stable. The amplitude Bode plot of the result is given in Figure 2.

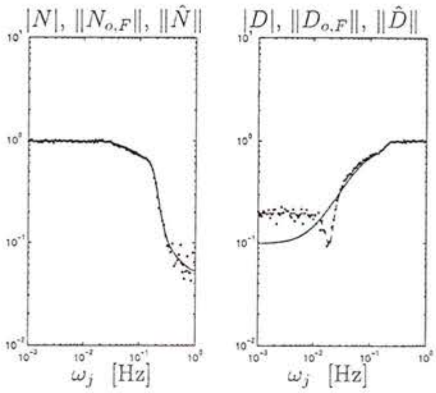


Fig. 2: Amplitude Bode plot of data $(N(e^{i\omega_j}), D(e^{i\omega_j}))$ (dotted), factorization $(N_{o,F}(e^{i\omega_j}), D_{o,F}(e^{i\omega_j}))$ of the plant $P_o$ (dashed) and 3rd order estimate $(\hat{N}(e^{i\omega_j}), \hat{D}(e^{i\omega_j}))$ (solid)

To illustrate the character of the minimization (18), a plot of the weighted error

$$\bar{\sigma}\{\left(\begin{bmatrix} N(e^{i\omega_j}) \\ D(e^{i\omega_j}) \end{bmatrix} - \begin{bmatrix} \hat{N}(e^{i\omega_j}) \\ \hat{D}(e^{i\omega_j}) \end{bmatrix}\right) F(e^{i\omega_j}) \begin{bmatrix} C(e^{i\omega_j}) & I \end{bmatrix}\} \tag{26}$$

is depicted in Figure 3. Clearly, this plot indicates the objective to minimize pointwise the maximum value of the error (26) along the available frequency grid.

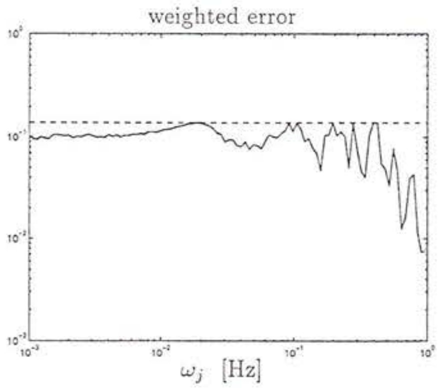


Fig. 3: Amplitude plot of the weighted error given in (26)

For completeness, a Bode plot of the resulting estimate $\hat{P} := \hat{N}\hat{D}^{-1}$ is depicted in Figure 4. From this picture it can be seen that a good estimate is obtained around the cross-over frequency of 0.2Hz. Moreover, it can be verified that the model $\hat{P}$ is also stabilized by the controller given in (25).

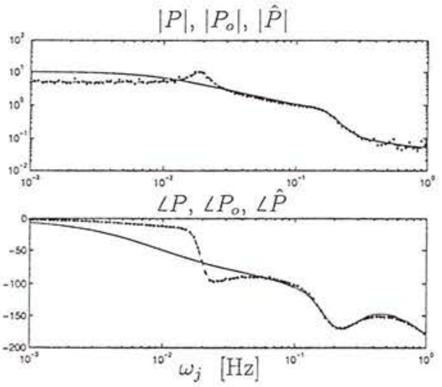


Fig. 4: Bode plot of data $P(e^{i\omega_j})$ (dotted), plant $P_o(e^{i\omega_j})$ (dashed) and 3rd order estimate $\hat{P}(e^{i\omega_j})$ (solid)

## 6 Conclusions

In this paper a procedure is presented to estimate a possibly unstable and feedback controlled multivariable plant by using an $H_\infty$-norm based feedback-relevant identification criterion. To handle the $H_\infty$-norm based criterion, (noisy) closed loop frequency domain measurements of the plant are used to fit a model, parametrized in a stable factorization of a prespecified McMillan degree, by a pointwise evaluation of the error along a prespecified frequency grid. To ensure stability of the factorization being estimated, a parametrization that enforces a stable, minimal and balanced state space representation is being used. With this parametrization, the $H_\infty$-norm based identification criterion can be solved by a non-linear constrained optimization.

## References


[1] D.S. Bayard, Y. Yam, and E. Mettler. A criterion for joint optimization of identification and robust control. *IEEE Trans. Autom. Contr., AC-37*, pp. 986–991, 1992.

[2] P.M.M. Bongers and O.H. Bosgra. Low order robust $H_\infty$ controller synthesis. In *Proc. 29th IEEE Conf. on Decision and Contr.*, pp. 194–199, Hawaii, USA, 1990.

[3] R.A. de Callafon and P.M.J. Van den Hof. Filtering and parametrization issues in feedback relevant identification based on fractional model representations. In *Proc. European Contr. Conf.*, Rome, Italy, 1995.

[4] C.T. Chou. *Geometry of Linear Systems and Identification.* PhD thesis, Univ. of Cambridge, Dep. of Engineering, 1994.

[5] J.C. Doyle, B.A. Francis, and A.R. Tannenbaum. *Feedback Control Theory.* MacMillan Publ., NY, USA, 1992.

[6] J.H. Friedman and P.P. Khargonekar. Identification of lightly damped systems using frequency domain techniques. In *Prepr. 10th IFAC Symp. on System Identification,* Vol. *2*, pp. 201–206, Denmark, 1994.

[7] M. Gevers. Towards a joint design of identification and control? *Essays on Control: Perspectives in the Theory and its Applications*, pp. 111–151, 1993.

[8] G.X. Gu and P.P. Khargonekar. A class of algorithms for identification in $H_\infty$. *Automatica,* Vol. *28*, pp. 299–312, 1992.

[9] R.G. Hakvoort, R.J.P. Schrama, and P.M.J. Van den Hof. Approximate identification with closed–loop performance criterion and application to LQG feedback design. *Automatica,* Vol. *30*, pp. 679–690, 1994.

[10] R.G. Hakvoort and P.M.J. Van den Hof. Frequency domain curve fitting with maximum amplitude criterion and guaranteed stability. *Int. Journal on Contr.*, Vol. *60*, No. *5*, pp. 809–825, 1994.

[11] A.J. Helmicki, C.A. Jacobson, and C.N. Nett. Fundamentals of control-oriented system identification and their applications for identification in $H_\infty$. In *Proc. Amer. Contr. Conf.*, pp. 89–99, Boston, USA, 1991.

[12] H. Hjalmarsson, M. Gevers, F. de Bruyne, and J. Leblond. Identification for control: closing the loop gives more accurate controllers. In *Proc. 33rd IEEE Conf. on Decision and Contr.*, pp. 4150–4155, Lake Buena Vista, USA, 1994.

[13] H. Hjalmarsson, S. Gunnarsson, and M. Gevers. A convergent iterative restricted complexity control design scheme. In *Proc. 33rd IEEE Conf. on Decision and Contr.*, pp. 1735–1740, Lake Buena Vista, USA, 1994.

[14] W.S. Lee, B.D.O. Anderson, R.L. Kosut, and I.M.Y. Mareels. A new approach to adaptive control. *Int. Journal of Adaptive Contr. and Signal Processing,* Vol. *7*, pp. 183–211, 1993.

[15] L. Ljung. *System Identification: Theory for the User.* Prentice Hall Inc., Inform. and Syst. Sciences Series, Englewood Cliffs, 1987.

[16] MatLab. *Matlab User's Guide for the Optimization Toolbox, version 1.02.* The Mathworks Inc., South Natick, USA, 1994.

[17] D. McFarlane and K. Glover. *Robust Controller Design using Normalized Coprime Factor Plant Descriptions.* Springer Verlag, Berlin, 1990.

[18] R. Ober. Balanced parametrizations of classes of linear systems. *SIAM Journal on Contr. and Optimization,* Vol. *29*, No. *6*, pp. 1251–1287, 1991.

[19] R.J.P. Schrama. Accurate identification for control design: the necessity of an iterative scheme. *IEEE Trans. Autom. Contr., AC-37*, pp. 991–994, 1992.

[20] R.J.P. Schrama and O.H. Bosgra. Adaptive performance enhancement by iterative identification and control design. *Int. Journal of Adaptive Contr. and Signal Processing,* Vol. *7*, pp. 475–487, 1993.

[21] P.M.J. Van den Hof and R.J.P. Schrama. Identification and control – closed loop issues. In *Prepr. 10th IFAC Symp. on System Identification,* Vol. *2*, pp. 1–13, Denmark, 1994.

[22] P.M.J. Van den Hof, R.J.P. Schrama, R.A. de Callafon, and O.H. Bosgra. Identification of normalized coprime plant factors from closed loop experimental data. *European Journal of Contr.,* Vol. *1*, No. *1*, 1995.

[23] M. Vidyasagar. *Control System Synthesis: A Factorization Approach.* Cambridge, MIT Press, 1985.

[24] Z. Zang, R.R. Bitmead, and M. Gevers. Disturbance rejection: on-line refinement of controllers by closed loop modelling. In *Proc. Amer. Contr. Conf.*, pp. 2829–2833, Chicago, USA, 1992.