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Model Sets and Parametrizations for Identification of Multivariable Equation Error Models*

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Equation error model sets are required to satisfy specific restrictions in non-trivial identification problems. An extensive characterization of this freedom shows close relationships with the construction of uniquely identifiable (pseudo-canonical) parametrizations.

Key Words—System identification; identifiability; canonical forms; least squares; equation error models; linear regression; parametrization.

Abstract—Equation error (or linear regression) models are known to inherently require the *a priori* choice for specific signal variables to be considered as regressand and/or regressor. This implies that a model set should be—*a priori*—restricted in some way in order to define an acceptable identification problem. In the case of approximate identification (i.e. the system to be modelled is not contained in the model set), this restriction acts as a design variable, with the identified models being dependent on its specific choice.

In this paper the necessity of this restriction is quantified by the property of discriminability, i.e. the ability of an identification criterion to distinguish between all the different models in a model set. Employing a deterministic, signal-oriented framework, several sets of sufficient conditions are derived for model sets to be discriminable by a least squares identification criterion. To this end use is made of polynomial model representations in two shift operators.

Although it is of a different nature, the problem discussed is shown to be closely related to the problem of constructing identifiable parametrizations for sets of rational transfer functions. It is shown that the pseudo-canonical or overlapping parametrization of all transfer functions with fixed McMillan degree constitutes a nonoverlapping set of equation error models that is discriminable by a least squares identification criterion.

1. INTRODUCTION

THE USE OF EQUATION error models, often also denoted by linear regression models, is widespread in issues of modelling and identification of dynamical systems. The essential characteristic of the linear regression model is that a residual component e is defined which is a linear function of the unknown model coefficients. In the SISO (single-input single-output) situation we can write:

$$e(t,\theta) = a_0 y(t) + a_1 y(t-1) + \cdots + a_{n_a} y(t-n_a) - b_0 u(t) - b_1 u(t-1) - \cdots b_{n_b} u(t-n_b), \quad (1)$$

with $\theta = (a_0, \ldots, a_{n_a}, b_0, \ldots, b_{n_b})$ the unknown coefficient vector, $y(t)$ the output signal, and $u(t)$ the input signal of the model. With the restriction $a_0 = 1$, the above model is known as the ARX model (Ljung, 1987).

The use of these kinds of models in estimation and identification problems is essentially motivated by the fact that a least squares identification criterion, i.e. minimizing the sum of squared residuals $e(t)$ over the time-interval of interest, is a convex optimization problem that is analytically solvable, and thus leads to fast solutions for the coefficients to be estimated.

Without any additional restriction on the model structure (1) these equation error models however are unsuitable for using in a least squares identification problem. Note that without any restriction on the coefficient vector θ in (1), a least squares identification will yield the trivial solution $\hat{\theta} = 0$ and a corresponding least squares criterion $\frac{1}{N}\sum_{t=0}^{N-1} e^2(t,\hat{\theta}) = 0$, irrespective of the data $\{u(t), y(t)\}_{t=1,\ldots,N}$ that have been measured. Consequently some restriction has to

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be imposed on (1) in order to guarantee that the identification problem is a 'sensible' one. The restriction $a_0 = 1$ leads to the so-called ARX model structure, and an interpretation of $e(t, \theta)$ as a one-step-ahead prediction error. However there are many different choices of restrictions, as e.g. any other coefficient being fixed to one and many alternatives employing some kind of scaling (see e.g. De Moor *et al.*, 1991).

Now the question arises whether it is important which kind of restriction on (1) is chosen; the answer to this question is affirmative. As has been advocated by many people, e.g. Kalman (1982), De Moor and Vandewalle (1990), the identified least squares model is essentially dependent on the type of restriction that is chosen. Formulated in terms of linear regression: the outcome of a least squares optimization is dependent on which variable has been chosen to be the regressand and which variables determine the regressors. This means that different choices of restrictions not only lead to different values of the estimated parameter $\hat{\theta}$, but also to different dynamics in the identified model, as e.g. reflected in its i/o transfer function

$$H_{yu}(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_{n_b} z^{-n_b}}{a_0 + a_1 z^{-1} + \cdots + a_{n_a} z^{-n_a}}.$$

When this choice of restriction is essentially influencing the dynamics of the identified model, this phenomenon should be considered as a design variable to be chosen with care. An interesting question now becomes which kind of restrictions are necessary and/or sufficient to guarantee a—what could be called—sensible identification problem. In this paper we will address the following problem.

Consider a set of multivariable equation error models, e.g. of the form,

$$e(t, \theta) = P_\theta(\sigma, \sigma^{-1})y(t) - Q_\theta(\sigma, \sigma^{-1})u(t), \quad (2)$$

with $\theta \in \Theta \subset \mathbb{R}^d$, P, Q polynomial matrices in two indeterminates where σ, σ^{-1} reflect the forward, c.q. backward shift operator. (Further notation will be specified in the next sections.)

Problem (i). How can we formalize and quantify the property of the model set that it induces a 'sensible' identification problem, i.e. that a least squares optimization does not always deliver a trivial solution, as discussed above; and

Problem (ii). Which conditions on a model set are sufficient to guarantee a 'sensible' identification problem, as meant in Problem (i).

In providing a solution for Problem (i) we will employ the notion of discriminability of a model set, as introduced in Van den Hof (1989a). This notion appears to be appropriate for formalizing the situation as described above. It will be discussed in Section 4. In Section 5 we will present different solutions to Problem (ii).

There is another problem that is closely related to Problem (ii) but that is of quite a different nature. It will appear to be important to distinguish it from Problem (ii).

Problem (iii). Under which conditions on a set of polynomial matrices $\{P_\theta, Q_\theta, \theta \in \Theta \subset \mathbb{R}^d\}$, will two different values of θ lead to two different i/o transfer functions, i.e.

$$(P_{\theta_1}^{-1} Q_{\theta_1} = P_{\theta_2}^{-1} Q_{\theta_2}) \Rightarrow \theta_1 = \theta_2. \quad (3)$$

This problem is studied extensively in the literature, directed towards the construction of identifiable parametrizations of dynamical i/o systems and of constructing sets of canonical forms for polynomial matrices, see e.g. Guidorzi (1981), Corrêa and Glover (1984), Gevers and Wertz (1987). Problem (iii) is not directly related to a least squares identification problem. It is a parametrization-type problem which in this paper will be clearly distinguished from Problem (ii), which is a problem of constructing model sets. Although Problems (ii) and (iii) are of a different nature, in some situations their solutions appear to be closely related. We will pay attention to this in Section 5, where existing solutions to problem (iii) are shown to constitute solutions to Problem (ii), and new solutions to (ii) appear to constitute generalized solutions to Problem (iii).

In Section 6 we will discuss some invariance properties of identified models under different choices of model sets. In other words, we will formulate conditions under which two different model sets will lead to essentially the same identified models. Especially in the multivariable case, Problems (i) and (ii) are not often addressed in the identification literature. This is due to the fact that in the classical statistical literature the additional assumption is often made that the residual signal $\{e(t)\}$ is a white noise signal, which causes the identified model to be invariant under different choices of restrictions. However, in almost all situations it appears to be rather unrealistic to assume that we can construct models on the basis of measurement data in such a way that the residual signal is a white noise signal (i.e. the data generating system is assumed to be in the model set). Therefore in approximate identification, Problems (i) and (ii) become specifically relevant.

Equation error models have been analyzed before in the context of approximate identification of dynamical systems. Mullis and Roberts

(1976), Inouye (1983) and Van den Hof and Janssen (1987) have shown properties of least squares identified equation error models in an approximative sense, formulated in terms of the Markov parameters of the models. For single-input, single-output systems, a frequency domain formulation of properties in the approximative situation is given in Wahlberg and Ljung (1986), while De Moor, Gevers and Goodwin (1991) give frequency domain results for the consequences of choosing different coefficient restrictions.

In this paper we will adopt part of the signal-based framework of Willems (1986, 1988) for this discussion, as this framework is especially appropriate for formulating the problem of identification as well as of clearly distinguishing the problems of identification (cf. Problem (ii)) and parametrization (cf. Problem (iii)). This refers to a clear distinction between choices that really affect the identified models (choice of the model set), and choices that only refer to matters of representation (choice of parametrization). We will use a generalized form of polynomial matrices to parametrize systems, dealing with both forward and backward shift operators. This enables us to use one framework for discussing the problems of parametrization, often stated in terms of polynomial matrices in the forward shift operator, and problems of identification, often stated in terms of polynomial matrices in the backward shift operator.

The main contribution of this paper appears to be the formal characterization of the freedom that is present in the construction of equation error model sets, and the formulation of two sets of sufficient conditions for arriving at model sets that are discriminable by a least squares identification criterion. Additionally the close relationship with parametrization problems is shown, e.g. in the interesting result that the set of polynomial matrices that constitute a collection of pseudo-canonical (overlapping) parametrization of all transfer functions with prespecified McMillan degree, actually constitutes a set of non-overlapping discriminable model sets when considered as multivariable equation error models in identification.

Preliminary work on the subject of this paper has been published in Van den Hof (1989a–d), while the reader is also referred to Janssen (1988), where similar problems are touched upon but from a different point of view. Structural properties of systems represented by polynomial matrices in two shift operators are discussed in Van den Hof (1992).

The proofs of all results are collected in an appendix.

2. NOTATION

In order to be able to deal with both forward and backward time shift operations in one model representation, we have to consider polynomial matrices in two indeterminates, (sometimes called binomials). Consider a polynomial matrix $T \in \mathbb{R}^{p\times q}[z, z^{-1}]$. We will denote:

$T_{i*}(T_{*j})$:= the ith row (jth column) of T;

$\nu_i^{(u)}(T)(\nu_i^{(l)}(T))$:= the maximum (minimum) power of z in T_{i*}, upper (lower) row degree;

$\mu_i^{(u)}(T)(\mu_i^{(l)}(T))$:= the maximum (minimum) power of z in T_{*i}, upper (lower) column degree;

$\Gamma_{hr}(\Gamma_{lr})$:= the leading (trailing) row coefficient matrix of T, i.e. the coefficient matrix related to the highest (lowest) row degree terms in T;

$\Gamma_{hc}(\Gamma_{lc})$:= the leading (trailing) column coefficient matrix of T;

$\Gamma_{c,(n_1,\ldots,n_q)}(T)$:= the column coefficient matrix of T related to the column degrees $n_1, \ldots, n_q$.

Note that the integer indices $\nu_i^{(u)}$, $\nu_i^{(l)}$, $\mu_i^{(u)}$ and $\mu_i^{(l)}$ are either positive or negative and that $\nu_i^{(u)} \geq \nu_i^{(l)}$, and $\mu_i^{(u)} \geq \mu_i^{(l)}$.

We will use $\mathbf{Z}$ for the set of integer numbers, $\mathbb{R}(z)$ for the field of rational functions, $\mathbb{R}[z, z^{-1}]$ for the ring of polynomials in two indeterminates. $T \in \mathbb{R}^{p\times p}[z, z^{-1}]$ is unimodular if its inverse is polynomial, i.e. $\det T = cz^d$, with $c \neq 0$ and $d \in \mathbf{Z}$. For polynomial or rational matrices T, the notation $\det T$, and rank T, will refer to the determinant and the rank of T taken over the field of rational functions $\mathbb{R}(z)$, while $\|T\|_2$ refers to the L_2-norm of T, i.e. $1/2\pi \int_{-\pi}^{\pi} T(e^{j\omega})^* T(e^{j\omega})\,d\omega$, and $(\cdot)^*$ the complex conjugate transpose. The notation $M_1 = \tilde{M}_p(T_1)$ refers to the dynamical system M_1 that is being induced by the polynomial matrix (autoregressive) representation T_1 (see Section 3). The shift operators σ, σ^{-1} are defined by: $(\sigma w)(t) = w(t+1)$, $t \in \mathbf{Z}$ and $(\sigma^{-1})w(t) = w(t-1)$, $t \in \mathbf{Z}$.

3. DYNAMICAL SYSTEMS AND EQUATION ERROR MODELS

In common terms, the typical character of an equation error model is that it has a form as in (2) with P, Q being polynomial, or alternatively:

$$y(t) = P^{-1}Qu(t) + P^{-1}e(t) \tag{4}$$

a typical property being that the transfer functions $H_{yu}(z) = P^{-1}Q$ and $H_{ye}(z) = P^{-1}$ have common dynamics.

Having specified the number of inputs (m), outputs (p) and residuals (p), we will refer to a model M as determined by its behaviour $\mathscr{B}(M)$ being defined as the collection of signal trajectories $\{w(t)\}_{t\in\mathbf{Z}} := \{y(t), u(t), e(t)\}_{t\in\mathbf{Z}}$ that are admissible by the model, see Willems (1986). The class of models $\hat{\Sigma}_{p,m}$ is the class of input–output-processing-residual (i/o/pr)-models (Van den Hof, 1989a, b), characterized by the fact that for any $M \in \hat{\Sigma}_{p,m}$ there exists a full row rank polynomial matrix $T = [P \mid -Q \mid -R] \in \mathbb{R}^{p\times(p+m+p)}[z, z^{-1}]$ with $\det P \neq 0$, and $\det R \neq 0$ such that the behaviour of the model is determined by all trajectories $\{w(t)\}$ that satisfy $T(\sigma, \sigma^{-1})w = 0$, or equivalently

$$P(\sigma, \sigma^{-1})y - Q(\sigma, \sigma^{-1})u - R(\sigma, \sigma^{-1})e = 0. \quad (5)$$

Two models are equal, i.e. their behaviours are the same, if and only if the corresponding polynomial matrices T are related through unimodular premultiplication. This brings us to the formal definition of an equation error model, as in (2).

Definition 3.1. An equation error model is an (i/o/pr)-model that satisfies the additional property that the residual e is observable from (y, u), or in other words:

$$\{(y, u, e_1) \in \mathscr{B}(M) \wedge (y, u, e_2) \in \mathscr{B}(M)\} \Rightarrow \{e_1 = e_2\}. \quad \square$$

The notion of observability of signals is defined in Willems (1986).

The definition implies that in the polynomial representation, the matrix R is restricted to be unimodular. Since unimodular premultiplication does not change the behaviour, this means that a linear regression model can always be represented by a full row rank polynomial matrix $T = [P \mid -Q \mid -I_p]$, with I_p the $p \times p$ identity matrix, as presented in (2). The set of all equation error models within $\hat{\Sigma}_{p,m}$ will be denoted by $\bar{\Sigma}_{p,m}$.

Taking a closer look at the polynomial matrix T that induces a model in the class $\hat{\Sigma}_{p,m}$, it follows that T induces two rational matrices, that can be considered as transfer functions of the corresponding model. These two transfer functions are denoted by:

$$H_y(z) = [H_{yu}(z) \mid H_{ye}(z)] := [P^{-1}Q \mid P^{-1}R], \text{ and} \quad (6)$$

$$H_e(z) = [H_{ey}(z) \mid H_{eu}(z)] := [R^{-1}P \mid -R^{-1}Q]. \quad (7)$$

The behaviour of an equation-error model $M \in \bar{\Sigma}_{p,m}$ is completely characterized by either one of the two transfer functions H_y, H_e. Note that for equation error models, the transfer function $H_e(z)$ is polynomial.

The i/o-part of an (i/o/pr)-model will also be specified as $\mathscr{B}^{io}(M)$ being defined as the collection of all signal trajectories $\{v(t)\} := \{y(t), u(t)\}$ that are admissible when $e = 0$. If a model M is defined by the behaviour $\mathscr{B}(M)$, then its i/o-part M^{io} is defined by $\mathscr{B}^{io}(M)$. For evaluation of the i/o-part of a (i/o/pr)-model, the residual component in the model is simply discarded. We will denote the notion of i/o-equivalence, $M_1 \underset{io}{\sim} M_2$ as $\mathscr{B}^{io}(M_1) = \mathscr{B}^{io}(M_2)$. We will also refer to the so called i/o-transfer-equivalence relation, $M_1 \overset{t}{\underset{io}{\sim}} M_2$, being defined by $H_{yu}^{(1)} = H_{yu}^{(2)}$. M^{io} is controllable (Willems, 1988) if the corresponding matrix $[P \mid -Q]$ is left coprime with respect to $\mathbb{R}[z, z^{-1}]$, i.e. $\operatorname{rank}[P(\lambda, \lambda^{-1}) \mid Q(\lambda, \lambda^{-1})] = p$ for all $\lambda \in \mathbf{C}\setminus\{0\}$.

Restrictions of signal variables and behaviours to the time set $\mathbf{Z}_+$ will be denoted by w^+ and $\mathscr{B}^+$.

4. MODELLING ON THE BASIS OF DATA

In the problem of modelling dynamical systems on the basis of input–output data, three central aspects have to be distinguished:

(1) the set of models $\mathscr{M}$ that is considered;
(2) a parametrization $\bar{M}$, representing the models in the model set with (real valued) parameters; and
(3) an identification criterion J that selects 'best' or 'optimal' models from the set of models, given the measured data.

Given the measurement data, the models that are finally obtained as a result of the modelling procedure should be determined by the set of models taken into account and by the identification criterion, and should not be dependent on other choices, like e.g. the parametrization of the model set. The parametrization acts as a tool for representing the models by real-valued parameter values in order to apply identification algorithms. In this paper we will consider as a set of models $\mathscr{M}$ a set of equation error models in $\bar{\Sigma}_{p,m}$. Note that a model is characterized in terms of its behaviour. A parametrization of a model set $\mathscr{M}$ is defined as a surjective mapping $\bar{M}: \Theta \to \mathscr{M}$, with $\Theta \subset \mathbb{R}^d$ the parameter set, and $\mathscr{M}$ the parametrized set of models. Parametrizations will be considered in terms of the polynomial representations T and/or H_e discussed before, with parameters being defined through the coefficients of the polynomials.

As identification criterion we consider the standard least squares criterion

$$J(v^+, \mathcal{M}) = \arg\min_{M \in \mathcal{M}} l(e^+) \quad \text{with}$$

$$l(e^+) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=0}^{N-1} e^T(t)e(t), \quad (8)$$

where the minimization is done over all residual signals e^+ that together with v^+ constitute an admissible trajectory $w^+ \in \mathcal{B}^+(M)$.

Actually, the identification criterion is defined as a selection rule. Given a time series and a model set, the criterion J selects one or more 'optimal' models from the model set. Note that J contains models (i.e. behaviours) and not estimated parameters.

Now referring to Problem (i) in Section 1, we want to formalize the situation that a model set $\mathcal{M}$ and the identification criterion J together yield a sensible identification problem. Analysing the example shown in Section 1, it appeared that without any additional restrictions on $\mathcal{M}$, the identification result $J(v^+, \mathcal{M})$ would be independent of v^+, a situation which is of course highly undesirable. Apparently what happens in that case is that the different models in the model set cannot be distinguished by the identification criterion. The ability to distinguish the different models in a model set during identification experiments will now be formalized through the notion of discriminability, as was introduced in Van den Hof (1989a).

Definition 4.1. A model M is called discriminable within $\mathcal{M}$ by an identification criterion J, defined on $\mathcal{M}$, if there exists a data sequence v^+ such that $J(v^+, \mathcal{M}) = \{M\}$. □

When all models in a model set $\mathcal{M}$ are discriminable, the model set will be called discriminable by J, and J will be called discriminating on $\mathcal{M}$. An identification criterion that is discriminating on a model set can distinguish between the different models in this set. In general terms it can be stated that if a model set is not discriminable by J, discriminability can be obtained by making a restriction to a discriminable subset $\mathcal{M}_1 \subset \mathcal{M}$. The main goal of this paper is to discuss how to construct model sets that have this discriminability property (cf. Problem (ii)).

Remark 4.2. The concept of discriminability is closely related to the more conventional concept of system identifiability. However system identifiability very often refers to consistency properties of identification methods. In order to stress that we consider a different non-statistical situation, we have chosen to denote the property of discriminability.

Remark 4.3. The definition of a discriminable model as presented in Definition 4.1 also shows the aspect of experimental conditions under which the model is discriminable. In Van den Hof (1991) the notion of discriminability under closed-loop experimental conditions is discussed.

It has to be stressed that the concept of discriminability, as discussed, is a property of a model set in conjunction with an identification criterion; it is not related to any parametrization of the set of (i/o/pr)-models. It can be illustrated that a restriction like $a_0 = 1$ on (1) is a matter of restricting the model set, rather than a matter of parametrization, by realizing that (a) each model is uniquely determined by the transfer function $H_e(z)$ as denoted in (7), and (b) $a_0 = 1$ directly restricts the transfer function $H_{ey}(z)$ within $H_e(z)$.

Remark 4.4. One could argue whether discriminability is necessary for obtaining a sensible identification problem. Without going into a detailed discussion, at this moment we note that we use discriminability as a sufficient condition for avoiding the undesirable situation of identifying $\hat{\theta} = 0$ irrespective of the measured data, as discussed in the introduction.

5. DISCRIMINABILITY OF EQUATION ERROR MODELS IN LS-IDENTIFICATION

5.1. *General results*

Having formalized the notion of discriminability, we have constructed a solution to Problem (i). Consequently Problem (ii) now becomes simply the problem of finding sufficient conditions on a model set $\mathcal{M}$ to guarantee that it is discriminable by the least squares identification criterion J. In the next two theorems we will formulate two different sets of sufficient conditions for discriminability of model sets. In two subsequent sections we will analyze the consequences of both types of conditions.

Theorem 5.1. Let $\mathcal{M} \subset \bar{\Sigma}_{p,m}$ be a set of equation error models. If for all $M_1, M_2 \in \mathcal{M}$:

$$M_1 \overset{t}{\underset{io}{\sim}} M_2 \Rightarrow M_1 = M_2,$$

then $\mathcal{M}$ is discriminable by J. □

The theorem shows that absence of distinct models that are i/o transfer-equivalent, is sufficient for guaranteeing discriminability of the model set by J. In other words, a sufficient condition for $\mathcal{M}$ to be discriminable by J, is that $\mathcal{M}$ is a set of canonical forms under the equivalence relation $\overset{t}{\underset{io}{\sim}}$. Note that this equivalence relation $M_1 \overset{t}{\underset{io}{\sim}} M_2$ is defined by $H_{yu}^{(1)}(z) = H_{yu}^{(2)}(z)$ or equivalently $P_1^{-1}Q_1 = P_2^{-1}Q_2$ in any full row rank polynomial representation of M_1, M_2.

A second set of sufficient conditions, being less restrictive than the condition of Theorem 5.1, is formulated in the following theorem.

Theorem 5.2. Let $\mathcal{M} \subset \bar{\Sigma}_{p,m}$ be a set of equation error models. Then $\mathcal{M}$ is discriminable by J if for any two models $M_1, M_2 \in \mathcal{M}$ satisfying $M_1 \overset{t}{\underset{io}{\sim}} M_2$ and $H_{e_2e_1} := H_{ey}^{(2)}(H_{ey}^{(1)})^{-1}$ a stable rational matrix*, it holds that

$$\|H_{e_2e_1}(z)\|_2 \geq p$$

with equality if and only if

$$M_1 = M_2.$$

The sufficient condition stated in this theorem is less restrictive than the condition of Theorem 5.1; whereas in Theorem 5.1 distinct i/o-transfer-equivalent models are abandoned, in Theorem 5.2 restrictions are formulated on distinct i/o-transfer-equivalent models. When discussing the consequences of these theorems, we will most often deal with the polynomial transfer functions $H_e(z)$ of the models, see (7). Note that this polynomial transfer function uniquely relates to (the behaviour of) the model.

In the next two subsections we will analyze the consequences of the sufficient conditions as formulated in the above theorems separately.

5.2. *Removing i/o-transfer-equivalent models*

The most straightforward way to construct model sets that satisfy the conditions of Theorem 5.1 is to restrict to models that have the property that M^{io} is controllable. In that case, $M_1 \overset{t}{\underset{io}{\sim}} M_2$ implies $M_1^{io} = M_2^{io}$, being a first step towards achieving the conditions of the theorem. This leads to the following corollary.

Corollary 5.3. Let $\mathcal{M}$ be a set of equation error models, $\mathcal{M} \subset \bar{\Sigma}_{p,m}$, such that for all models $M \in \mathcal{M}$, M^{io} is controllable. Then $\mathcal{M}$ is discriminable by J if for all $M_1, M_2 \in \mathcal{M}$ with corresponding transfer functions $H_e^{(1)}, H_e^{(2)} \in \mathbb{R}^{p\times(p+m)}[z, z^{-1}]$:

$$\{H_e^{(1)} = UH_e^{(2)} \text{ with } U \in \mathbb{R}^{p\times p}[z, z^{-1}], \text{ unimodular}\} \Rightarrow \{U = I\}. \quad (9)$$

* A rational matrix is called stable if it is analytic in $|z| \geq 1$, except possibly in $z = \infty$.

The corollary clearly reflects the type of problem that is concerned. In this setting the construction of a discriminable model set comes down to the construction of a set of canonical forms for polynomial matrices based on the equivalence relation of unimodular premultiplication (unimodularity here has to be considered with respect to the ring $\mathbb{R}^{p\times(p+m)}[z, z^{-1}]$). This directly refers to a parametrization problem as Problem (iii), as discussed in Guidorzi (1981) and Gevers and Wertz (1987) for polynomials in one indeterminate, and in Heij (1989) for the binomial case. However note again that the problem of constructing discriminable model sets (Problem (ii)) as such is not a parametrization problem, and $J(v^+, \mathcal{M})$ will essentially be dependent on the choice of $\mathcal{M}$.

Next we will formulate a set of conditions on an equation error model set that guarantees discriminability according to Corollary 5.3. Moreover the condition will appear to generalize the results known from the literature as solutions to the parametrization-type Problem (iii), as will be shown later on. The conditions on the model set again are formulated in terms of the polynomial transfer functions $H_e(z)$ that uniquely represent the models.

Corollary 5.4. Let $\mathcal{M}$ be a set of equation error models, $\mathcal{M} \subset \bar{\Sigma}_{p,m}$, such that for all models $M \in \mathcal{M}$, M^{io} is controllable.

Then $\mathcal{M}$ is discriminable by J if there exist polynomial matrices $K \in \mathbb{R}^{(p+m)\times q_1}[z, z^{-1}]$, $N \in \mathbb{R}^{(p+m)\times q_2}[z, z^{-1}]$, and $L \in \mathbb{R}^{(p+m)\times q_3}[z, z^{-1}]$, and integers $m_1, \ldots, m_{q_1}$, $n_1, \ldots, n_{q_2}$, with $q_1, q_2, q_3 \geq p$, such that for all $M \in \mathcal{M}$ with corresponding transfer functions H_e the following conditions are satisfied:

(i) $m_j \geq \mu_j^{(u)}(H_eK)$, and rank $\Gamma_{c,(m_1,\ldots,m_{q_1})}(H_eK) = p$;
(ii) $n_j \leq \mu_j^{(l)}(H_eN)$, and rank $\Gamma_{c,(n_1,\ldots,n_{q_2})}(H_eN) = p$; and
(iii) $H_eL = \sum_{k=s}^{t} G_k z^k$, for some $s \leq t$, and G_0 has rank p and is equal for all $M \in \mathcal{M}$. □

In this corollary there are three types of conditions on $H_e(z)$ that have to be satisfied; one rank condition on an upper column degree matrix (i), one rank condition on a lower column degree matrix (ii), and a surjective coefficient matrix that has to be fixed over the model set (iii). The prespecified polynomial matrices K, N, L determine the way in which the three conditions are enforced on $H_e(z)$.

Note that in the situation $q_1 = p$, implying that H_eK is a square matrix, the integers $m_1, \ldots, m_p$

are the upper column degrees of H_eK. Similarly in the situation $q_2 = p$, H_eN is square, and the integers $n_1, \ldots, n_p$ are the lower column degrees of H_eN.

Specific choices for matrices K and N can be made, e.g. in order to select columns within H_e on which the restrictions formulated in conditions (i), (ii) should apply to. Consider for instance $K = N = \begin{bmatrix} I_p \\ 0 \end{bmatrix}$, leading to the situation that the restrictions operate on the polynomial matrices H_{ey}. Condition (iii) states that one of the coefficient matrices (Markov parameters) of H_eL should be fixed to a (prespecified) surjective matrix.

Next we will show that several solutions to Problem (iii) that are known in the literature occur as special cases in this corollary, through specific choices of matrices K, L and N.

(1) Monic (full polynomial) ARMA form, or prescribed maximum lag form (e.g. Hannan, 1969; Deistler, 1983).

These identifiable parametrizations in terms of polynomial matrices in the backward shift operator, concern sets of polynomial matrices $H_e = [P \mid Q] \in \mathbb{R}^{p\times(p+m)}[z^{-1}]$, with

$$\begin{aligned} P &= P_0 + P_1z^{-1} + \cdots + P_rz^{-r}, \\ Q &= Q_0 + Q_1z^{-1} + \cdots + Q_rz^{-r} \end{aligned} \tag{10}$$

and $P_0 = I$, $\det P \neq 0$ and (P, Q) left coprime with respect to $\mathbb{R}[z^{-1}]$.

Condition (i) of the corollary is satisfied with $K = \begin{bmatrix} I_p \\ 0 \end{bmatrix}$, $m_i = \mu_i^{(u)}(P) = 0$ and $\operatorname{rank}\Gamma_{c,(0,\ldots,0)}(P) = \operatorname{rank} P_0 = p$.

Condition (ii) is satisfied with $N = I_{p+m}$, and an additional condition on the—so called—column end matrix, formulated as

$$\operatorname{rank}\Gamma_{c,(n_1,n_2,\ldots,n_{p+m})}(H_e) = p \quad \text{for} \quad n_j = \mu_i^{(l)}(H_e).$$

Condition (iii) is satisfied with $L = \begin{bmatrix} I_p \\ 0 \end{bmatrix}$, since $P = \sum_{k=0}^{r} P_kz^{-k}$ and $P_0 = I$ is fixed over the model set.

(2) Canonical and pseudo-canonical (overlapping) observability forms (Guidorzi, 1981; Gevers and Wertz, 1987).

These identifiable parametrizations in terms of polynomial matrices in the forward shift operator, concern sets of polynomial matrices $H_e = [P \mid Q] \in \mathbb{R}^{p\times(p+m)}[z]$, with

$$\begin{aligned} P_{ij} &= p_{ij,r_{ij}}z^{r_{ij}} + \cdots + p_{ij,0} \\ Q_{ik} &= q_{ik,s_{ik}}z^{s_{ik}} + \cdots + q_{ik,0}, \end{aligned} \tag{11}$$

r_{ij}, $s_{ik} \geq 0$, and $\det P \neq 0$, (P, Q) left coprime with respect to $\mathbb{R}[z]$, and additionally $\Gamma_{hc}(P) = I$. The degrees of the polynomial entrees in P, Q satisfy the following rules:

(a) Canonical observability form:

$$\begin{aligned} r_{ij} &= \min(r_{ii}+1, r_{jj}) \quad i > j \\ &= \min(r_{ii}, r_{jj}) \quad i \leq j \\ s_{ik} &= r_{ii} + 1. \end{aligned}$$

(b) Pseudo-canonical (overlapping) form:

$$\begin{aligned} r_{ij} &= r_{jj} \\ r_{\max} &:= \max_i (r_{ii}) \\ s_{ik} &= r_{\max} + 1 \quad \text{if } r_{ii} = r_{\max} \\ &= r_{\max} \quad \text{if } r_{ii} < r_{\max}. \end{aligned} \tag{12}$$

The conditions of the corollary can be shown to be satisfied as follows:

(i) With $K = \begin{bmatrix} I_p \\ 0 \end{bmatrix}$, $m_j = \mu_j^{(u)}(P) = r_{jj}$ and $\Gamma_{c,(m_1,\ldots,m_p)}(P) = \Gamma_{hc}(P) = I$.

(ii) With $N = I_{p+m}$ and $n_j = 0 \leq \mu_j^{(l)}(H_e)$, this condition reads $\operatorname{rank}[P(z) \mid Q(z)]_{z=0} = p$, which is ascertained by left coprimeness of (P, Q).

(iii) With $L = [\operatorname{diag}\{z^{-r_{jj}}, j = 1, \ldots, p\}\ 0]^T$ it follows that $H_eL = \sum_{k=-d}^{0} G_kz^k$ for some $d > 0$, with $G_0 = \Gamma_{hc}(P) = I$.

Any set of polynomial matrices $H_e \in \mathbb{R}^{p\times(p+m)}[z, z^{-1}]$ that satisfies the conditions of Corollary 5.4 constitutes a set of canonical forms under the equivalence relation of unimodular premultiplication. Exactly this property is employed by the two classes of identifiable parametrizations mentioned above. In these identifiable parametrizations a set $\mathcal{T}$ of polynomial matrices $[P \mid Q]$ is determined in order to uniquely parametrize a set $\mathcal{G}$ of i/o transfer functions $G = P^{-1}Q$. The set $\mathcal{T}$ constitutes a set of canonical forms, and in terms of the parametrization problem (Problem (iii)) any other set of polynomial matrices that uniquely parametrizes the same set $\mathcal{G}$ through an alternative set of canonical forms, would solve the same problem. However in terms of the construction of discriminable model sets (Problem (ii)) any two different sets of polynomial matrices satisfying the conditions of Corollary 5.4 constitute different discriminable model sets and as a result they will generally lead to different identified models $J(v^+, \mathcal{M})$.

Remark 5.5. Conditions (ii), (iii) of Corollary 5.4 have been formulated in terms of column degree and column coefficient properties of polynomial matrices. Similar statements can also be made based on row degree and row coefficient properties, however only in the situation that all upper (c.q. lower) row degrees are equal.

5.3. *Restricting i/o-transfer-equivalent models*

When we analyze the results of Theorem 5.2, we come to a second set of sufficient conditions for obtaining discriminability. We will first present the formal corollary, and subsequently discuss its implications.

Corollary 5.6. Let $\mathcal{M}$ be a set of equation error models, $\mathcal{M} \subset \bar{\Sigma}_{p,m}$. Then $\mathcal{M}$ is discriminable by J if there exist polynomial matrices $K, N \in \mathbb{R}^{(p+m)\times q}[z, z^{-1}]$, $q \geq p$, such that for all $M_1, M_2 \in \mathcal{M}$ with corresponding polynomial transfer functions $H_e^{(1)}$, $H_e^{(2)}$, either one of the following conditions (i) to (iv) is satisfied.

(i) (a) $\Gamma_{hr}(H_e^{(1)}K) = \Gamma_{hr}(H_e^{(2)}K)$, having rank p, and
(b) $\nu_i^{(u)}(H_e^{(1)}K) = \nu_i^{(u)}(H_e^{(2)}K)$
for $i = 1, \ldots, p$;

(ii) (a) $\Gamma_{hr}(H_e^{(1)}K) = \Gamma_{hr}(H_e^{(2)}K)$, having rank p, and
(b) $\sum_{i=1}^{p} \nu_i^{(u)}(H_e^{(1)}K) = \sum_{i=1}^{p} \nu_i^{(u)}(H_e^{(2)}K)$; and
(c) $\nu_i^{(l)}(H_e^{(1)}N) = \nu_i^{(l)}(H_e^{(2)}N)$
for $i = 1, \ldots, p$, and
(d) $\operatorname{rank} \Gamma_{lr}(H_e^{(1)}N) = \operatorname{rank} \Gamma_{lr}(H_e^{(2)}N) = p$;

(iii) there exist integers $m_1, \ldots, m_q$, satisfying $m_j \geq \mu_j^{(u)}(H_eK)$, $j = 1, \ldots, q$, for all H_e, such that
$\Gamma_{c,(m_1,\ldots,m_q)}(H_e^{(1)}K) = \Gamma_{c,(m_1,\ldots,m_q)}(H_e^{(2)}K)$, having rank p;

(iv) (a) $q = p$;
(b) $\Gamma_{hc}(H_e^{(1)}K) = \Gamma_{hc}(H_e^{(2)}K)$, having rank p, and
(c) $\sum_{i=1}^{p} \mu_i^{(u)}(H_e^{(1)}K) = \sum_{i=1}^{p} \mu_i^{(u)}(H_e^{(2)}K)$; and
(d) $\nu_i^{(l)}(H_e^{(1)}N) = \nu_i^{(l)}(H_e^{(2)}N)$, for $i = 1, \ldots, p$ and
(e) $\operatorname{rank} \Gamma_{lr}(H_e^{(1)}N) = \operatorname{rank} \Gamma_{lr}(H_e^{(2)}N) = p$. □

Especially if we take a closer look at situations (i) and (iii) of the corollary, the conditions that have to be imposed on the model set are less restrictive than the conditions formulated in the previous subsection. In situations (ii) and (iv) even the row (column) degrees do not have to be specified, only the sum of the row ((ii)b) or column ((iv)c) degrees. Moreover note that, in contrast with the situation of the previous subsection, it is not required that M^{io} is controllable for all models in the model set.

An additional difference between the conditions of Corollaries 5.4 and 5.6, is that in Corollary 5.4 a surjective matrix is fixed (part (iii)) that is one of the coefficient matrices of a polynomial matrix, whereas in Corollary 5.6 the matrix to be fixed is specifically required to be a surjective leading row ((i)a,(ii)a) or column ((iii),(iv)b) coefficient matrix.

This mechanism is briefly illustrated in the following example.

Example 5.7. Consider two equation error models M_1, M_2 induced by $H_e^{(1)}(z)$, $H_e^{(2)}(z)$, with

$$H_e^{(1)}(z) = [(z-a)(z-c) \mid k(z-b)(z-c)] \quad (13)$$

and

$$H_e^{(2)}(z) = [(z-a)(z-d) \mid k(z-b)(z-d)], \quad (14)$$

with $a, b, c, d, k \in \mathbb{R}\setminus\{0\}$. Consequently

$$H_{yu}^{(1)}(z) = H_{yu}^{(2)}(z) = k\frac{z-b}{z-a}.$$

Note that, since there appear common factors in $H_e^{(i)}(z)$, M_1^{io}, M_2^{io} are not controllable. It follows that $H_{e_2e_1}(z) = (z-d)/(z-c)$. Since the highest powers of z in $H_{ey}^{(1)}$, $H_{ey}^{(2)}$ have been prespecified to one, it follows that $H_{e_2e_1}(z)$ is proper with $\lim_{z\to\infty} H_{e_2e_1} = 1$. Consequently $\|H_{e_2e_1}\|_2 \geq 1$. It is easily verified that e.g. conditions (i) of the Corollary are satisfied with $K = [1\ 0]^T$. If, alternatively, the coefficient of any of the other powers of z had been fixed in H_{ey}, as e.g. the coefficient of z^0 being related to the lower column/row degree, then $\|H_{e_2e_1}\|_2 \geq 1$ would not be guaranteed, and lack of discriminability would be possible.

An implication of this result is that for an equation error model set (1), discriminability can simply be obtained for the restrictions $a_0 = 1$ or $b_0 = 1$, but that for a restriction as e.g. $a_1 = 1$ additionally controllability of M^{io} has to be required, i.e. pole-zero cancellations in the transfer function $H_{yu}(z)$ have to be excluded. □

We specifically want to pay some more attention to situation (iv) of the corollary. To this end we isolate a set of restrictions (Set A) that is the result of the choices $K = \begin{bmatrix} I_p \\ 0 \end{bmatrix}$, $N = I_{p+m}$ in (iv). We are going to compare this set with a set $\mathcal{T}$ of polynomial matrices $[P(z) \mid Q(z)]$ that represents the pseudo-canonical (overlapping) parametrization of all proper i/o transfer functions $P^{-1}Q$ having a prespecified McMillan degree n, as discussed e.g. in Beghelli and Guidorzi (1983), Corrêa and Glover (1984), Gevers and Wertz (1984, 1987).

This second set of polynomial matrices is denoted as Set B.

Set A	Set B
• $H_e = [H_{ey} \mid H_{eu}] \in \mathbb{R}^{p\times(p+m)}[z]$	• $T_{PQ} = [P \mid Q] \in \mathbb{R}^{p\times(p+m)}[z]$
• $\Gamma_{hc}(H_{ey}) = I$	• $\Gamma_{hc}(P) = I$
• $\sum_{i=1}^{p} \mu_i^{(u)}(H_{ey}) = n$, fixed	• $\sum_{i=1}^{p} \mu_i^{(u)}(P) = n$, fixed
• $v_i^{(l)}(H_e) = 0$, fixed	• (P, Q) left coprime with respect to $\mathbb{R}[z]$
• rank $\Gamma_{lr}(H_e) = p$.	• $P^{-1}Q$ proper.

It can be verified that left coprimeness of (P, Q) with respect to $\mathbb{R}[z]$ implies that rank $[P(z) \mid Q(z)]_{z=0} = p$ which implies the last two conditions of Set A, i.e. $v_i^{(l)}(H_e) = 0$, and rank $\Gamma_{lr}(H_e) = p$. As a result the two sets of polynomial matrices, $\{H_e\}_{\text{Set A}}$ and $\{T_{PQ}\}_{\text{Set B}}$ as described above, are related as $\{T_{PQ}\}_{\text{Set B}} \subset \{H_e\}_{\text{Set A}}$. Consequently the set of polynomials that constitute the overlapping parametrization induce a discriminable set of equation error models. Note also that in the overlapping parametrization it requires the choice of a specific set of (pseudo-observability-) integer indices $\mu_1^{(u)}, \ldots, \mu_p^{(u)}$ summing up to n, to construct a parametrization that is uniquely identifiable. The finite collection of all sequences $\mu_1^{(u)}, \ldots, \mu_p^{(u)}$ that sum up to n then constitute the (overlapping) parametrization of all proper i/o transfer functions with McMillan degree n. In $\{H_e\}_{\text{Set A}}$ every separate element induces an equation error model that can be discriminated from all the other models in the set. In other words, when considering $\{T_{PQ}\}_{\text{Set B}}$ for pre-specified n as an overlapping parametrization of i/o transfer functions, it constitutes a finite collection of overlapping sets of transfer functions. However when considering the set as a representation of equation error models, it constitutes a nonoverlapping set of models that is discriminable by a least squares identification criterion.

One of the consequences of this situation is the following. When choosing for a specific set of integer indices $\mu_1^{(u)}, \ldots, \mu_p^{(u)}$, and performing an (approximate) identification with a corresponding model set, different dynamical models will be identified, dependent on the specific structure indices that have been chosen (but still summing up to n). This above observation was already pointed out in Van den Hof (1989d) and an illustrative example was presented in Van den Hof (1989c). This result is also supported in Van den Hof and Janssen (1987) where properties are derived of approximately identified equation error models, being dependent on the specific sequence of integer indices chosen in an overlapping parametrization.

Remark 5.8. The above result applies e.g. to the following multivariable identification procedure as employed by Corrêa and Glover (1982) and De Mathelin and Bodson (1990). First a (pseudo) canonical form is constructed for a set of i/o transfer functions with a fixed McMillan degree. Consecutively this form is used to identify an equation error model. Based on the results presented above, such a procedure will yield different identified models if an alternative set of canonical forms were chosen for the same set of i/o transfer functions. This implies that the identified models are—to some extent—arbitrary, a fact which is not recognized in the current literature.

6. SOME INVARIANCE PROPERTIES IN LS-IDENTIFICATION

In the situations discussed in the previous section, discriminability of model sets is achieved by restricting a constant surjective matrix to be fixed over the set of models. In Corollary 5.4 this refers to the Markov parameter with index 0 of a specifically constructed polynomial matrix, whereas in Corollary 5.6 the restriction is made concerning a leading row or column coefficient matrix. In the sequel of this section we will refer to such a prescribed matrix as the matrix L while we will restrict attention to the case where $L \in \mathbb{R}^{p\times p}$.

It has been mentioned that different choices for this matrix L generally lead to different properties of the selected models, i.e. to different $J(v^+, \mathcal{M})$. However for specific relations between two different choices of L, some invariance properties exist. If L is 'scaled' in some sense, the resulting identification procedure might generate 'scaled' models, that e.g. remain to have the same input–output part. A similar property of invariance is bound to hold for a shift-operation on the residual signals. In order to analyze these invariance properties we introduce the following two notions.

Definition 6.1. Two models $M_1, M_2 \in \hat{\Sigma}_{p,m}$ are scaling-equivalent with respect to l, denoted by $M_1 \overset{\mathcal{L}}{\sim} M_2$, if there exists a constant $c \in \mathbb{R}\setminus\{0\}$ such that

$$\{(y, u, e_1) \in \mathcal{B}(M_1) \wedge (y, u, e_2) \in \mathcal{B}(M_2)\} \Rightarrow l(e_2^+) = c.l(e_1^+).$$

The definition states that two scaling-equivalent models always show a fixed constant scaling factor between the corresponding values of the criterion function l, irrespective of the data (y, u). A similar definition is formulated for model sets.

Definition 6.2. Two model sets $\mathcal{M}_1, \mathcal{M}_2 \subset \hat{\Sigma}_{p,m}$ are scaling-equivalent with respect to l, denoted by $\mathcal{M}_1 \overset{\mathcal{L}}{\sim} \mathcal{M}_2$, if there exists a constant $c \in \mathbb{R} \setminus \{0\}$ such that

for all $M_1 \in \mathcal{M}_1$ there exists a $M_2 \in \mathcal{M}_2$, and for all $M_2 \in \mathcal{M}_2$ there exists a $M_1 \in \mathcal{M}_1$, such that
$\{(y, u, e_1) \in \mathcal{B}(M_1) \wedge (y, u, e_2) \in \mathcal{B}(M_2)\}$
$\Rightarrow l(e_2^+) = c.l(e_1^+)$.

A characterization of the property of scaling-equivalence of two equation error models is presented in the following proposition.

Proposition 6.3. Two equation error models $M_1, M_2 \in \hat{\Sigma}_{p,m}$ are scaling-equivalent with respect to l, if and only if there exists a polynomial matrix $U \in \mathbb{R}^{p \times p}[z, z^{-1}]$ such that the corresponding polynomial transfer functions $H_e^{(1)}$, $H_e^{(2)}$ are related by $H_e^{(1)} = UH_e^{(2)}$ with
(i) U unimodular with respect to $\mathbb{R}[z, z^{-1}]$; and
(ii) $U^T(z^{-1}, z)U(z, z^{-1}) = cI$ with $c \in \mathbb{R} \setminus \{0\}$. □

The operation of premultiplication of $H_e(z)$ as mentioned in the proposition both reflects the possibility of shifting the residuals over a specific time (e.g. $U(z) = \text{diag}(z^{n_1}, \ldots, z^{n_p})$, with $n_i \in \mathbf{Z}$, $i = 1, \ldots, p$) and of changing the constant matrix L as meant in the introduction of this section, as far as this constant matrix refers to a fixed Markov parameter (as in Corollary 5.4) or a fixed column coefficient matrix (as in Corollary 5.6(iii), (iv)), through application of a constant matrix ($U^TU = cI$). The situation of fixed row coefficient matrices is excluded here, because premultiplication of a polynomial matrix with a constant (nonsingular) matrix will generally affect the row degrees of the matrix. Note that from the corollary follows that $M_1 \overset{\mathcal{L}}{\sim} M_2$ implies $M_1 \underset{io}{\sim} M_2$.

The consequences of choosing scaling-equivalent model sets is presented in the following proposition.

Proposition 6.4. Consider two equation error model sets $\mathcal{M}_1, \mathcal{M}_2 \subset \hat{\Sigma}_{p,m}$. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are scaling-equivalent with respect to l, then for all data sequences v^+:

$$J(v^+, \mathcal{M}_1) \overset{\mathcal{L}}{\sim} J(v^+, \mathcal{M}_2).$$

Proof. The proposition follows directly from the appropriate definitions; if $M_1 \in J(v^+, \mathcal{M}_1)$ then automatically $M_2 \in J(v^+, \mathcal{M}_2)$ with $M_1 \overset{\mathcal{L}}{\sim} M_2$, and *vice versa*. □

The proposition shows that applying scaling equivalent model sets, leads to scaling equivalent identification results. This actually means that in essence the identification result is not affected by an operation of scaling-equivalence to the model set. Since scaling-equivalence of two models has been shown to imply i/o-equivalence of the models, the input–output properties of the selected (identified) models in $\mathcal{M}_1$ and $\mathcal{M}_2$ will be invariant.

Remark 6.5. For the special situation that an equation error model set can be parametrized by a set of polynomial matrices H_e that have the additional property that all rows of H_e are parametrized independently, i/o-equivalence of identified models can be shown for a larger class of scaling-equivalent model sets than presented in Definition 6.2. In that situation the restriction on the matrix U in Proposition 6.3 can be relaxed to

$$U^T(z^{-1}, z)U(z, z^{-1}) = \text{diag}(c_1, \ldots, c_p) \text{ with } c_i \in \mathbb{R} \setminus \{0\}, \quad i = 1, \ldots, p,$$

see Van den Hof (1989a).

CONCLUSIONS

Equation error models are known to inherently require the *a priori* choice for specific signal variables to be considered as regressands and/or regressors. The choice for a different regressand will influence the models that will be obtained by applying a least squares identification criterion in an approximative sense. We have generalized this notion to the multivariable case, by quantifying the restrictions that can be laid upon the model set, in order to guarantee that all models in a model set can be distinguished by a least squares identification criterion. This property, denoted by 'discriminability of the model set by the identification criterion' has been analyzed, and different sets of sufficient conditions have been formulated, employing system representations dealing with both forward and backward shift operators.

The first set reflects a problem of constructing sets of canonical forms for polynomial matrices given the equivalence relation of unimodular premultiplication. The result presented directly resembles and generalizes results that are obtained in a related problem of constructing identifiable parametrizations.

The second set of sufficient conditions is less restrictive, and does not directly refer to such a parametrization problem. Nevertheless the result has close connections to parametrization issues, and it is shown that the pseudo-canonical or overlapping parametrization of all dynamical systems with prespecified McMillan degree, actually constitutes a nonoverlapping set of models that is discriminable by a least squares identification criterion.

A notion of scaling-equivalence of model sets is introduced, that is shown to guarantee invariance properties for identified models.

REFERENCES

Beghelli, S. and R. P. Guidorzi (1983). Transformations between input–output multistructural models: properties and applications. *Int. J. Control*, **37**, 1385–1400.

Corrêa, G. O. and K. Glover (1984). Pseudo-canonical forms, identifiable parametrizations and simple parameter estimation for linear multivariable systems: input–output models. *Automatica*, **20**, 429–442.

Deistler, M. (1983). The properties of the parametrization of ARMAX systems and their relevance for structural estimation. *Econometrica*, **51**, 1187–1207.

De Mathelin, M. and M. Bodson (1990). Frequency domain conditions for parameter convergence in multivariable recursive identification. *Automatica*, **26**, 757–767.

De Moor, B. and J. Vandewalle (1990). A unifying theorem for linear and total linear least squares. *IEEE Trans. Automat. Contr.*, **AC-35**, 563–566.

De Moor, B., M. Gevers and G. C. Goodwin (1991). Estimation of transfer functions: overbiased, underbiased and unbiased identification schemes. *Proc. 9th IFAC/IFORS Symposium on Identif. and System Param. Estim.*, Budapest, Hungary.

Gevers, M. and V. Wertz (1984). Uniquely identifiable state-space and ARMA parametrizations for multivariable linear systems. *Automatica*, **20**, 333–347.

Gevers, M. and V. Wertz (1987). On identifiable parametrizations for multivariable linear systems. In C. T. Leondes (Ed.), *Control and Dynamic Systems*, Vol. 26, pp. 35–86. Academic Press, NY.

Guidorzi, R. P. (1981). Invariants and canonical forms for systems structural and parametric identification. *Automatica*, **17**, 117–133.

Hannan, E. J. (1969). The identification of vector mixed autoregressive moving-average systems. *Biometrika*, **56**, 223–225.

Heij, Ch. (1989). Deterministic Identification of Dynamical Systems. Lecture Notes in *Control and Inform. Sciences*, Vol. 127. Springer Verlag, Berlin.

Inouye, Y. (1983). Approximation of multivariable linear systems with impulse response and autocorrelation sequences. *Automatica*, **19**, 265–277.

Janssen, P. H. M. (1988). On Model Parametrization and Model Structure Selection for Identification of MIMO-Systems. Dr Dissertation, Dept. Electr. Eng., Eindhoven Univ. Technology.

Kalman, R. E. (1982). System identification from noisy data. In A. R. Bednarek and L. Cesari (Eds), *Dynamical Systems II*, pp. 135–164. Academic Press, NY.

Ljung, L. (1976). On the consistency of prediction error identification methods. In R. K. Mehra and D. G. Lainiotis (Eds), *System Identification, Advances and Case Studies*, pp. 121–164. Academic Press, NY.

Ljung, L. (1987). *System Identification: Theory for the User*. Prentice-Hall, Englewood Cliffs, NJ.

Mullis, C. T. and R. A. Roberts (1976). The use of second order information in the approximation of discrete time linear systems. *IEEE Trans. Acoust. Speech, Signal Processing*, **ASSP-24**, 226–238.

Van den Hof, P. M. J. (1989a). On Residual-Based Parametrization and Identification of Multivariable Systems. Dr Dissertation, Dept. Electrical Engineering, Eindhoven University of Technology, The Netherlands.

Van den Hof, P. M. J. (1989b). A deterministic approach to approximate modelling of input–output data. *Proc. 28th IEEE Conf. Decision and Control*, pp. 659–664. Tampa, FL.

Van den Hof, P. M. J. (1989c). A criterion-based approach to parametrization and identification of multivariable systems. In Han-Fu Chen (Ed.), *Identification and System Parameter Estimation 1988.* IFAC Proc. Series 1989, No. 8, pp. 715–720. Proc. 8th IFAC/IFORS Symp. Ident. and Syst. Param. Estim., 1988, Beijing, P.R. China.

Van den Hof, P. M. J. (1989d). Criterion-based equivalence for equation error models. *IEEE Trans. Automat. Contr.*, **AC-34**, 191–193.

Van den Hof, P. M. J. (1991). Discriminability of linear models under closed loop observations. In H. Kimura and S. Kodama (Eds), *Recent Advances in Mathematical Theory of Systems, Control, Networks and Signal Processing I*, pp. 233–238. MITA-Press, Tokyo, Japan.

Van den Hof, P. M. J. (1992). On system order and structure indices of linear systems in polynomial form. *Int. J. Control*, **55**, 1471–1490.

Van den Hof, P. M. J. and P. H. M. Janssen (1987). Some asymptotic properties of multivariable models identified by equation error techniques. *IEEE Trans. Automat. Contr.*, **AC-32**, 89–92.

Wahlberg, B. and L. Ljung (1986). Design variables for bias distribution in transfer function estimation. *IEEE Trans. Automat. Contr.*, **AC-31**, 134–144.

Willems, J. C. (1986). From time series to linear system—part I: finite dimensional linear time-invariant systems. *Automatica*, **22**, 561–580.

Willems, J. C. (1988). Models for dynamics, In Kirchgraber, U. and H. O. Walther (Eds), *Dynamics Reported*, Vol. 2, pp. 171–269. Wiley and Teubner, NY.

APPENDIX

Lemma A.1. Consider matrices $V_1, V_2 \in \mathbb{R}^{p\times p}$, $L_1, L_2 \in \mathbb{R}^{p\times q}$, with $\operatorname{rank}[V_1\ \ V_2] = \operatorname{rank} L_1 = \operatorname{rank} L_2 = p$. Then $[V_1\ \ V_2]\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = 0$ implies $\operatorname{rank} V_1 = \operatorname{rank} V_2 = p$. □

Proof. Since $\operatorname{rank}[V_1\ \ V_2] = \operatorname{rank}(L_1) = \operatorname{rank}(L_2) = p$, it follows with Sylvester's inequality that $\operatorname{rank}\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = p$. As a consequence there exists a nonsingular matrix $A \in \mathbb{R}^{p\times p}$ such that $L_2 = AL_1$. If $[V_1\ \ V_2]\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = 0$ then $[V_1\ \ V_2]\begin{bmatrix} I \\ A \end{bmatrix} L_1 = 0$. Since $\operatorname{rank} L_1 = p$, it follows that $[V_1\ \ V_2]\begin{bmatrix} I \\ A \end{bmatrix} = 0$, and consequently $V_1 = -AV_2$.

Since $\operatorname{rank}[V_1\ \ V_2] = p$, this implies that $\operatorname{rank} V_1 = \operatorname{rank} V_2 = p$, which proves the lemma. □

Lemma A.2. Let M_1, M_2 be two controllable (i/o/pr)-models $M_1, M_2 \in \hat{\Sigma}_{p,m}$, $M_1 = \tilde{M}_p(T_1)$, $M_2 = \tilde{M}_p(T_2)$, with $T_i = [P_i \mid -Q_i \mid -R_i]$ for $i = 1, 2$. Denote

$$\mathcal{R}(M_1, M_2) := \{(e_1, e_2) \mid \exists v, (v, e_1) \in \mathcal{B}(M_1), (v, e_2) \in \mathfrak{B}(M_2)\}.$$

Then

(a) $\mathcal{R}(M_1, M_2) = \{(e_1, e_2) \mid \exists u, (u, e_1, e_2) \in \mathcal{B}_c(\tilde{M}_p(T_e))\}$, where $T_e \in \mathbb{R}^{p\times(m+p+p)}[z, z^{-1}]$ is defined by $T_e = [-Q_e \mid -V_1R_1 \mid -V_2R_2]$, with $V_1, V_2 \in \mathbb{R}^{p\times p}[z, z^{-1}]$, nonsingular and left coprime, satisfying $V_1P_1 + V_2P_2 = 0$, and $Q_e = V_1Q_1 + V_2Q_2$;

(b) In matrix T_e, $Q_e = 0$ if and only if $M_1 \overset{L}{\underset{io}{}} M_2$, or equivalently $P_1^{-1}Q_1 = P_2^{-1}Q_2$.

Proof. Consider two models $M_1, M_2 \in \hat{\Sigma}_{p,m}$ with $M_i = \tilde{M}_p(T_i)$, $T_i = [P_i \mid -Q_i \mid -R_i]$, $i = 1, 2$. Define the combined dynamical system $M_{1,2}$ with variables (y, u, e_1, e_2) such that

$$(v, e_1, e_2) \in \mathcal{B}(M_{1,2}) \Leftrightarrow \{(v, e_1) \in \mathcal{B}(M_1) \wedge (v, e_2) \in \mathcal{B}(M_2)\}. \quad (A.1)$$

Consequently

$$\mathcal{R}(M_1, M_2) = \{(e_1, e_2) \mid \exists v, (v, e_1, e_2) \in \mathcal{B}(M_{1,2})\},$$

and $M_{1,2} = \tilde{M}_p(T_3)$ with

$$T_3 = \begin{bmatrix} P_1 & -Q_1 & -R_1 & 0 \\ P_2 & -Q_2 & 0 & -R_2 \end{bmatrix}.$$

Now consider two matrices $V_1, V_2 \in \mathbb{R}^{p\times p}[z, z^{-1}]$, having rank p, that are left coprime over $\mathbb{R}[z, z^{-1}]$, such that $V_1P_1 + V_2P_2 = 0$. By premultiplication of T_3 with the matrix $\begin{bmatrix} I & 0 \\ V_1 & V_2 \end{bmatrix}$ we construct T_4, and as a result $\mathcal{B}(M_{1,2}) = \mathcal{B}_c(\tilde{M}_p(T_4))$ while

$$T_4 = \begin{bmatrix} P_1 & -Q_1 & -R_1 & 0 \\ 0 & -Q_e & -V_1R_1 & -V_2R_2 \end{bmatrix}, \text{ and } Q_e = V_1Q_1 + V_2Q_2.$$

We have to consider the controllable behaviour $\mathcal{B}_c(\tilde{M}_p(T_4))$, since the premultiplication of T_3 as mentioned is not necessarily unimodular.

This T_4 shows that

$$\{(y, u, e_1, e_2) \in \mathcal{B}(M_{1,2})\} \Leftrightarrow \{(y, u, e_1) \in \mathcal{B}(M_1) \wedge (u, e_1, e_2) \in \mathcal{B}_c(M_e)\}, \quad (A.2)$$

$$\text{with } M_e := \tilde{M}_p([-Q_e \mid -V_1R_1 \mid V_2R_2]). \quad (A.3)$$

Since e_1 is free* in $\mathcal{B}(M_1)$, the condition $(y, u, e_1) \in \mathcal{B}(M_1)$ is not any restriction on e_1. As a result it follows that $(e_1, e_2) \in \mathcal{R}(M_1, M_2)$ if and only if there exists a u such that $(u, e_1, e_2) \in \mathcal{B}_c(M_e)$. This proves the result of part (a). The result of part (b) follows directly by combining $Q_e = V_1Q_1 + V_2Q_2$ and $V_1P_1 + V_2P_2 = 0$. □

Lemma A.3. Consider two equation error models $M_1, M_2 \in \hat{\Sigma}_{p,m}$ with corresponding transfer functions $H_e^{(1)}, H_e^{(2)}$ and $M_1 \stackrel{\perp}{io} M_2$.

Consider a polynomial matrix $K \in \mathbb{R}^{(p+m)\times q}[z, z^{-1}]$, $q \geq p$, such that

(i) $v_i^{(l)}(H_e^{(1)}K) = v_i^{(l)}(H_e^{(2)}K) =: n$, for $i = 1, \ldots, p$ and

(ii) rank $\Gamma_{lr}(H_e^{(1)}K) = \text{rank } \Gamma_{lr}(H_e^{(2)}K) = p$.

Then $H_{ey}^{(2)}(H_{ey}^{(1)})^{-1}$ and $H_{ey}^{(1)}(H_{ey}^{(2)})^{-1}$ are analytic in $z = \infty$. □

Proof. Since $M_1 \stackrel{\perp}{io} M_2$ there exist matrices $V_1, V_2 \in R^{p\times p}[z^{-1}]$ and left coprime with respect to $\mathbb{R}[z^{-1}]$, such that

$$V_1H_e^{(1)} + V_2H_e^{(2)} = 0, \quad \text{implying} \quad (A.4)$$

$$V_1H_e^{(1)}K + V_2H_e^{(2)}K = 0, \quad (A.5)$$

while $H_{ey}^{(2)}(H_{ey}^{(1)})^{-1} = -(V_2)^{-1}V_1$.

We write $H_e^{(i)}K = z^nL_i + \bar{H}_i(z)$ with $L_i = \Gamma_{lr}(H_e^{(i)}K)$ having rank p.

Using (A.5) shows:

$$V_1[L_1 + z^{-n}\bar{H}_1(z)] + V_2[L_2 + z^{-n}\bar{H}_2(z)] = 0. \quad (A.6)$$

The polynomials $z^{-n}\bar{H}_i(z)$ will be polynomials in z, satisfying $[z^{-n}\bar{H}_i(z)]_{z=0} = 0$. Now define $W_i(z) := V(z^{-1})$ being left coprime over $\mathbb{R}[z]$. Evaluating (A.6) for $z = 0$ now shows

$$[W_1(0) \;\; W_2(0)]\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = 0.$$

Consequently rank $[W_1(0) \;\; W_2(0)] = p$, and with Lemma A.1 it follows that rank $W_1(0) = \text{rank } W_2(0) = p$, which proves the result. □

* The variable e with signal set $\mathbb{R}^p$ is called free when for all $e \in (\mathbb{R}^p)^{\mathbb{Z}}$ there exists a trajectory $(y, u, e) \in \mathcal{B}(M)$; Willems (1988).

Proposition A.4. Two models $M_1, M_2 \in \hat{\Sigma}_{p,m}$ are scaling-equivalent with respect to the residual function l, if and only if:

(i) $M_1 \stackrel{\perp}{io} M_2$

(ii) $H_{e_1e_2}(z)$ and $H_{e_2e_1}(z)$ are polynomial and unimodular with respect to $\mathbb{R}[z, z^{-1}]$; and

(iii) $H_{e_1e_2}^T(z^{-1})H_{e_1e_2}(z) = c^{-1}I$ and $H_{e_2e_1}^T(z^{-1})H_{e_2e_1}(z) = cI$, with $c \in \mathbb{R}\setminus\{0\}$. □

Proof. The proof of this proposition is based on the following considerations. Let M_e be a dynamical system $M_e \in \Sigma_{p,p}$ with variables $e_1, e_2 \in (\mathbb{R}^p)^{\mathbb{Z}}$, and $M_e = \tilde{M}_p(T)$ with $T = [R_{e_1} \mid R_{e_2}]$, $R_{e_1}, R_{e_2} \in \mathbb{R}^{p\times p}[z, z^{-1}]$, having rank p. If $H(z) = R_{e_2}^{-1}R_{e_1}(z)$ is stable, and if $(e_1, e_2) \in \mathcal{B}(M_e)$ then

$$(\|H^{-1}(z)\|_{L^\infty})^{-1}l^+(e_1) \leq l^+(e_2) \leq \|H(z)\|_{L^\infty} \cdot l^+(e_1), \quad (A.7)$$

where

$$\|H(z)\|_{L^\infty} = \sup_{\omega\in(-\pi,\pi]} \bar{\sigma}(|H(e^{j\omega})|),$$

and

$$(\|H^{-1}(z)\|_{L^\infty})^{-1} = \inf_{\omega\in(-\pi,\pi]} \underline{\sigma}(|H(e^{j\omega})|),$$

with $\bar{\sigma}$, $\underline{\sigma}$ the maximum, minimum, singular value of a real constant matrix.

If in such a situation $l^+(e_2) = cl^+(e_1)$ for all e_1, and e_1 is free, then consequently $(\|H^{-1}(z)\|_{L^\infty})^{-1} = \|H(z)\|_{L^\infty} = c$, which leads to the situation that $\sigma_i(|H(e^{j\omega})|) = c$ for all singular values σ_i, $i = 1, \ldots, p$, and for all $\omega \in (-\pi, \pi]$. Consequently $H^T(e^{-j\omega})H(e^{j\omega}) = cI$. These expressions will be used in the sequel of this proof.

Conditions (i), (ii) and (iii) are sufficient for the scaling-equivalence of M_1 and M_2 since (i) guarantees that there indeed exists a transfer function between e_1 and e_2; (ii) guarantees that the transfer functions $H_{e_1e_2}(z)$ and $H_{e_2e_1}(z)$ both are stable, and (iii) guarantees the proper constant quotient of $l^+(e_2)$ and $l^+(e_1)$.

The necessity of Condition (i) follows from the fact that the condition $l^+(e_2) = cl^+(e_1)$ for all possible data, implies that $\mathcal{R}(M_1, M_2)$ is independent of u. In the notation of Lemma A.2 this means that $Q_e = 0$, which is equivalent with $M_1 \stackrel{\perp}{io} M_2$, and equivalent with Condition (i) in Proposition A.4.

The necessity of Condition (ii) follows from the fact that in order to satisfy $l^+(e_2) = cl^+(e_1)$ for all data, both transfer functions $H_{e_1e_2}(z)$ and $H_{e_2e_1}(z)$ have to satisfy the requirement that they are stable, and moreover that

$$H_{e_2e_1}^T(z^{-1})H_{e_2e_1}(z) = cI \quad \text{and} \quad H_{e_1e_2}^T(z^{-1})H_{e_1e_2}(z) = c^{-1}I.$$

From these requirements it follows that both transfer functions are not allowed to have any finite poles, except in $z = 0$. Necessity of Condition (iii) follows directly from the remarks made in the beginning of this proof. □

Proof of Theorem 5.1.

Consider any element $M_0 \in \mathcal{M}$, and a signal $w = (v, e_0) \in \mathcal{B}(M_0)$, with $e_0 = 0$, $v = (y, u)$, and u satisfying $\lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} u(t)u^T(t+\tau) = I_m\delta(\tau)$, with I_m the $m \times m$-identity matrix, and $\delta(\tau)$ the Kronecker delta-function. Consequently for any other model $M_1 \in \mathcal{M}$ it follows that

$$M_1 \in J(v^+, \mathcal{M}) \Leftrightarrow \{\text{there exists } e_1 \text{, such that } (e_1^+, e_0^+) \in \mathcal{R}^+(M_1, M_0) \text{ and } l(e_1) = 0\}.$$

Because of the persistently exciting character of u, the right-hand side of this equivalence relation can only be achieved if $\mathcal{R}^+(M_1, M_0)$ is not dependent on u. With Lemma A.2 this is equivalent to $M_1 \stackrel{\perp}{io} M_0$. Under the condition as formulated in the theorem, it follows that $\{M_0\} = J(v^+, \mathcal{M})$.

Since M_0 can be chosen any element of $\mathcal{M}$, this proves discriminability of $\mathcal{M}$ by J. □

Proof of Theorem 5.2

Consider two models $M_1, M_2 \in \mathcal{M}$. According to Theorem 5.1 absence of discriminability of $\mathcal{M}$ can only be caused by models satisfying $M_1 \overset{L}{\underset{io}{\sim}} M_2$. This can be understood by noting that when two models M_1, M_2 do not satisfy $M_1 \overset{L}{\underset{io}{\sim}} M_2$, they can always be discriminated from each other through employing a data sequence $v^+ \in \mathcal{B}^{io}(M_1)$ or $v^+ \in \mathcal{B}^{io}(M_2)$, with an input signal that is sufficiently exciting.

Consequently for formulating conditions for obtaining discriminability, we only have to consider models that satisfy $M_1 \overset{L}{\underset{io}{\sim}} M_2$.

Let $M_i = \bar{M}_p(T_i)$, $T_i = [P_i \mid -Q_i \mid -R_i]$, for $i = 1, 2$. For M_i it follows that $H_{ey}^{(i)} = R_i^{-1}P_i$.

Since $M_1 \overset{L}{\underset{io}{\sim}} M_2$ it follows from Lemma A.2 that

$$\{(e_1, e_2) \in \mathcal{R}(M_1, M_2)\} \Leftrightarrow \{(e_1, e_2) \in \mathcal{B}_c(\bar{M}_p(T_x))\}, \quad (A.8)$$

$$T_x = [-V_1R_1 \mid -V_2R_2], \quad (A.9)$$

with $V_1, V_2 \in \mathbb{R}^{p\times p}[z, z^{-1}]$ having rank p and being left coprime, and $V_1P_1 + V_2P_2 = 0$. Consequently $\mathcal{R}(M_1, M_2)$ is a controllable behaviour induced by the transfer function

$$H_{e_2e_1}(z) = -R_2^{-1}V_2^{-1}V_1R_1(z) = -R_2^{-1}P_2P_1^{-1}R_1(z),$$

which equals $H_{ey}^{(2)}(H_{ey}^{(1)})^{-1}$.

Now construct a data sequence v that satisfies $(v, e_1) \in \mathcal{B}(M_1)$, with u and e_1 satisfying:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} u(t)u^T(t+\tau) = I_m\delta(\tau), \quad \tau \in \mathbf{Z},$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} e_1(t)e_1^T(t+\tau) = I_p\delta(\tau) \quad \tau \in \mathbf{Z}, \quad (A.10)$$

$$\text{and } \lim_{N\to\infty} \frac{1}{N}\sum_{t=0}^{N-1} u(t)e_1^T(t+\tau) = 0 \quad \tau \in \mathbf{Z}.$$

Since u and e_1 are free in M_1, such a data sequence can always be constructed. Given the special character of e_1 it follows that $l^+(e_1) = p$.

If $H_{e_2e_1}(z)$ is stable then it can easily be verified that

$$(e_1, e_2) \in \mathcal{R}(M_1, M_2) \Rightarrow \{l^+(e_2) = \|H_{e_2e_1}(z)\|_2\}.$$

The condition stated in the theorem now implies

$\|H_{e_2e_1}(z)\|_2 \geq l^+(e_1)$ with equality if and only if $M_1 = M_2$.

This implies that for $(e_1, e_2) \in \mathcal{R}(M_1, M_2)$:

$l^+(e_2) \geq l^+(e_1)$ with equality if and only if $M_1 = M_2$.

If this condition holds for all $M_2 \in \mathcal{M}$ it follows that for this data sequence v^+, $J(v^+, \mathcal{M}) = \{M_1\}$. Since M_1 can be taken any element of $\mathcal{M}$ it follows that $\mathcal{M}$ is discriminable by J. □

Proof of Corollary 5.3.

From results in Willems (1988), it follows that $\{M_1^{io}$ and M_2^{io} controllable, and $M_1 \overset{L}{\underset{io}{\sim}} M_2\}$ implies $\{H_e^{(1)}(z) = U(z)H_e^{(2)}(z)$, with $U(z)$ unimodular$\}$.

Given this implication, and knowing that $\{M_1 = M_2\} \Leftrightarrow \{H_e^{(1)}(z) = H_e^{(2)}(z)\}$, the corollary directly follows form Theorem 5.1. □

Proof of Corollary 5.4

The corollary is going to be proved by showing that under the given conditions

$$\{H_e^{(1)} = UH_e^{(2)},\ U \in \mathbb{R}^{p\times p}[z, z^{-1}]\} \Rightarrow \{H_e^{(1)} = H_e^{(2)}\},$$

from which discriminability follows via Corollary 5.3.

We write $U \in \mathbb{R}^{p\times p}[z, z^{-1}]$ as

$$U(z, z^{-1}) = U_sz^s + \cdots + U_tz^t \quad s, t \in \mathbf{Z}; s \leq t. \quad (A.11)$$

Step 1. Using Condition (i) we can write

$$H_e^{(i)}K = \Gamma_i\Lambda(z) + T_i(z),$$

with

$\Delta(z) = \text{diag}\{z^{m_j}\}$, $j = 1, \ldots, q_1$, $\Gamma_i = \Gamma_{c,(m_1,\ldots,m_{q_1})}(H_e^{(i)}K)$,

and $T_i(z)$ satisfying $\mu_j^{(u)}(T_i) < m_j$. If $H_e^{(1)} = UH_e^{(2)}$ then $H_e^{(1)}K = UH_e^{(2)}K$, leading to

$$\begin{aligned}\Gamma_1\Lambda(z) + T_1(z) &= U[\Gamma_2\Lambda(z) + T_2(z)] \text{ and} \\ \Gamma_1 + T_1(z)\Lambda^{-1}(z) &= U[\Gamma_2 + T_2(z)\Lambda^{-1}(z)],\end{aligned} \quad (A.12)$$

with $T_i(z)\Lambda^{-1}(z)$ strictly proper, $i = 1, 2$.

Equating the coefficients of z^j, $j > 0$ in (A.12), using (A.11), shows that $U_j\Gamma_2 = 0$, $j > 0$. Since Γ_2 is a surjective matrix it follows that $U_j = 0$, $j > 0$, and consequently that $U \in \mathbb{R}^{p\times p}[z^{-1}]$.

Step 2. Similar reasoning as in Step 1, now directed to the dual situation using $H_e^{(i)}N$, employing the lower column degree conditions (ii), shows that $U \in \mathbb{R}^{p\times p}[z]$, which together with the result in Step 1 leads to $U \in \mathbb{R}^{p\times p}$.

Step 3. Employing Condition (iii) leads to $\{H_e^{(1)} = UH_e^{(2)}\} \Rightarrow \{H_e^{(1)}L = UH_e^{(2)}L\}$, and since $U \in \mathbb{R}^{p\times p}$, the polynomial terms of $H_e^{(i)}L$ show that $G_0^{(1)} = UG_0^{(2)}$ with $G_0^{(1)} = G_0^{(2)}$ a surjective matrix. Consequently $U = I_p$. □

Proof of Corollary 5.6.

This corollary follows from Theorem 5.2. Consider two equation error models M_1, M_2 with $M_1 \overset{L}{\underset{io}{\sim}} M_2$ and corresponding polynomial transfer functions $H_e^{(i)}(z)$, $i = 1, 2$. Since $M_1 \overset{L}{\underset{io}{\sim}} M_2$ there exist matrices V_1, $V_2 \in R^{p\times p}[z]$ and left coprime, such that $V_1H_e^{(1)} + V_2H_e^{(2)} = 0$. For any matrix K as specified, it follows that

$$V_1H_e^{(1)}K + V_2H_e^{(2)}K = 0, \quad (A.13)$$

while $H_{e_2e_1}(z) = -(V_2)^{-1}V_1 = H_e^{(2)}KU(H_e^{(1)}KU)^{-1}$, for any $U \in \mathbb{R}^{p\times p}(z)$ such that the inverted product indeed is invertible.

Part (i) and (ii).

Since $\Gamma_{hr}(H_e^{(1)}K) = \Gamma_{hr}(H_e^{(2)}K) = L$, there can be written:

$$H_e^{(j)}K(z) = \Lambda_j(z)L + \bar{H}_j(z) \quad j = 1, 2, \quad (A.14)$$

$$\text{with } \Lambda_j(z) = \text{diag}\,(z^{n_1^{(j)}}, \ldots, z^{n_p^{(j)}}), \quad (A.15)$$

$$n_i^{(j)} = v_i^{(u)}(H_e^{(j)}K) \text{ and } v_i^{(u)}(\bar{H}_j) < v_i^{(u)}(H_e^{(j)}K), \quad i = 1, \ldots, p. \quad (A.16)$$

Since rank $L = p$, there exists a $U \in \mathbb{R}^{q\times p}$ such that $LU = I_p$, and consequently:

$$H_e^{(j)}K(z)U = \Lambda_j(z)(I + \Lambda(z)^{-1}\bar{H}_j(z)U), \quad j = 1, 2, \quad (A.17)$$

with $\Lambda_j(z)^{-1}\bar{H}_j(z)U$ is strictly proper (according to (A.16)).

Using the above expression for $H_{e_2e_1}(z)$, shows

$$H_{e_2e_1}(z) = \Lambda_2(z)B(z)\Lambda_1(z)^{-1}, \quad (A.18)$$

$$\text{with } B(z) = [I + \Lambda_2(z)^{-1}\bar{H}_2(z)U][I + \Lambda_1(z)^{-1}\bar{H}_1(z)U]^{-1}. \quad (A.19)$$

It follows that $B(z)$ is proper with $\lim_{z\to\infty} B(z) = I$, and consequently $\|B(z)\|_2 \geq p$. Because of the structure of $\Lambda_i(z)$ it follows from (A.18) that $\|H_{e_2e_1}\|_2 = \|B\|_2$. As a result $\|H_{e_2e_1}\|_2 \geq p$ if and only if $\|B\|_2 \geq p$, which indeed is satisfied.

Now it remains to formulate conditions for $\|H_{e_2e_1}\|_2 = p$, to hold true if and only if $M_1 = M_2$, or equivalently $H_{e_2e_1}(z) = I$.

Because $\lim_{z\to\infty} B(z) = I$, $\|B\|_2 = p$ if and only if $B(z) = I$, or equivalently (with (A.18)),

$$H_{e_2e_1}(z) = \Lambda_2(z)\Lambda_1(z)^{-1}. \quad (A.20)$$

In order to prove discriminability we now have to formulate conditions such that $\Lambda_2(z)\Lambda_1(z)^{-1} = I$, in the situation that (A.20) holds.

(i) If $v_i^{(u)}(H_e^{(1)}K) = v_i^{(u)}(H_e^{(2)}K)$ for $i = 1, \ldots, p$, then $\Lambda_2(z) = \Lambda_1(z)$ and $H_{e_2e_1}(z) = I$. This proves part (i) of the theorem.

(ii) If

$$\sum_{i=1}^{p} \nu_i^{(u)}(H_e^{(1)}K) = \sum_{i=1}^{p} \nu_i^{(u)}(H_e^{(2)}K)$$

then $\det(\Lambda_2(z)\Lambda_1(z)^{-1}) = 1$. Conditions (c) and (d) show with Lemma A.3 that $H_{e_2e_1}(z)$ is analytic in $z=\infty$. As a result, $H_{e_2e_1}(z) = I$. This proves part (ii) of the theorem.

Part (iii).

Using the notation

$$\Gamma_{c,(n_1,\ldots,n_q)}(H_e^{(1)}K) = \Gamma_{c,(n_1,\ldots,n_q)}(H_e^{(2)}K) = L,$$

there can be written:

$$H_e^{(j)}K(z) = L\Lambda(z) + \bar{H}_j(z) \quad j=1,2, \tag{A.21}$$

$$\text{with } \Lambda(z) = \mathrm{diag}\,(z^{n_1}, \ldots, z^{n_q}), \tag{A.22}$$

and since $n_i \geq \mu_i^{(u)}(H_e^{(j)}K)$,

$$\mu_i^{(u)}(\bar{H}_j) < \mu_i^{(u)}(H_e^{(j)}K), \quad i=1,\ldots,q. \tag{A.23}$$

Since rank $L=p$, there exists a $U \in \mathbb{R}^{q\times p}$ such that $LU = I_p$. It follows that $H_{e_2e_1}(z)$ can be written as:

$$H_{e_2e_1}(z) = H_e^{(2)}K(z)\Lambda(z)^{-1}U[H_e^{(1)}K(z)\Lambda(z)^{-1}U]^{-1}, \tag{A.24}$$

$$= [I + \bar{H}_2(z)\Lambda(z)^{-1}U][I + \bar{H}_1(z)\Lambda(z)^{-1}U]^{-1}, \tag{A.25}$$

with $\bar{H}_j(z)\Lambda(z)^{-1}U$ strictly proper.

Since $\lim_{z\to\infty} H_{e_2e_1}(z) = I$, it follows that $\|H_{e_2e_1}\|_2 \geq p$, with equality if and only if $H_{e_2e_1}(z) = I$, or equivalently $M_1 = M_2$.

Part (iv).

In similar notation as in Part (iii), we can now write:

$$H_e^{(j)}K(z) = L\Lambda_j(z) + \bar{H}_j(z) \quad j=1,2, \tag{A.26}$$

$$\text{with } \Lambda_j(z) = \mathrm{diag}\,(z^{n_1^{(j)}}, \ldots, z^{n_q^{(j)}}), \tag{A.27}$$

$$n_i^{(j)} = \mu_i^{(u)}(H_e^{(j)}K) \text{ and } \mu_i^{(u)}(\bar{H}_j) < \mu_i^{(u)}(H_e^{(j)}K), \quad i=1,\ldots,p. \tag{A.28}$$

$$\text{and } L = \Gamma_{hc}(H_e^{(1)}K) = \Gamma_{hc}(H_e^{(2)}K). \tag{A.29}$$

Since $H_e^{(i)}K$ are square matrices, it follows that

$$H_{e_2e_1}(z) = [L\Lambda_2(z) + \bar{H}_2(z)][L\Lambda_1(z) + \bar{H}_1(z)]^{-1}$$

$$= B_2(z)\Lambda_2(z)\Lambda_1(z)^{-1}B_1(z), \tag{A.30}$$

with $B_2(z) = [L + \bar{H}_2(z)\Lambda_2(z)^{-1}]$ and $B_1(z) = [L + \bar{H}_1(z)\Lambda_1(z)^{-1}]^{-1}$. Since $\bar{H}_j(z)\Lambda_j(z)^{-1}$ is strictly proper for $j=1,2$, it follows that $B_1(z)$ and $B_2(z)$ are proper with $\lim_{z\to\infty} B_1(z) = L^{-1}$ and $\lim_{z\to\infty} B_2(z) = L$, and consequently $B_2(z)B_1(z)$ is proper with $\lim_{z\to\infty} B_2(z)B_1(z) = I$.

If

$$\sum_{i=1}^{p} \mu_i^{(u)}(H_e^{(1)}K) = \sum_{i=1}^{p} \mu_i^{(u)}(H_e^{(2)}K)$$

then $\det(\Lambda_2(z)\Lambda_1(z)^{-1}) = 1$. Conditions (c) and (d), as in Part (ii), imply that $H_{e_2e_1}(z)$ is analytic in $z=\infty$. Since $B_1(z)$ and $B_2(z)$ have no zeros nor poles in $z=\infty$, it follows that analyticy of $H_{e_2e_1}(z)$ in $z=\infty$, directly implies that $\Lambda_2(z)\Lambda_1(z)^{-1}$ is analytic in $z=\infty$. Given the structure of $\Lambda_j(z)$ and the fact that $\det(\Lambda_2(z)\Lambda_1(z)^{-1}) = 1$, it follows that $\Lambda_1(z) = \Lambda_2(z)$ and a similar situation occurs as in Part (iii). □

Proof of Proposition 6.3.

Sufficiency of the conditions formulated in the corollary, follows by inspection, showing that the conditions of Proposition A.4 are satisfied. Necessity is going to be proved.

Consider two models $M_1, M_2 \in \hat{\Sigma}_{p,m}$, with $M_1 = \bar{M}_p(T_1)$ and $M_2 = \bar{M}_p(T_2)$, where $T_1 = [P_1 \mid -Q_1 \mid -R_1]$ and $T_2 = [P_2 \mid -Q_2 \mid -R_2]$.

Since $M_1 \overset{\ell}{\underset{io}{\sim}} M_2$ it follows from Lemma A.2 that

$$\{(e_1, e_2) \in \mathcal{R}(M_1, M_2)\} \Leftrightarrow \{(e_1, e_2) \in \mathcal{B}_c(\bar{M}_p(T_x))\},$$

with $T_x = [-V_1R_1 \mid V_2R_2]$, $V_1P_1 + V_2P_2 = 0$, and $V_1, V_2 \in \mathbb{R}^{p\times p}[z, z^{-1}]$ having rank p and being left coprime. Consequently $H_{e_2e_1}(z) = -R_2^{-1}V_2^{-1}V_1R_1(z)$.

Therefore Conditions (i) and (ii) of Proposition A.4 imply:

$$V_1P_1 + V_2P_2 = 0 \text{ with } V_1, V_2 \in \mathbb{R}^{p\times p}[z, z^{-1}], \text{ having rank } p, \text{ and} \tag{A.31}$$

$$V_1Q_1 + V_2Q_2 = 0, \tag{A.32}$$

$$V_1R_1 + V_2R_2U = 0 \text{ with } U \in \mathbb{R}^{p\times p}[z, z^{-1}] \text{ a unimodular matrix.} \tag{A.33}$$

Combining these three equations shows that

$$[P_1 \mid -Q_1 \mid -R_1] = -V_1^{-1}V_2[P_2 \mid -Q_2 \mid -R_2U]$$

and since the models M_1, M_2 are controllable it follows that $V_1^{-1}V_2$ is polynomial and unimodular showing that $M_1 \underset{io}{\approx} M_2$.

This means that P_2, Q_2 can be chosen equal to P_1, Q_1 and as a result $H_e^{(1)} = UH_e^{(2)}$. Condition (ii) now follows directly from Condition (iii) in Propositon A.4. □