MULTIVARIABLE PARTIAL REALIZATION AND THE ASPECT OF UNIQUENESS

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Abstract. This paper deals with the minimal partial realization problem for a multivariable power series of finite length.
Specifically, the problem of the uniqueness of the extension sequence is considered and a fundamental criterion is given which is necessary and sufficient for the existence of a unique extension.
A specific class of formal power series is introduced (finite generic sequences) for which the criterion mentioned above takes a very simple form. This class is of interest when considering the availability of a sequence of noise disturbed Markov parameters of a multivariable system.
Finally the approximate partial realization problem is defined.

Keywords: Approximation theory; linear systems; Markov parameters; minimal realization; stochastic realization; modelling; multivariable systems; system order reduction.

1 INTRODUCTION

The problem of realization of a finite sequence of Markov parameters $\{M(i)\}_L$ (L indicating the number of Markov parameters M(i)), by a state space model $\{A,B,C\}$ has been dealt with by several authors in different ways. One of the reasons for this divergency in strategy is the fact that the problem can be considered from different points of view, depending on the application of the results.
We can distinguish between an exact realization, where a given sequence of Markov parameters has to be reconstructed exactly by a corresponding state space model, and an approximate realization, where the sequence has to be approximated in some specific sense. The first direction, analysed by Kalman (1969), Tether (1970) and continued by Roman and Bullock (1975), Kalman (1979), Bosgra/v.d. Weijden (1980), Bistritz (1983) and Damen and others (1984) is of mainly theoretical importance; exact realization of a finite sequence $\{M(i)\}_L$ will generally lead to state space models of high dimensions (depending on L). In the field of identification, the problem of approximate realization is much more interesting: a sequence of Markov parameters has been measured or estimated, i.e. considered to be noise corrupted. One is interested in a low dimensional approximation of the given sequence, leading to a reduced order model.
This problem has been worked upon by Zeiger/McEwen (1974), Kung (1979), v. Zee/Bosgra (1979), Staar/Vandewalle/Wemans (1981), Damen/Hajdasinski (1982), Damen/Van den Hof/Hajdasinski (1982) and others.

While for the exact realization a strict formulation of the problem has been stated by Kalman (1979), a proper definition of the approximate realization has not yet been given.
The absence of a proper definition is one of the reasons why the present solutions to the approximation problem are ill-defined, and therefore still far from optimal.
(see a companion paper Hajdasinski/Damen/Van den Hof (1984)).

In this paper an attempt will be made to bridge the gap between the two subjects by clearly posing the problems in a comparable way.
Corresponding to the minimal partial realization problem (MPR), as formulated by Kalman (1979), the approximate partial realization problem (APR) will be defined, and it will be shown that in a specific situation, an optimal solution to this latter problem can be found. (section 4).
One of the important aspects of both problems: the uniqueness of the extension of the finite sequence $\{M(i)\}_L$, will be given special attention in section 3.

2 MINIMAL PARTIAL REALIZATION

A multivariable formal power series is given by an infinite sequence of $[q \times p]$ matrices,

$$\{M(i)\}_\infty = M(1),\ M(2), \ldots \qquad (1)$$

which can be represented by

$$M(i) = CA^{i-1}B \qquad \forall\ i \in N^+ \qquad (2)$$

where A,B,C are matrices of sizes $[n \times n]$, $[n \times p]$, $[q \times n]$ and n is the dimension of this realization. N^+ is the set of all positive integers.
The triplet $\{A,B,C\}$ is called a minimal realization if and only if the dimension n is minimal among all possible $\{A',B',C'\}$ fulfilling (2). A necessary and a sufficient condition for the existence of a minimal realization is given by the realizability criterion:

There is an integer N_o and a block Hankel matrix H such that

$$\rho H[N_o,N_o] = \rho H[N_o+i,\ N_o+j] = n \quad \forall i,j \quad N^+ \qquad (3)$$

where $\rho H[a,b]$ is the rank of the block Hankel matrix of block dimension $[a \times b]$.

If we are dealing with a sequence of finite length L:

$$\{M(i)\}_L = M(1), M(2), \ldots, M(L) \qquad L \in N^+ \qquad (4)$$

we can always find a triplet $\{A,B,C\}$ with a finite dimensional A, such that:

$$M(i) = CA^{i-1}B \quad i = 1,2,\ldots,L \qquad (5)$$

and consequently the dimension n of this sequence will always be finite. The crucial question is whether the continuation $M(i) = CA^{i-1}B$ for i > L is unique for all triplets $\{A,B,C\}$ satisfying (5).

Considering the definition of Kalman (1979) we can state the problem as follows:

Definition 1: Minimal Partial Realization Problem (MPR)

a. Find all exact realizations of sequence $\{M(i)\}_L$ whose dimension n is minimal over the family of all possible realizations for this sequence.

b. Give the necessary and sufficient conditions under which there is only one minimal partial realization. ('uniqueness' in the sense of the continuation of $\{M(i)\}_L$, which means that the result forms an equivalence class modulo the choice of basis in the state space).

c. Where non-uniqueness occurs, parametrize all minimal partial realizations.

In this paper we will be dealing with aspects a. and b. Results on analysis of aspect c can be found in e.g. Bosgra/v.d. Weiden (1980), Bosgra (1983), Bistritz (1983) and Damen and others (1984).

The minimal dimension of the realization of $\{M(i)\}_L$ can be found by means of the rank of the partial behaviour matrix, defined as follows:

The partial behaviour matrix associated to the finite sequence $\{M(i)\}_L$ is the following (block) Hankel matrix:

$B_L = B(M(1),\ldots,M(L))$:

$$\begin{bmatrix} M(1) & M(2) & \cdots & \cdots & M(L) \\ M(2) & M(3) & \cdots & \cdots & ? \\ \vdots & & M(L) & & \vdots \\ M(L-1) & & ? & & \vdots \\ M(L) & ? & \cdots & \cdots & ? \end{bmatrix}$$

where the elements denoted by ? correspond to values of the sequence which are not part of the data given for the partial realization problem. The rank of this partial behaviour matrix is the minimal possible rank obtained by proper choice of the elements indicated by the question marks.
As has been shown by Kalman (1969) and Tether (1970), and is quite straightforward, additional Markov parameters M(L+1), M(L+2),... can always be found in such a way that the rank of the partial behaviour matrix does not increase.

Both authors mentioned above, have analysed the MPR-problem by looking for situations where the Ho-Kalman algorithm could be applied, leading to an exact realization. For this purpose the next criterion has been introduced, which is a reflection of the realizability criterion (3).

Definition 2: Partial Realizability Criterion (PRC)

There exist positive integers N and N' such that:
a. $N + N' = L$, and
b. $\rho H[N',N] = \rho H[N'+1,N] = \rho H[N',N+1]$ (=n) (7)

Note that $\rho H[N',N]$ is the rank of the Hankel matrix of size $[qN' \times pN]$ whose elements are belonging to $\{M(i)\}_{L-1}$.
If the partial realizability criterion is fulfilled, the Ho-Kalman algorithm can be applied (Ho/Kalman(1966)) leading to a realization. It is easy to prove that this realization realizes the sequence $\{M(i)\}_L$ exactly and moreover that this realization is minimal and unique (see Tether (1970), and Damen and others. (1984)).

We will now work out an example, illustrating the effect of this criterion:

Example

Let $\{M(i)\}$ be the Markov parameters of a single input single output system (SISO), which consists of two delay lines of 1 and 5 samples:
$\{M(i)\} = 1,0,0,0,1,0,0 \ldots..$
The infinite Hankel matrix is then given by:

$$\begin{matrix} 1&0&0&0&1&0&0&0&0&0&\cdots \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&. \\ . & & & & & & & & & . \\ . \end{matrix}$$

and it is easy to show that:

For $2 \leq L \leq 4$ - the rank will be 1
- the PRC is satisfied for all N,N' (N+N'=L)
- there is a unique MPR with transfer function z^{-1} and e.g. A = 0, B = C = 1

For $5 \leq L \leq 7$ - the rank n will be 4
- the PRC is not satisfied for any N,N' (N+N'=L), for N,N' ≥ 4 will require L ≥ 8
- there is not a unique MPR

For L = 8
- the rank n equals 4
- the PRC is fulfilled for N=N'=4
- there is a unique MPR with transfer function

$$z^{-1} + z^{-5} + z^{-9} + \ldots. = \frac{z^3}{z^4-1}$$

for example, $C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For L = 9
- the rank n equals 5
- the PRC is not satisfied for any N,N' (N+N'=L), for N,N' ≥ 5 will require L ≥ 10.
- there is not a unique MPR

For L ≥ 10
- the rank n equals 5
- the PRC is satisfied for N ≥ 5 N' ≥ 5 (N+N'=L)
- there exists a unique MPR with transfer function

$$z^{-1} + z^{-5} = \frac{z^4+1}{z^5}\text{, for example}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Until now nothing has been said about the minimal partial realization when this partial realizability criterion is not fulfilled. This aspect, together with the uniqueness of the MPR will be dealt with in the next section.

3 UNIQUENESS OF MINIMAL PARTIAL REALIZATIONS; FINITE GENERIC SEQUENCES

In the previous section it has been shown that in a specific situation it is possible to find a unique minimal partial realization by means of the Ho-Kalman algorithm. Both Tether (1970) and Kalman (1971) have given this result before. However if the partial realizability criterion is not fulfilled, the existence of a unique minimal partial realization has never been discussed for the general situation. With respect to this we present the next result:

Theorem 1: A finite sequence $\{M(i)\}_L$ has a unique minimal partial realization if and only if the partial realizability criterion is fulfilled.

A proof of this theorem, using results of Bosgra/v.d. Weiden (1980) is given in the appendix.

Now formally the necessary and sufficient condition for a unique MPR is stated by the PRC. In practice this criterion shows two disadvantages:

1. It is a rank test on at least three different matrices, which implies quite a lot of work. 'At least' is used as we do not know beforehand N and N', which could fulfil PRC. Therefore we have to check all possible pairs (N,N') as long as no pair is found that satisfies the PRC.

2. A rank test becomes rather indefinite when it concerns inaccurate data.

In cases where one is dealing with data which is not disturbed by noise, the second disadvantage may be overcome, but numerical (truncational) errors may still be troublesome. If we are dealing with noisy data, the exact minimal partial realization will be of a high dimension and will also incorporate the noise contribution. We will now show that the partial realizability criterion can be substantially simplified for such "noisy sequences". A "noisy sequence" will belong to the following class of sequences:

Definition 3: Finite Generic Sequence (FGS)

A finite generic sequence $\{\tilde{M}(i)\}_L$ is a sequence of $[q \times p]$ matrices for which every Hankel matrix constructed from this sequence has full rank.

Comment: In practice it concerns a "generic" sequence, which has been deteriorated by (independent) broad-banded noise. Only in the exceptional case of certain deterministic (particular, singular) sequences, dependencies may appear among the rows (or columns) of the Hankel matrices concerned. Consequently the probability is one that a random sequence will be an FGS.

Because of the special properties of a FGS, a concrete statement can be made with respect to the minimal dimension of the corresponding partial realization, reflected in the next theorem:

Theorem 2: The minimal dimension $\tilde{n}$ of a finite generic sequence is the rank of the partial behaviour matrix and equals the dimension of the greatest square submatrix of known elements.

The verification of this theorem is easy and will be left to the reader.

For a finite generic sequence the partial realizability criterion cannot be satisfied by linear dependency between the elements in the rows/columns in the Hankel matrix; the stochastic nature of the sequence will prevent this. Therefore the only situation in which the PRC can be satisfied, is when the rank of an enlarged Hankel matrix cannot increase because it is restricted by the smallest dimension of H. This result is stated in the next lemma.

Lemma: For a finite generic sequence the partial realizability criterion is satisfied if and only if there exist integers N',N such that N'+N = L and H[N',N] is a square matrix.

The proof of this lemma is given in Damen and others (1984).
If the condition mentioned in this lemma is fulfilled, the rank of the square Hankel matrix will equal the minimal dimension $\tilde{n}$ of the finite generic sequence.
The results above give us instruments for developing a very concrete and manageable criterion for the uniqueness of a minimal partial realization of a FGS:

Theorem 3: A finite generic sequence has a unique minimal partial realization if and only if:

$$L = \frac{a(p+q)}{k} \quad \text{where } a \in N^+, \text{ k is the greatest common divisor (GCD) of } (p,q) \tag{8}$$

Proof: In theorem 1 it has been proved that the PRC is a necessary and sufficient condition for the existence of a unique MPR which, of course, also holds for a FGS.
According to the lemma, the fulfilment of the PRC is equivalent to the existence of a square Hankel matrix of size qN' x pN.

Consequently:

$$\left| \begin{array}{l} L = \frac{\tilde{n}}{q} + \frac{\tilde{n}}{p} \\ \tilde{n} = \frac{apq}{k} \quad a \in N^+ \\ \qquad\qquad k = GCD(p,q) \end{array} \right.$$

because all variables are positive integers.
This is equivalent to

$$L = \frac{a(p+q)}{k} \qquad \tilde{n} = L\,\frac{pq}{p+q} \qquad \text{Q.E.D}$$

Comment:
- This criterion for uniqueness is a very suitable, simply decountable tool.
- The rank of the partial behaviour matrix was $\tilde{n}$. Consequently the degrees of freedom for a state space description equal $\tilde{n}(p+q)$. By putting this equal to the number of elements Lpq in the FGS $\{\tilde{M}(i)\}_L$ the same condition (8) can be derived.

Finally, if, for a finite generic sequence the partial realizability criterion is fulfilled or equivalently L satisfies theorem 3 the realization can easily be found, e.g. by means of the Ho-Kalman algorithm. If the PRC is not fulfilled random (indeterminate) matrices $\tilde{M}(i)$ have to be joined until L satisfies the given condition and again Ho-Kalman can be applied. Another approach is, of course, to neglect the last number of matrices until L satisfies the given condition. However, in this case these neglected matrices will not be realized exactly and therefore the solution to the MPR problem will not be correct in the sense of definition 1.

The results, as presented in this section, can be considered as a starting point to deal with the problem of approximation as illustrated in the introduction. This will be handled in the next section.

4 THE APPROXIMATE PARTIAL REALIZATION PROBLEM

As we have seen in theorem 2 the minimal dimension $\tilde{n}$ of a finite generic sequence equals the dimension of the greatest square submatrix of known elements in the partial behaviour matrix.
This greatest square submatrix of B_L has to contain either an integer number of block rows or an integer number of block columns or both. Based on this statement an expression for the minimal dimension $\tilde{n}$ can be derived:

$$\tilde{n} = \max\left\{q \text{ entier}\left[(L+1)\frac{p}{p+q}\right], p \text{ entier}\left[(L+1)\frac{q}{p+q}\right]\right\} \tag{9}$$

(see also Damen and others (1984)).

If a proper value of L is chosen, corresponding with the uniqueness condition (8), this expression (9) again becomes $\tilde{n} = \frac{apq}{k}$.

Whether the MPR is unique or not, the dimension $\tilde{n}$ generally given by expression (9) becomes extremely high with increasing L. This is due to the fact that the realization represents both the corrupting noise and the underlying deterministic system, which generated the sequence. Under these circumstances one wants to eliminate the noise part by choosing a proper norm in the reduction of the dimension. This brings us to the problem of approximate realization.

Definition 4: Approximate Partial Realization Problem (APR).

Given the sequence $\{M(i)\}_L$ and the dimension n of the corresponding MPR, where $L \in N^+$ (possibly infinite)

a. Find all realizations of dimension $n_r < n$, where a prescribed norm is minimized.
b. Give the necessary and sufficient conditions under which the realization is unique.
c. In the case of non-uniqueness, parametrize all APR's.

Comment: Note that we defined the APR for the general case and not just for finite generic sequences. General sequences may be processed along the same lines e.g. if one wants to apply dimension reduction of known filters.

- Examples of norms:

1. $\| H(\{M(i)\}_L) - H(\{CA^{i-1}B\}_L) \|_S$

   where $\{A,B,C\}$ is a triplet representing a wanted partial realization.

   The matrix H is a block Hankel matrix of specific size built up by the sequence $\{M(i)\}_L$ and S stands for spectral norm.
   This norm particularly suits control purposes as it weighs the sequence as an operator on input signals:
   Let $\underline{u}^-$ be the vector of all inputs prior to moment k, then
   $\underline{y}^+ = H(\{M(i)\}_L)\underline{u}^-$ is the prediction of the outputs after moment k and
   $\underline{\varepsilon}^+ = [H(\{M(i)\}_L) - H(\{CA^{i-1}B\})_L]\underline{u}^-$ is a measure for the misfit, if a dimension reduction to n_r has been applied. Then:
   $\|\varepsilon^+\|_F \leq \|H(\{M(i)\}_L) - H(\{CA^{i-1}B\}_L)\|_S$
   $\forall \underline{u}^-$: $\|\underline{u}^-\|_F = 1$, where F stands for Frobenius norm.

2. If we are dealing with a parameter estimation problem, where our goal is to estimate the actual Markov parameters from the given noise disturbed sequence $\{\tilde{M}(i)\}_L$, the sequence itself is an object and no longer seen as an operator. For a least squares estimate the norm would then simply be the sum of squares of all elements in the sequence (Frobenius norm). This conforms with the following norm:
   $\| P(\{M(i)\}_L) - P(\{CA^{i-1}B\}_L) \|_F$
   where P stands for Page matrix, whose entries consist of the matrices M(i), such that each M(i) is used only once. More information can be found in Damen/Van den Hof/Hajdasinski (1982).

In the above examples of norms it has been suggested that the state space representation is directly parametrized. Most authors dealing with this topic attacked the problem, by initially estimating a sequence $\{\hat{M}(i)\}_L$, which was conditioned to have a unique realization of order n_r, e.g. Zeiger/McEwen (1974), van Zee/Bosgra (1979), Kung (1979), Staar/Vandewalle/Wemans (1981), Damen/Hajdasinski (1982), Damen/Van den Hof/Hajdasinski (1982), Van den Hof (1984).

No author, however, has so far succeeded in finding an optimal solution in the sense of minimizing a prescribed norm.

Following the opposite line, where from the very beginning a state space representation is used, fruitful use may be made of the Adamjan/Arov/Krein algorithm (see Kung/Lin, 1981). This algorithm is capable of finding a realization of prescribed dimension n_r according to the minimization of the spectral norm of the infinite Hankel matrix, once a partial realization for the original sequence $\{\tilde{M}(i)\}_L$ has been found.
Problems arise in the case that this original MPR is non-unique because it cannot be stated that the non-uniqueness (indefinite part) will be eliminated during the dimension reduction phase. Consequently the condition of uniqueness remains an important one. This is precisely the reason why we were so concerned with the uniqueness of the MPR in section 3.
However, even in cases where a unique MPR can be determined, the Adamjan/Arov/Krein approach may lead to ill-defined solutions. If a MPR of the original sequence $\{\tilde{M}(i)\}_L$ has been determined, this sequence can be extended to an infinite sequence $\{\tilde{M}(i)\}_\infty$. In the algorithm mentioned above the infinite Hankel matrix serves in the criterion from which the approximation of the original sequence has been derived theoretically. Dependent on its index i the Markov parameter $\tilde{M}(i)$

appears more frequently in this infinite Hankel matrix, which implies that in the dimension reduction procedure the extension of the sequence $\{\tilde{M}(i)\}_L$ is much more weighed than the original sequence itself. This aspect becomes important in cases where the MPR of the original sequence contains (almost) non-stable poles. If we are dealing with a sequence of Markov parameters disturbed by noise, this situation will very likely happen, caused by the (random) effects of the noise. As a result these non-stable poles will be considered to be dominant because of their heavy influence on the extension of the sequence $\{\tilde{M}(i)\}_L$; therefore in the dimension reduction phase these poles will not be eliminated, in contrast with the really relevant poles of this sequence. For this moment we will only give two suggestions that may serve as a remedy for this problem.

1. Construct a new sequence $\{M'(i)\}$ in such a way that $M'(i) = \tilde{M}(i) \quad i=1,..,L$
   $M'(i) \equiv 0 \quad i \geq L+1$
   Find a unique MPR of $\{M'(i)\}_\infty$ and apply the Adamjan/Arov/Krein algorithm to this MPR to construct an approximate partial realization. Because the last part of the sequence $\{M'(i)\}$ is fixed to zero, a MPR will be found without any non-stable poles, thus circumventing the problem described above.
2. Find a unique MPR of $\{\tilde{M}(i)\}_L$ as before and use the sum of squared contributions of the different poles of this MPR present in the original FGS, as a criterion to distinguish between relevant and non relevant poles. This directly leads to a APR of desired dimension n_r.

5 CONCLUSIONS

The realization of a finite sequence of Markov parameters is handled in this paper from two different points of view: exact and approximate realization. The notion of uniqueness of the extension of the given sequence plays a very important role. If the extension and, consequently, the realization is non-unique, it will lead to results which contain indefinite information. Depending on the application of the results, this indefinite part may cause problems, e.g. in dimension reduction. For the partial realization problem, a necessary and sufficient condition has been introduced for the existence of a unique realization which can always be evaluated using the Ho-Kalman algorithm. A class of finite generic sequences (FGS) is defined containing sequences that are constructed from noisy data.
When such a sequence is considered, the criterion for uniqueness of a minimal partial realization takes a very simple form. A straightforward algebraic test on the length of the available sequence will show uniqueness or non-uniqueness of the extension sequence. This unique extension, in turn, is important when trying to find an approximate realization of prescribed low dimension, which fits the original sequence as well as possible.
This problem of approximate partial realization is defined, and it is shown how a unique solution to this problem may be obtained.

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APPENDIX

Proof of theorem 1

It has to be proved that the partial realizability criterion (PRC) is a necessary and sufficient condition for the existence of a unique minimal partial realization (MPR).

For this purpose the following results of Bosgra/-v.d. Weiden (1980) and Bosgra (1983) are of interest: Given a multivariable formal power series $\{M_{ij}(k)\}_{k=1,\dots,L;\ i=1,\dots q;\ j=1,\dots,p'}$ with ij indicating element ij in matrix M(k).
Suppose $I_n = \{\nu_1,\nu_2,\dots,\nu_q\}$ and $J_n = \{\mu_1,\mu_2,\dots,\mu_p\}$ represent resp. structural row and column indices, defining a nice selection of independent rows and columns of the partial behaviour matrix B_L.
It can be proved that $\{I_n,\nu_n,G\}$ with

$$G = \{M_{ij}(k)\ k=1,2,\dots,\nu_i+\mu_j\} \text{ for } i=1,\dots,q;$$

$j=1,\dots,p$ constitutes a complete set of independent invariants under state-coordinate transformations for triples (A,B,C). Note that according to the definition of a nice selection by Bosgra/v.d. Weiden (1980) structural indices are allowed to be zero.
If we consider a set of structural indices (I_n', J_n') defining a Kronecker selection, the result mentioned above is still valid and the set $\{I_n',J_n',G'\}$ is uniquely determined.
From this statement it follows immediately that a unique MPR of a formal power series exists <u>if and only if</u> this basis of invariants is known.
Furthermore, (see Bosgra (1983)), this basis of invariants is known <u>if and only if</u>

$$\max_{ij} (\nu_i' + \mu_j') \leq L \tag{A1}$$

For the proof of theorem 1, the equivalence of this condition with the PRC has to be shown:

<u>Necessity</u>: Assume that the PRC is fulfilled. For all positive ℓ, block rows with block row index $N'+\ell$ as well as block columns with block column index $N+\ell$ in B_L will only contain dependent rows/columns.

As a result $\max_i \nu_i' \leq N'$ and $\max_j \mu_j' \leq N$ which means that (A1) is fulfilled, because $N'+N = L$.

<u>Sufficiency</u>: Assume (A1) is fulfilled.

Define $\max_i \nu_i' = \nu'$ and $\max_j \mu_j' = \mu'$.
Because of (A1) $\nu'+\mu' \leq L$.
It follows directly that the PRC is satisfied with e.g. $N' = \nu'$ and $N = L-\nu' \geq \mu'$.