Approximate realization of noisy linear multivariable systems

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SUMMARY

When given a sequence of noise disturbed Markov parameters of a multivariable system, representing the respective impulse responses of the different transmittances of this system, one problem is to find a corresponding state space model.
This paper presents suboptimal solutions to this problem of approximate realization. Existing methods fail theoretically but appear to be practically quite useful. In an attempt to find a new approach to this problem a new method, based on a so called Page matrix, is described and results of simulations are presented.

1. Introduction

System identification, as a part of system theory, is dealing with the problem of finding a description of the behaviour of a specific system when only restricted information on the system is available. This information can be in the form of measurements of input- and/or output signals, of knowledge of the physics of the studied system, or of any other a priori information. In many practical situations, a combination of several types of knowledge will appear.
In engineering studies mathematical models are used for describing the system behaviour in a comprised form. Such models are used in many different situations, and for several purposes.
Some examples :
- identification of systems for predicting future behaviour (e.g. weather forecasts);
- identification for determining quantities that are not directly measureable; diagnosis (e.g. in biomedical applications);
- identification of systems for judging quality, reliability, for provoking some specific future behaviour by adjusting a proper input (e.g. the aging processes of machines as a function of the intensity of their operation);
- identification of systems for designing optimal process control.

Dealing with this last aspect, it must be observed that over the years models of processes that have to be controlled have become much more complicated. From the point of view of efficient, economical and proper operation (e.g. restricted use of energy and raw materials, diminishing of pollution) as well as competition, the requirements that have to be met by the process control have become heavier. Information on the process itself now becomes an important aspect when designing a control circuit, leading to the application of system identification techniques in control engineering.

Realization theory is providing us with instruments to construct a state space description of a system when some other type of model of the system has become available.
The study reported here, has been dealing with the realization of MIMO (multi-input multi-output) systems based on a truncated sequence of noise disturbed Markov parameters.
Markov parameters appear in an impulse response model of a system as a sequence of weighing matrices. In order to create a compact representation that is suited for various applications, our final goal is to determine a state space description of the system under consideration. Apart from the realization problem mentioned, the methods presented in this paper can also be applied e.g. in time series analysis (so-called stochastic realization problem) and in model reduction problems. For theoretical details one is referred to [1], [4].
After some introductory notes on linear models, the approximate realization problem will be defined. In sections 4 and 5 two methods will be presented for dealing with this problem. Simulations have been performed in order to evaluate the properties of both methods. Some results of these simulations are reported in section 6.

2. Linear models of multivariable systems

We will restrict ourselves to models showing linear relations between input- and output-variables of a system. Moreover, we will consider systems that are time-invariant and that are described as being discrete in time. This paper will deal with two kinds of models the impulse response model and the state space model.
* Impulse response model
The impulse response model of a strictly proper multivariable system (i.e. a system showing no static trans-

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mittance) can be written as

$$\underline{y}(k) = \sum_{i=1}^{\infty} M(i)\underline{u}(k-i) \qquad (1)$$

where $\underline{y}(k)$ is a q-vector of outputs at time instant k,
$\underline{u}(k)$ is a p-vector of inputs at time instant k,
and $\{M(i)\}_{i=1,\ldots}$ is a sequence of Markov parameters.

A Markov parameter M(k) is a [qxp]-matrix where element $M_{ij}(k)$ reflects the response at time instant k, of output i when input j has been excited with a pulse at time instant 0, if the other inputs have not been excited.
A sequence of Markov parameters of a system can be measured directly if the practical situation allows the excitation of the system with pulses. In many situations this will cause insuperable problems, and as a result the sequence of Markov parameters has to be estimated based on measurements of input- and output signals.
In case of systems with only outputs available (e.g. time series analysis) estimated covariance matrices of the outputs play the role of Markov parameters in the realization problem [2].
In both cases, measurement as well as estimation of Markov parameters, only a *truncated sequence* of parameters will become available. Because the number of Markov parameters in the model (1) is infinite, this always has to result in a negligence of the contribution of the 'tail' of the sequence.
In the theoretical case, when considering a system with finite dimension, the problem of the infinite number of parameters can be overcome; it can be proved [3] that an integer s and constants $\{a(i)\}_{i=1,\ldots,s}$ exist in such a way that

$$M(s+j) = \sum_{i=1}^{s} a(i)\, M(s+j-i) \qquad \forall j \geqslant 1 \qquad (2)$$

This means that the infinite sequence of Markov parameters can be described by a finite number of parameters: $\{a(i), M(i)\}_{i=1,\ldots,s}$

* State space model
A state space model of a strictly proper system is defined as :

$$\begin{aligned} \underline{x}(k+1) &= A\,\underline{x}(k) + B\,\underline{u}(k) \\ \underline{y}(k) &= C\,\underline{x}(k) \end{aligned} \qquad (3)$$

where $\underline{x}(k)$ is an n-vector of state variables and (A, B, C), a set of matrices with proper dimensions, is called a realization of dimension n.
Because it is only possible to identify the observable and controllable part of a system and due to a preference of a system representation as simply as possible, models with minimal dimension are constructed, corresponding to a *minimal realization* (A, B, C).
Contrary to an impulse response model, the parameters in a state space model cannot be measured directly. For determining these parameters, an estimation procedure has to be applied based on measured input and output signals. This can be performed in several ways, both with respect to the estimation technique as well as with respect to the parametrization of the model.
Matrices (A, B, C) can be fully parametrized, leading to n(n+p+q) parameters. However, this number of parameters is excessive, caused by the fact that the representation (A, B, C) is not unique.
It is well known that, given a realization (A, B, C), any realization (TAT^{-1}, TB, CT^{-1}), T being a non-singular matrix, will show the same input-output behaviour, leading to an equivalence class of triples (A, B, C) showing the same input-output behaviour.
In order to restrict the number of parameters and to attain a unique parametrization, very often use is made of canonical forms : special structures are imposed on the matrices (A, B, C) in such a way that any system within this structure has only one representation.
This approach is still largely in development.

3. The approximate realization problem

Realization theory is dealing with the problem of constructing a state space description of a process, when given some other kind of model of this process.
In this paper the situation is considered that a finite sequence of Markov parameters $\{M(i)\}_N$ is available, obtained by measurement or estimation, and that a state space model has to be constructed.
With respect to this problem two situations can be distinguished :

1. deterministic case; Markov parameters of a finite dimensional system are available in an exact form. Furthermore the number N indicating the length of the sequence is large enough such that (2) holds for this sequence and that a unique extension of the sequence can be found in such a way that (2) remains valid.
2. noise disturbed case; the Markov parameters available are noise disturbed c.q. corresponding to an infinite dimensional system.

In the deterministic case, realization theory offers an elaborate solution to the problem (see e.g. [3], [7]), and finding a minimal realization is quite straightforward.
In practical situations however, deterministic data will very seldom be available. Measured signals will mostly be disturbed by noise, thus influencing the Markov parameters in both situations of direct measurement or estimation.
In the case of a finite sequence of noise disturbed Markov parameters, like in the deterministic case, it is possible to construct an extension of this sequence in such a way that the criterion (2) is fulfilled. However the corresponding value of s will be very large and will be related to the number of available Markov parameters, rather than to the dynamics of the original system. Therefore any corresponding realization will show a very large dimension and will not be suited for practical use.
As a result, in the noise disturbed case, an exact description of restricted dimension does not exist, and therefore, an approximation has to be constructed, leading to an *approximate realization*.
For every state space model a corresponding sequence

of Markov parameters can be generated by

$$M(k) = CA^{k-1}B \qquad k \geqslant 1 \tag{4}$$

The goal is to find a realization (A, B, C) with predetermined dimension n in such a way that the sequence of Markov parameters corresponding to this realization is 'in some sense' as close as possible to the original available noise disturbed sequence $\{\tilde{M}(i)\}_N$. This result will be called an approximate realization. There are several criteria one can use when constructing an approximation. From an identification point of view, a least squares criterion on the sequence of Markov parameters is a proper one if the sequence of Markov parameters is supposed to be disturbed by additive white noise, independent on all entries.
This criterion leads to an error function

$$F = \sum_{k=1}^{N} \operatorname{tr}\{[\tilde{M}(k) - CA^{k-1}B]^T [\tilde{M}(k) - CA^{k-1}B]\} \tag{5}$$

summing the squared errors on all entries in the sequence of Markov parameters.
Finding a well-defined approximate realization, i.e. minimizing any prescribed criterion, is still a very difficult problem. In practice a number of methods has been proposed – however none of them leading to a well defined solution.
In the following sections a well-known realization method for deterministic data will be recalled (so-called Ho-Kalman algorithm) and a method to be applied in the noisy case, derived from this algorithm, will be presented.

4. The Hankel matrix approach

The realization problem has been attacked by many authors; the algorithm introduced by Ho/Kalman [3] forms a basis for many methods, mainly showing minor differences.
This algorithm is centered around a block Hankel matrix of Markov parameters :

$$H[i,j] = \begin{bmatrix} M(1) & M(2) & . & . & M(j) \\ M(2) & . & . & . & M(j+1) \\ & & & & \\ M(i) & . & . & . & M(i+j-1) \end{bmatrix} \tag{6}$$

The central aspect of the method is reflected in the next theorem [6] :

Theorem 1

If, for a finite sequence of Markov parameters $\{M(k)\}_N$, integers i and j exist in such a way that $i+j = N$ and

$$\operatorname{rank} H[i,j] = \operatorname{rank} H[i+1,j] = \operatorname{rank} H[i,j+1] \;(=n) \tag{7}$$

the extension of the sequence $\{M(k)\}$ for $k > N$ is unique and the Ho-Kalman algorithm will lead to an exact realization of the given sequence with minimal dimension n.
Suppose that this condition is fulfilled and as a result a realization of dimension n exists. With (4) a decomposition of the Hankel matrix can now be written as :

$$H[i,j] = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^{i-1} \end{bmatrix} [B \; AB \; A^2B \; .... \; A^{j-1}B] = \Gamma . \Delta \tag{8}$$

If we are dealing with a minimal realization (observable and controllable) the rank of both matrices Γ and Δ will equal n.
The triplet (A,B,C) can be obtained from (Γ, Δ) as follows : the matrices B and C can be recognized as the first blocks in Δ and Γ respectively. In order to obtain matrix A, we need a shifted matrix, which we indicate by an arrow. A vertically pointing arrow indicates a shift of one block row, whereas a horizontally pointing arrow denotes a shift of one block column :

$$\overleftarrow{H}[i,j] = \begin{bmatrix} M(2) & M(3) & . & . & M(j+1) \\ M(3) & . & . & . & M(j+2) \\ . & & & & \\ . & . & . & & \\ M(i+1). & . & . & . & M(i+j) \end{bmatrix}$$

Because of the block symmetric structure of H we can write :

$$\overleftarrow{H} = H\!\uparrow\, = \Gamma . A . \Delta \tag{9}$$

and as Γ and Δ have a full rank n,

$$A = \Gamma^+ H\!\uparrow\! \Delta^+ \tag{10}$$

where $^+$ stands for the Moore-Penrose pseudo inverse.
Each decomposition of H into two matrices of full rank will lead to a triplet (A,B,C) in the same equivalence class (see also [1]).
A numerically stable decomposition of H is offered by the singular value decomposition, given by

$$H = W_n \Sigma_n V_n^T \tag{11}$$

where Σ is a diagonal matrix of n positive singular values and W_n and V_n are matrices consisting of n orthonormal columns.
A decomposition with proper rank, is now given by

$$H = \Gamma . \Delta = (W_n \Sigma_n^{1/2}) . (\Sigma_n^{1/2} V_n^T) \tag{12}$$

In the situation of noise disturbed Markov parameters, generally a unique extension sequence leading to a minimal realization will not exist; moreover, unique or not unique, any extension sequence will correspond to a realization of very high dimension. Actually, this is due to the fact that one tries to incorporate the contribution of the noise in the model, while the purpose should be to filter out this noise contribution.
This problem can be dealt with by approximating the available Hankel matrix with a Hankel matrix of restricted rank. When applying a singular value decomposition to the Hankel matrix and fixing the smallest singular values to zero, both an order test is performed (by choosing the number of singular values taken into account) and a simple least squares approximation of the Hankel matrix is constructed with reduced rank [7]. Applying the Ho-Kalman algorithm to this rank reduced matrix will

lead to an approximate realization of the sequence of Markov parameters.
However, this approach suffers from the following problems :
— the approximating matrix lacks a block symmetric structure and therefore it is not a Hankel matrix anymore. This implies that the resulting [q×p]-parameters in the approximating matrix can neither be considered as a unique sequence of Markov parameters, nor do they represent the outcome of an n^{th} order state space representation;
— the least squares criterion (Euclidean or Frobenius norm of H) causes an unequal weighing of Markov parameters. This is due to the fact that, depending on their index, the Markov parameters appear in the Hankel matrix with different frequencies.
As a result, the outcome of this method will not be well-defined, in the sense that it will not be the result of minimizing a criterion like (5).

5. The Page matrix approach

In an attempt to find a new approach to the realization problem that overcomes the problems mentioned above, a new matrix is introduced that represents the Markov parameters in a simple way [1].
A Page matrix is defined as :

$$P[\eta,\mu] = \begin{bmatrix} M(1) & M(2) & \cdot & \cdot & \cdot & M(\mu) \\ M(\mu+1) & M(\mu+2) & \cdot & \cdot & \cdot & M(2\mu) \\ \cdot & & & & & \\ \cdot & & & & & \\ M((\eta-1)\mu+1) & \cdot & & & & M(\eta.\mu) \end{bmatrix} \tag{13}$$

If a finite dimensional realization of the sequence of Markov parameters exists, the Page matrix can be decomposed in a way similar to the Hankel matrix :

$$P[\eta,\mu] = \Gamma_{\mu}.\Delta = \begin{bmatrix} C \\ CA^{\mu} \\ CA^{2\mu} \\ \vdots \\ CA^{(\eta-1)\mu} \end{bmatrix} . [B\ AB\ A^2B \ldots A^{\mu-1}B] \tag{14}$$

The central theorem with respect to this approach is stated as follows :

Theorem 2
If the dimensions of a Page matrix for a system which has a minimal realization of dimension n, are chosen large enough (we take $\mu, \eta \geq n$ as a sufficient condition), and if (C, A^{μ}) is a completely observable couple, it holds that

$$\text{rank } P = n \tag{15}$$

and any decomposition in Γ_{μ} and Δ of full rank n will lead to a minimal realization which can be found by applying the Ho-Kalman algorithm with Γ_{μ} and $\overleftarrow{P}$ substituted for respectively Γ and $\overleftarrow{H}$.

$$\overleftarrow{P} = \Gamma_{\mu}.A.\Delta = \begin{bmatrix} M(2) & \cdot & \cdot & \cdot & M(\mu+1) \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ M((\eta-1)\mu+2) & \cdot & & \cdot & M(\eta\mu+1) \end{bmatrix} \tag{16}$$

As discussed in [4], cases of non-observability of the couple (C, A^{μ}) may occur, while the couple (C,A) is conditioned to be observable.
Further on, we shall assume that all precautions have been taken in order to be sure that the rank criterion of the Page matrix simply defines the dimension of the system, and a realization (A, B, C) can be obtained in a manner similar to the method used for the Hankel matrix.
When given N, the number of Markov parameters, several combinations η, μ may be possible to construct P. In [1] it has been derived that under some conditions a Page matrix chosen as square as possible will lead to optimal results with respect to the filtering of the noise.
It can be expected that the introduction of the Page matrix will be fruitful in by-passing the previously mentioned problems, when finding an approximate realization. Indeed, in the Page matrix all Markov parameters appear only once which means that, when reducing the rank by means of singular value decomposition, there is an equally balanced filtering over the parameters.
For the same reason a Page matrix of reduced rank provides a unique sequence of Markov parameters, contrary to the Hankel matrix approach where filtered Markov parameters with equal index appear at different positions.
Moreover, dimensions of a Page matrix are much smaller than a Hankel matrix considering a fixed number of parameters. This leads directly to faster calculations and less memory space requirement.
On the other hand however, also this method will not lead to well-defined results. As there is no condition of structure on the approximating Page matrix, there is no guarantee that the elements in it represent a state space model with dimension n even if rank P equals n. This problem also occurred in the Hankel matrix approach, but could immediately be recognized by the lack of Hankel block structure.
As a result, both the Hankel and Page matrix approach fail in finding a well-defined solution to the problem, though in practice they appear to produce quite acceptable suboptimal solutions, as will be illustrated in the next chapter.

6. Results of simulations

In order to test the properties of both methods, different systems have been simulated. Simulations show the advantage of giving the freedom to influence certain quantities relevant for the realization algorithm, such as
— number N of available Markov parameters
— level of noise disturbance
— eigenvalues of the system.
In this paper only a few results will be presented. An extended analysis can be found in [4].
Fig. 1 shows a block diagram of the operations :

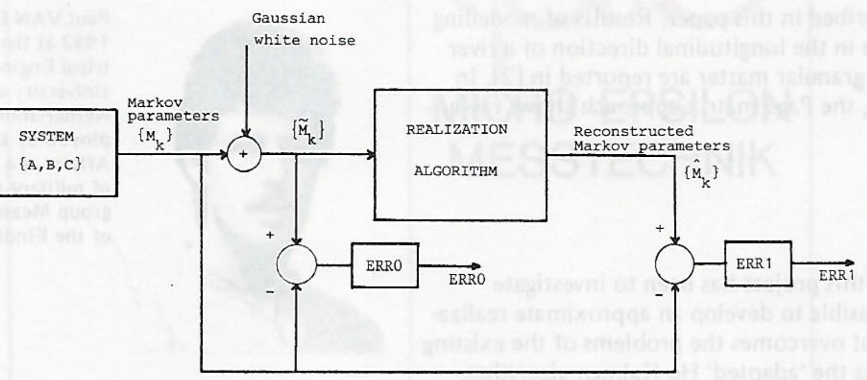

Fig. 1. Simulation and processing of Markov parameters.

The applied noise disturbance is an additive white Gaussian noise, independent on the different entries in Markov parameters. The standard deviation σ is chosen 0.25.

The results for two systems will be presented, both having 3 inputs, 2 outputs and dimension 4.

In state space form :

$$B = \begin{bmatrix} 1 & 0 & 1 \\ -1 & .5 & .5 \\ 0 & 1. & -.5 \\ 0 & .5 & -1. \end{bmatrix} \qquad C = \begin{bmatrix} .5 & 0 & 1 & 1 \\ 1 & -.5 & .5 & -.5 \end{bmatrix} \tag{17}$$

$$\text{SYSA}: \quad A = \text{diag}(0.4, 0.3, 0.2, 0.1) \tag{18}$$

$$\text{SYSB}: \quad A = \text{diag}(0.6, 0.5, 0.9, 0.7) \tag{19}$$

The relative errors ERRO and ERRI are defined as the quotients of the energy in the error and the energy in the deterministic sequence of Markov parameters :

$$\text{ERRO} = \frac{\sum_{k=1}^{N} \| \tilde{M}(k) - M(k) \|_E^2}{\sum_{k=1}^{N} \|M(k)\|_E^2} \tag{20}$$

$$\text{ERRI} = \frac{\sum_{k=1}^{N} \|M(k) - M(k)\|_E^2}{\sum_{k=1}^{N} \|M(k)\|_E^2} \tag{21}$$

where $\| \ \|_E$ stands for Euclidean (Frobenius) norm, and $\tilde{M}(k)$, $\hat{M}(k)$ are Markov parameters from the noise disturbed, resp. reconstructed sequence.

ERRO is a measure for the disturbance of the available Markov parameters, ERRI is a measure for the error in the result of the approximate realization.

The results of the simulations are presented in Fig. 2.

For SYSA, a system with relatively small eigenvalues, the Page matrix approach shows considerably better results than the Hankel method. Apparently the unequal weighing of Markov parameters in the other algorithm disturbs the results quite heavily. For SYSB with relatively large eigenvalues the performance of both methods is different too : a Hankel matrix approach shows better results.

Apart from these simulations, both methods have also been applied to a stochastic realization problem. In this problem stochastic signals are modelled as filtered white noise sequences of Gaussian distributions, while covariance matrices take the position of the Markov

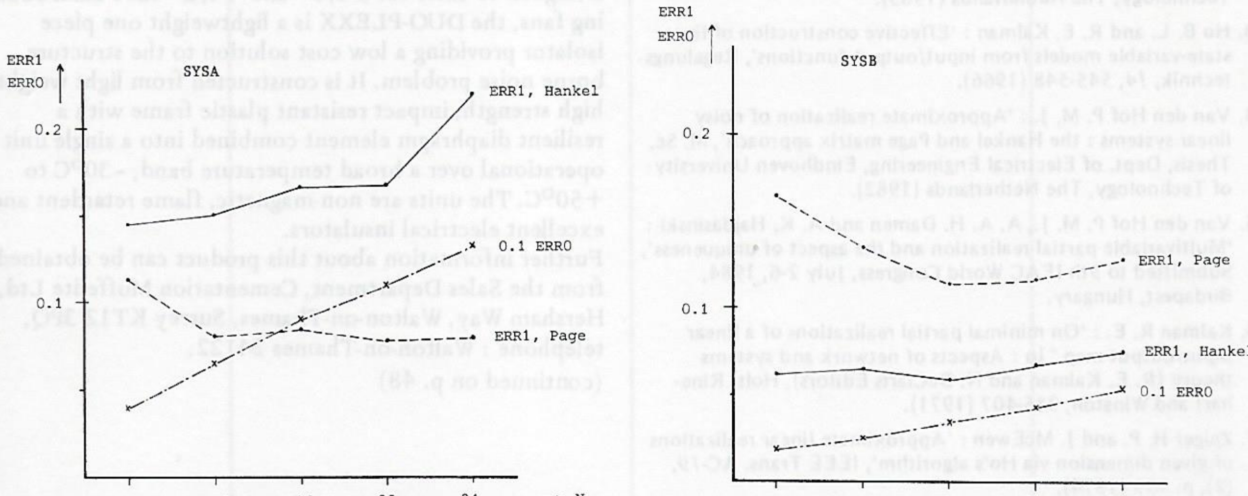

Fig. 2. Relative error in reconstructed and input Markov parameters, measured over N parameters
N = 8-24, $\sigma = 0.25$
Results are averaged over 3 noise series
Fig. 2a: SYSA, Fig. 2b: SYSB

parameters described in this paper. Results of modelling the dune profile in the longitudinal direction of a river in a riverbed of granular matter are reported in [2]. In this application, the Page matrix approach shows favourable results.

7. Conclusions

The purpose of this project has been to investigate whether it is possible to develop an approximate realization method that overcomes the problems of the existing methods, such as the 'adapted' Ho-Kalman algorithm. Within this scope, the Page matrix approach has been introduced and analysed.
Both methods, though formally incorrect, provide us with reasonable low order models of multivariable systems.
In situations where the signal to noise ratio is bad due to low sample frequency (small eigenvalues) and/or high noise level, the Page matrix approach shows better results.
Moreover, this latter algorithm is favourable if computation time and memory space cause problems.

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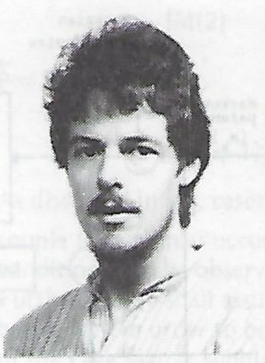

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SHORT COMMUNICATIONS

A BREAKTHROUGH IN NOISE REDUCTION

Cementation Muffelite, a Trafalgar House company, announce a brand new product to help solve the problem of fan induced structure borne noise.
'Today, every office contains a quantity of noisy, fan cooled electronic equipment – computers, word processors, photo-copiers, disc-drive units and electronic equipment in cabinets – all producing structure borne noise from cooling fan mechanisms,' says Marketing Director, Tony Jones.
'We are pleased to be the first UK producers of a mounting which can reduce these noise levels from between 3-16 decibels depending on the type of fan, housing and cabinet structure. We call it the DUO-PLEXX and it provides the most efficient and cost effective cure for cooling fan noise in electronic equipment; a breakthrough in noise suppression technology,' he said.
Designed to cater for 3 1/8″ and 4 1/2″ tube axial cooling fans, the DUO-PLEXX is a lightweight one piece isolator providing a low cost solution to the structure borne noise problem. It is constructed from light weight, high strength, impact resistant plastic frame with a resilient diaphragm element combined into a single unit operational over a broad temperature band, $-30^{o}C$ to $+50^{o}C$. The units are non-magnetic, flame retardent and excellent electrical insulators.
Further information about this product can be obtained from the Sales Department, Cementation Muffelite Ltd, Hersham Way, Walton-on-Thames, Surrey KT12 3PQ, telephone : Walton-on-Thames 24122.
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