

Integrating data-informativity conditions in predictor models for single module identification in dynamic networks^{*}

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Abstract: For consistent identification of a target module in a dynamic network with the local direct method, basically two prime conditions need to be satisfied: (a) a set of structural conditions on the choice of the predictor model, i.e. a set of input and output node variables, and (b) conditions on data-informativity. While for conditions (a) constructive algorithms for node selection have been presented that appropriately guarantee that the identified object can indeed reveal the target module, the requirements for satisfying (b) have not yet been integrated fully. In this paper, we will present simplified path-based results for generic data-informativity, and show how they can be integrated in constructive algorithms for predictor model selection that provide consistent target module estimates. It is shown that data-informativity not only requires a sufficient number of external excitation signals to be present in the network, but also puts restrictions on the structure of the predictor model, i.e. the selection of input and output node variables. Some examples are presented to illustrate the new results.

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1. INTRODUCTION

Large-scale data analytics is currently present in a growing number of technology areas, where data-driven diagnostics and model-based control require the handling of dynamic systems that are composed of interconnected subsystems, while the interconnections are characterized by a topology, that is often specified up front. Performing data-driven dynamic modeling in these systems has been given attention in the literature over the last fifteen years, see e.g. Gonçalves and Warnick (2008); Materassi and Innocenti (2010); Haber and Verhaegen (2014); Van den Hof et al. (2013); Hendrickx et al. (2019).

The particular problem of identifying a single module in such networks, of which the topology is known, has been addressed in e.g. Dankers et al. (2016); Gevers et al. (2018); Bazanella et al. (2019); Materassi and Salapaka (2020); Jahandari and Materassi (2022); Shi et al. (2023). In this problem, selecting the set of node variables that should be available from measurements, together with an appropriate identification method are the key issues.

In Ramaswamy and Van den Hof (2021) a local direct method has been presented in a prediction error context, that provides the conditions under which a target module can be identified with maximum likelihood properties, i.e., consistent and with minimum variance asymptotically in the number of data. Relative to indirect methods, as e.g.

Gevers et al. (2018), it can exploit the excitation properties of nonmeasured disturbance signals for satisfying the required data-informativity conditions, and therefore allows for less expensive experiments (with less external excitation signals).

The core conditions in Ramaswamy and Van den Hof (2021) for arriving at a consistent estimate of the target module are covered by two sets: (a) a set of graph-based conditions for constructing an appropriate predictor model, specified by inputs and outputs; and (b) a spectral condition on data-informativity of the concerned signals in the predictor model. For satisfying the conditions of type (a), several algorithms were presented. While preliminary work has shown that also the conditions for (b) can be phrased as graph-based conditions, provided that we accept that the conditions hold generically (Van den Hof and Ramaswamy (2020)), they were not in a form that could be easily integrated into the conditions of type (a), and thus into the algorithms for constructing the predictor model. In this paper we are going to show that the data-informativity conditions can be phrased in a graph-based setting, and can be integrated into the predictor model construction algorithms. While one could have the impression that any predictor model satisfying the type (a) conditions could be made data-informative by adding a sufficient number of external excitation signals, it will be shown that this is not the case, and that including the type (b) data-informativity conditions can lead to differently structured predictor models.

After presenting the network setup and the prime result of Ramaswamy and Van den Hof (2021) in Sections 2 and

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3, new results on the data-informativity conditions will be presented in Sections 4 and 5. Next the available full input and minimum input algorithms for predictor model construction are extended to include the data-informativity conditions, and the consequences are illustrated with a 4-node example.

2. NETWORK SETUP

We consider a dynamic network in the so-called module framework where node variables w are interconnected through transfer functions. A compact representation of the network is given by

$$w(t) = G(q)w(t) + H(q)e(t) + u(t) \quad (1)$$

with w an L -dimensional vector, $G(q)$ a hollow rational transfer function matrix, i.e. with zeros on the diagonal entries, in the the time shift operator q , i.e. $q^{-1}w(t) = w(t-1)$, e an L -dimensional vector of white noise processes, $H(q)$ the rational disturbance model, and $u(t) = R \cdot r(t)$, accounting for the effect of measured external excitation signals r on the network, with R a binary matrix. A non-zero element $G_{k\ell}(q)$ in $G(q)$ is referred to as a *module*.

It is further assumed that the network is stable and well-posed Van den Hof et al. (2013). In this paper it is also assumed for simplicity that all elements in $G(q)$ are strictly proper.

After selecting a target module $G_{ji}(q)$ with input w_i and output w_j , the single module identification problem comes down to selecting a set of measured node variables w , that compose a predictor model with predictor inputs and predicted outputs, on the basis of which a (consistent) estimate of $G_{ji}(q)$ can be obtained through a prediction error identification method. In this problem it is assumed that the network topology, i.e. the zero-pattern in G , H and R is known.

3. CONSTRUCTING A PREDICTOR MODEL WITH THE LOCAL DIRECT METHOD

On the basis of a selected set of node variables w , a predictor model is constructed with inputs w_D and outputs w_Y , where $w_i \in w_D$ and $w_j \in w_Y$, that is formalized as follows:

$$w_Y(t) = \bar{G}(q)w_D(t) + \bar{H}(q)\xi_Y(t) + \underbrace{\bar{J}(q)u_\kappa(t) + \bar{S}u_p(t)}_{w_Y(t)} \quad (2)$$

with ξ_Y a vector white noise process, and the effect of external excitation signals, $u_Y(t)$ is decomposed in two contributions, where u_κ and u_p are distinct sets of components of the excitation signal u , $\bar{J}(q)$ is a dynamic transfer function and \bar{S} is a binary matrix. The dynamic term $\bar{J}(q)$ typically occurs when there is an excitation signal u entering a node w that is not present in $w_Y \cup w_D$.

This predictor model is schematically depicted in Figure 1. There can be node signals w_Q with $Q = \mathcal{D} \cap \mathcal{Y}$ that appear in both input and output. This will be used to handle confounding variables¹ in the estimation problem, by

¹ A confounding variable is an unmeasured variable that has paths to both the input and output of an estimation problem Pearl (2000). The resulting paths are referred to as “confounding paths”. If one of the confounding paths passes through a node in w the confounding variable is called “indirect”, else it is called “direct” (Ramaswamy and Van den Hof (2021)).

modelling their effect through a multivariate disturbance model at the output, to be explained later on. Additional notation includes $w_\mathcal{U}$ with $\mathcal{U} = \mathcal{D} \setminus \mathcal{Q}$, while $w_o = w_j$ if $j \notin \mathcal{Q}$; else w_o is void. The node signals that do not appear in the predictor model are referred to as the unmeasured nodes and are collected in w_Z , i.e. $\mathcal{Z} := \{1, \dots, L\} \setminus \{\mathcal{Y} \cup \mathcal{U}\}$.

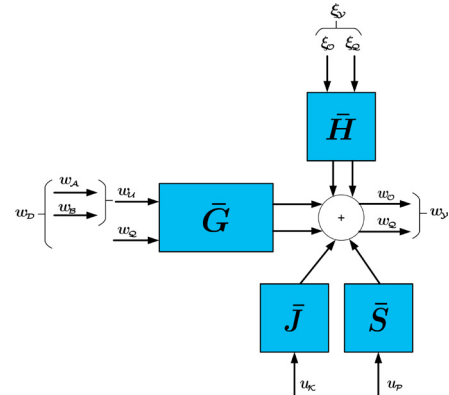


Fig. 1. Predictor model for the local direct method (Ramaswamy (2022)).

The predictor model (2) is the result of writing the network equation for the collected set of selected node signals in the structured form:

$$\begin{bmatrix} w_Y \\ w_\mathcal{U} \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{G} & 0 \\ \bar{G}_{\mathcal{U}\mathcal{D}} & \bar{G}_{\mathcal{U}\mathcal{O}} \end{bmatrix}}_{\bar{G}_m} \begin{bmatrix} w_D \\ w_o \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{H} & 0 \\ 0 & \bar{H}_{\mathcal{U}\mathcal{U}} \end{bmatrix}}_{\bar{H}_m} \begin{bmatrix} \xi_Y \\ \xi_\mathcal{U} \end{bmatrix} + \begin{bmatrix} u_Y \\ u_\mathcal{U} \end{bmatrix} \quad (3)$$

with ξ_m a white noise process, while \bar{H} is monic, stable and stably invertible, the components in \bar{G} are zero if it concerns a mapping between identical signals, and u_Y and $u_\mathcal{U}$ are appropriate components of the excitation signal u .

A network transformation to the form (3) can always be made (Proposition 1 in Ramaswamy and Van den Hof (2021)), and is attractive because it decorrelates the disturbances on the variables w_Y and $w_\mathcal{U}$, reflected in the block-diagonal structure of \bar{H}_m . The first block row of (3) then serves as a basis for the predictor model (2).

The predictor model leads to an identification of the target module $G_{ji}(q)$ through the one-step-ahead predictor (Ljung (1999)):

$$\hat{w}_Y(t, \theta) = \bar{H}(q, \theta)^{-1} [\bar{G}(q, \theta)w_D(t) + \bar{J}(q, \theta)u_\kappa(t)] + (I - \bar{H}(q, \theta)^{-1})[w_Y(t) - \bar{S}u_p(t)] \quad (4)$$

where θ represent the parameters of a parametrized model. Applying a quadratic identification criterion, according to

$$\theta^* = \arg \min_{\theta} \bar{\mathbb{E}}(w(t) - \hat{w}_Y(t, \theta))^T P (w(t) - \hat{w}_Y(t, \theta)) \quad (5)$$

with P a positive definite weighting matrix, then leads to an (asymptotically) estimated model $\bar{G}(q, \theta^*)$.

For recalling the consistency result of this local direct method for network identification, we first need to present the structural conditions for module invariance in (3).

Proposition 1. (Ramaswamy and Van den Hof (2021)).

Let G_{ji}^o be the target module. In the transformed system's equation (3) it holds that $\bar{G}_{ji} = G_{ji}^o$ under the following conditions

- (a) In the network (1), every path from w_i to w_j that does not pass through G_{ji} , and every loop around w_j pass through a node in $w_{\mathcal{D}}$ (Parallel path and loop (PPL) condition);
- (b) \mathcal{U} is decomposed in disjunct sets \mathcal{A} and \mathcal{B} , such that in (1) there are no confounding variables for the estimation problem $w_{\mathcal{A}} \rightarrow w_{\mathcal{B} \cup \mathcal{D}}$;
- (c) $i \in \{\mathcal{A} \cup \mathcal{Q}\}$, and
- (d) Every path from $\{w_i, w_j\}$ to $w_{\mathcal{B}}$ passes through a node in $w_{\mathcal{Y} \cup \mathcal{U}}$.

The interpretation of this result is that we can allow for input signals $w_{\mathcal{B}}$ that have confounding variables with the outputs $w_{\mathcal{Y}}$, as a result of which their corresponding modules in $\tilde{G}_{\mathcal{Y}\mathcal{B}}$ will typically be estimated incorrectly. However since $i \notin \mathcal{B}$ this is not a problem, while at the same time the nodes in $w_{\mathcal{B}}$ can help to satisfy the other conditions.

Theorem 2. (Ramaswamy and Van den Hof (2021)).

Consider a dynamic network represented by (1) and a predictor model according to (4). Then the local direct identification method according to (5), leads to a consistent estimation of G_{ji}^o , i.e. $\tilde{G}_{ji}(q, \theta^*) = G_{ji}^o$ if

- (1) \mathcal{M} is chosen to satisfy $\mathcal{S} \in \mathcal{M}$ (system in the model set);
- (2) The structural conditions for module variance of Proposition 1 are satisfied;
- (3) $\Phi_{\kappa}(\omega) \succ 0$ for almost all ω , with

$$\kappa(t) = \begin{bmatrix} w_{\mathcal{D}}(t) \\ \xi_{\mathcal{Y}}(t) \\ u_{\kappa}(t) \end{bmatrix} \quad (6)$$

(data-informativity condition).

The interpretation of the data-informativity condition is that actually $\xi_{\mathcal{Y}}$ and u_{κ} cannot be used to contribute to the excitation of $w_{\mathcal{D}}$. For $\xi_{\mathcal{Y}}$ this is due to the fact that it serves as excitation source for the noise model $\tilde{H}(q, \theta)$, while u_{κ} excites the parametrized transfer $\tilde{J}(q, \theta)$ in (4). In the next section we will particularly focus on this data-informativity condition, and analyze how it can be turned into graphical conditions that can be merged with the structural conditions for module invariance in Proposition 1.

4. EXCITATION SIGNALS IN THE PREDICTOR MODEL

For further analysis of the data-informativity conditions in Theorem 2, we need to specify how the signals u_{κ} and $u_{\mathcal{P}}$ are composed on the basis of excitation signals in the original network (1). To this end we formulate the following result.

Proposition 3. Consider a predictor model specified by the signal sets $\mathcal{A}, \mathcal{B}, \mathcal{Q}, \mathcal{D}, \mathcal{Y}$ as defined in Section 3. Then

- (1) The set of excitation signals $u_{\mathcal{P}}$ are a subset of $u_{\mathcal{Y}}$ and composed as follows:
 - For $\ell \in \mathcal{Q}$, $u_{\ell} \in u_{\mathcal{P}}$ if all loops around w_{ℓ} and all paths from w_{ℓ} to \mathcal{B} pass through a node in $\mathcal{Q} \cup \mathcal{A}$.
 - $u_o \in u_{\mathcal{P}}$ if all loops around w_o pass through a node in $\mathcal{Q} \cup \mathcal{A}$ and all paths from w_o to \mathcal{Q} pass through a node in \mathcal{A} .

- (2) The set of excitation signals u_{κ} is composed of
 - All signals $u_{\mathcal{B}}$, and
 - All signals $u_{\mathcal{Y}}$ that are not in $u_{\mathcal{P}}$, and
 - All signals $u_{\mathcal{Z}}$ of which the effect on $w_{\mathcal{Y}}$ in the predictor model is not covered by the predictor inputs $w_{\mathcal{D}}$.

Proof: See Appendix.

In Figure 2 it is illustrated how excitation signals on nodes in the original network, map to excitation signals in the predictor model. Here $\tilde{\mathcal{Z}}$ denotes the subset of immersed nodes (\mathcal{Z}) whose excitations do not explicitly appear in (2), but whose effects are incorporated in $w_{\mathcal{D}}$. It follows

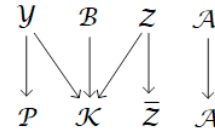


Fig. 2. Mapping of excitation signals in the original network (1) to their role in the predictor model (2).

that u_{κ} is composed of the excitation signals u that enter the network through (a selection of) node signals w_{ℓ} , with $\ell \in \mathcal{Y} \cup \mathcal{B} \cup \mathcal{Z}$, while $u_{\mathcal{P}}$ is composed of the excitation signals on a subset of the node signals $w_{\mathcal{Y}}$. In order to use these results for further specifying the data-informativity condition in (6), we first need to convert the spectrum condition into a path-based condition.

5. A PATH-BASED RESULT FOR GENERIC DATA-INFORMATIVITY

We follow the reasoning introduced in Van den Hof and Ramaswamy (2020), based on the graph-based results in Hendrickx et al. (2019), that the spectrum condition (6) can be rephrased as a full row rank condition on the mapping from all external signals in the network to the vector κ , together with a persistence of excitation condition on the external signals in the network. This can be turned into a graph-based condition as formulated next.

Proposition 4. (Van den Hof and Ramaswamy (2020)). The spectrum condition (6) holds generically if in the graph of the network model there exist $\dim(\kappa)$ vertex disjoint paths between the node sets (u, ξ) and $(w_{\mathcal{D}}, \xi_{\mathcal{Y}}, u_{\kappa})$, with u all measured external excitation signals in the network, and the signals in u are mutually independent and persistently exciting.

If the structural conditions for generic data-informativity can be turned into graph-based conditions, we can incorporate them during the construction of an appropriate predictor model, and thus guaranteeing that generic data-informativity can be realized by simply choosing excitation signals u that are persistently exciting.

Proposition 5. For the local direct method and predictor model (4), the structural conditions for data-informativity are satisfied if for all signals $w_k, k \in \mathcal{Q}$, every path from w_k to \mathcal{B} , and every loop around w_k passes through a node in $w_{\mathcal{A} \cup \mathcal{Q}}$.

Proof: The required condition for generic data informativity entails that there should be $\dim(\kappa)$ vertex disjoint

paths in the mapping $(u, \xi) \rightarrow (w_D, \xi_Y, u_K)$ or equivalently $(u_K, u_P, u_A, u_Z, \xi_I, \xi_Y) \rightarrow (u_I, w_Q, \xi_Y, u_K)$.

In the condition on vertex disjoint paths similar components on the left hand side and the right hand side can be removed, provided that we condition the remaining vertex disjoint paths not to pass through the removed components. This allows us to remove u_K and ξ_Y , while we also know that, due to the assumption of full rank disturbances in the network, ξ_I has $\dim(u_I)$ vertex disjoint paths to u_I . As a result the vertex disjoint path condition can be reformulated as the existence of $\dim(w_Q)$ vertex disjoint paths in the mapping $(u_P, u_A, u_Z) \rightarrow w_Q$, where paths are not allowed to pass through ξ_Y , u_I or u_K .

Since any path from u_A to w_Q passes through $w_A \subset u_I$, and all paths from u_Z to w_Y pass through a node in u_I , no paths from u_A and u_Z can be used, and therefore we consider the vertex disjoint paths between $u_P \rightarrow w_Q$, where paths are not allowed to pass through u_I . By the definition of the set \mathcal{P} , an excitation on the output w_o can only be an element of u_P if all paths from w_o to w_Q pass through a node in u_I , which in the above statement is not allowed. Therefore the result is that there should be $\dim(Q)$ vertex disjoint paths between $u_P \cap Q \rightarrow w_Q$, in other words: only excitation signals that directly excite w_Q can be used for satisfying the structural data-informativity conditions. With the definition of the set \mathcal{P} the result then follows directly. \square

The immediate implication of this result is that for satisfying the structural data-informativity conditions:

- (1) All nodes in w_Q need (independent) external excitation, and
- (2) All those excitations should satisfy: $u_Q \in u_P$.

The first condition can typically be satisfied by adding external excitations r to all nodes in w_Q . However the second condition is a structural condition on the graph of the network model, and is principally limited by the network topology. Whereas the construction of predictor models in Ramaswamy and Van den Hof (2021) was based on satisfying condition 2 of Theorem 2 only, we now have to extend these graphical conditions with the structural conditions for data-informativity, as formulated in Proposition 5. This is further explored and illustrated in the next Sections for two different algorithms presented in Ramaswamy and Van den Hof (2021), the full input algorithm and the minimum input algorithm.

6. FULL INPUT ALGORITHM

The original full input algorithm for predictor model construction according to Condition 2 of Theorem 2 is outlined below.

- (1) We start by selecting $w_D = w_A = w_i$ and $w_o = w_y = w_j$;
- (2) Extend \mathcal{A} in such a way that all w -inneighbors of w_y are included in w_A ;
- (3) For every node signal $w_k, k \in \mathcal{A}$ that has a confounding variable with w_y , move w_k to w_B if this is allowed by the conditions on \mathcal{B} ;
- (4) For every remaining node signal $w_k, k \in \mathcal{A}$ that has a direct confounding variable with w_y , copy w_k to the output w_y , move it to w_Q and return to Step (2).

- (5) For every remaining node signal $w_k, k \in \mathcal{A}$ that has an indirect confounding variable with w_y , find a new node signal to be added to w_B that satisfies the conditions on \mathcal{B} and blocks² the concerned confounding variable.
- (6) For every remaining node signal $w_k, k \in \mathcal{A}$ that has a confounding variable with w_y , copy w_k to w_y , move it to w_Q and return to Step (2).
- (8) Finally the predictor model is defined by the mapping $w_D \rightarrow w_y$, with $\mathcal{D} = Q \cup \mathcal{A} \cup \mathcal{B}$ and $\mathcal{Y} = Q \cup \{o\}$.

This algorithm can now be extended to also include the structural data-informativity condition as formulated in Proposition 5, as follows:

- In Steps (3) and (5), add a node to w_B only if the data-informativity conditions of Proposition 5 (DI conditions) are satisfied, and
 - At the end of the Algorithm, add an additional Step (7), as follows.
- (7) For every $w_k \in w_Q$ that does not satisfy the DI conditions:
 - (a) Verify if moving w_o to w_Q removes a violated condition for w_k ;
 - (b) Verify if moving one or more nodes in w_B to w_Q removes a violated condition for w_k ;
 - (c) Find an additional node w_ℓ , not present yet in $w_{Y \cup \mathcal{A}}$, to be added to either w_A or w_Q that solves the problem for w_k ;
 - (d) If in the previous steps a new node is added to w_Q , return to Step (2).

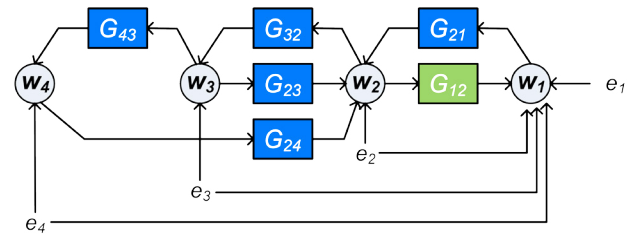


Fig. 3. Four node example with G_{12} as target module.

In the following example we illustrate how the original algorithm and its extension to include the structural data-informativity conditions differ when applied to the example sketched in Figure 3.

Example 1. (Full input algorithm). Consider the network in Figure 3 with G_{12} the target module we want to identify. The original full input algorithm, as presented in this section, starts with Step 1, including the input and output of the target module, i.e. $w_2 \rightarrow w_1$, with $w_D = w_A = w_2$; $w_Y = w_o = w_1$. Since w_2 is the only in-neighbor of the output w_1 , no further inputs need to be added in Step 2. Since w_2 has a direct confounding variable (e_2) and indirect confounding variables (e_3 and e_4) with w_1 these need to be blocked. In Step 3 we evaluated whether w_2 can be moved to w_B . This is not allowed because of the rule that $i \notin \mathcal{B}$. The direct confounding variable can only be blocked by copying w_2 to the output w_y (Step 4), such that it becomes an element of w_Q , resulting in the

² “Blocking” the confounding variable means that a measured node w_ℓ is added to w_B that is on the path from the confounding variable to w_k .

predictor model $w_2 \rightarrow (w_1, w_2)$ with $w_o = w_1$; $w_\mathcal{Q} = w_2$. Since the new output w_2 has been added we need to return to Step 2 and add all its inneighbors to $w_\mathcal{P}$. As a result w_3 and w_4 are added to $w_\mathcal{A}$ and w_1 (already present in $w_\mathcal{Y}$) is added to $w_\mathcal{Q}$. We are back in Step 3 now with $\mathcal{A} = \{3, 4\}$ and noticing that both nodes have confounding variables with output w_1 . We first consider node $w_3 \in w_\mathcal{A}$, that has a direct confounding variable (e_3) and an indirect confounding variable (e_4) with w_1 . Moving w_3 to $w_\mathcal{B}$ is not allowed since it does not satisfy Condition (d) of Proposition 1. Therefore, (Step 4) we copy w_3 to the output $w_\mathcal{Y}$, such that it becomes an element of $w_\mathcal{Q}$. The direct confounding variable between w_4 and w_1 is addressed (Step 3) by moving w_4 to $w_\mathcal{B}$. This results in the predictor model

$$(w_1, w_2, w_3, w_4) \rightarrow (w_1, w_2, w_3) \quad (7)$$

with $w_\mathcal{A} = \emptyset$; $w_\mathcal{B} = w_4$; $w_o = \emptyset$; $w_\mathcal{Q} = \{w_1, w_2, w_3\}$

and this concludes the original algorithm that satisfies Condition 2 of Theorem 2.

If we evaluate the DI conditions for this predictor model, it appears from the results of Section 5 that all nodes in $w_\mathcal{Q}$ (i.e. w_1, w_2, w_3) should have independent external excitations r , but at the same time also that the excitation on w_3 cannot be effectively used for data informativity, as it does not satisfy the Conditions of Proposition 5. w_3 has a path to $w_4 \in w_\mathcal{B}$ that is not blocked by a node in $w_\mathcal{A} \cup \mathcal{Q}$. As a result the selected predictor model can never satisfy the DI conditions. Even if we excite all nodes (w_1, w_2, w_3), the excitation on w_3 will appear in the predictor model as input to a parametrized transfer function, and therefore cannot be used for excitation of G_{12} .

Following the full-input algorithm now with the additional Step 7, shows that w_4 is not allowed to go to $w_\mathcal{B}$; it cannot be moved to $w_\mathcal{A}$ since there is a confounding variable then between $w_\mathcal{A}$ and $w_\mathcal{Y}$. As a result w_4 should go to $w_\mathcal{Q}$, leading to the predictor model:

$$(w_1, w_2, w_3, w_4) \rightarrow (w_1, w_2, w_3, w_4) \quad (8)$$

with all nodes in $w_\mathcal{Q} = \{w_1, w_2, w_3, w_4\}$. For the final predictor model it holds that all external signals u_1, u_2, u_3, u_4 belong to $u_{\mathcal{P} \cap \mathcal{Q}}$, and are required for satisfying the data-informativity conditions. \square

7. MINIMUM INPUT ALGORITHM

The minimum input algorithm for predictor model selection focuses on using a minimum number of measured node signals as predictor inputs, and, in its original form, follows the following strategy

- (1) We start by selecting $w_\mathcal{D} = w_\mathcal{A} = w_i$ and $w_o = w_\mathcal{Y} = w_j$;
- (2) Extend \mathcal{A} with a minimum number of w -inneighbors of $w_\mathcal{Y}$, such that the PPL condition (Condition(a) of Proposition 1) is satisfied;
- (3) For every node signal $w_k, k \in \mathcal{A}$ that has a confounding variable with $w_\mathcal{Y}$, copy w_k to $w_\mathcal{Y}$, move it to $w_\mathcal{Q}$;
- (5) Finally the predictor model is defined by the mapping $w_\mathcal{D} \rightarrow w_\mathcal{Y}$, with $\mathcal{D} = \mathcal{Q} \cup \mathcal{A}$, $\mathcal{Y} = \mathcal{Q} \cup \{o\}$ and $\mathcal{B} = \emptyset$.

This algorithm can now be extended to also include the structural data-informativity condition as formulated in Proposition 5, as follows:

- At the end of the Algorithm, add an additional Step (4), as follows.
- (4) For every $w_k \in w_\mathcal{Q}$ that does not satisfy the DI conditions:
 - (a) Verify if moving w_o to $w_\mathcal{Q}$ removes a violated condition for w_k ;
 - (b) Find an additional node w_ℓ , not present yet in $w_\mathcal{Y} \cup \mathcal{A}$, to be added to either $w_\mathcal{A}$ or $w_\mathcal{Q}$ that solves the problem for w_k ;

We will illustrate this algorithm on the Example presented in the previous Section.

Example 2. (Minimum input algorithm). Consider the same network as in Example 1 in Figure 3. The predictor model based on the original algorithm starts with the target module: $w_2 \rightarrow w_1$, with $w_2 \in w_\mathcal{A}$ and $w_1 = w_o$, which in line with Step 2, already satisfies the PPL condition (a) of Proposition 1. To deal with the confounding variable e_2 between w_2 and w_1 , we copy w_2 to the output and move it to $w_\mathcal{Q}$ (Step 3)

$$w_2 \rightarrow (w_1, w_2) \quad (9)$$

with $w_\mathcal{Q} = w_2, w_o = w_1$, after which the original algorithm finishes.

It can be observed that with this algorithm the predictor model is considerably smaller sized than with the full input algorithm. Applying the data-informativity analysis shows that $w_2 = w_\mathcal{Q}$ would need external excitation, but at the same time it also shows that any excitation on w_2 will not become a part of the excitation signals $u_\mathcal{P}$, as there is a non-blocked loop $w_2 \rightarrow w_3 \rightarrow w_2$ that is violating the conditions of Proposition 5. As a result the selected predictor model can never satisfy the DI conditions.

Continuing the minimum-input algorithm with the addition of Step 4, shows that with $w_2 \in w_\mathcal{Q}$, there are multiple loops around w_2 that need to be blocked. The loop $w_2 \rightarrow w_1 \rightarrow w_2$ can be blocked by moving the output w_1 to $w_\mathcal{Q}$ (Step 4(a)). Moreover, the remaining loops around w_2 pass through node w_3 , therefore we add w_3 to the predictor model (Step 4(b)) to block them. Because there is a confounding variable between w_3 and $w_\mathcal{Y}$, w_3 cannot go into $w_\mathcal{A}$ and so it has to go to $w_\mathcal{Q}$. The final solution of the extended algorithm now provides the predictor model:

$$(w_1, w_2, w_3) \rightarrow (w_1, w_2, w_3) \quad (10)$$

with all elements in $w_\mathcal{Q} = \{w_1, w_2, w_3\}$, and thus requires external excitation signals u_1, u_2, u_3 to guarantee data-informativity, where the required excitation signals automatically belong to $u_{\mathcal{P} \cap \mathcal{Q}}$ due to the additional step 4 in the algorithm. \square

The algorithms presented in this paper have been implemented in the MATLAB app SYSDYNET (Van den Hof, 2023).

Remark 1. Note that the example in the previous Sections has been selected to emphasize the role of data-informativity conditions for the local direct method in the presence of confounding variables. Alternatively, for the particular example network, an indirect identification method as in e.g. Gevers et al. (2018) could provide a consistent target module estimate on the basis of a single external excitation signal only, while refraining from extensive noise modeling.

8. CONCLUSIONS

We have addressed the construction of predictor models for consistent identification of a single module in a dynamic network, with the local direct prediction error method. The original structural conditions have been extended with appropriate graph-based data-informativity conditions that are shown to have a serious impact on the predictor models. These graph-based data-informativity conditions can be integrated in constructive algorithm for arriving at predictor models that guarantee consistent estimation of a local target module.

Appendix A. PROOF OF PROPOSITION 3.

Given size constraints we will only provide a sketch of proof, and for the details refer to Ramaswamy (2022). From (3) and (2), it follows that $u_{\mathcal{Y}}(t) = \bar{J}(q)u_{\mathcal{K}}(t) + \bar{S}u_{\mathcal{P}}(t)$. By writing $u_{\mathcal{Y}}(t) = \bar{R}_{\mathcal{Y}\star}(q)u(t)$, analyzing $\bar{R}_{\mathcal{Y}\star}(q)$ will show which external signals in u will appear in $u_{\mathcal{K}}$ or in $u_{\mathcal{P}}$. Under conditions (b) and (d) of Proposition 1, and the fact that w_o is a scalar signal, Lemma 4.1. in Ramaswamy (2022) provides the following results:

- (a) $\bar{R}_{k\ell} = 0$ with $k, \ell \in \mathcal{Q}$ and $k \neq \ell$;
- (b) $\bar{R}_{o\ell} = 0$ with $\ell \in \mathcal{Q}$;
- (c) $\bar{R}_{k\ell} = 0$ with $k \in \mathcal{Y}$ and $\ell \in \mathcal{A}$;
- (d) $\bar{R}_{kk} = 1, k \in \mathcal{Y}$, if:
 - all loops through w_k pass through a node in $w_{\mathcal{Q}\cup\mathcal{U}}$;
 - there exist no direct or unmeasured paths from w_k to any $w_m, m \in \mathcal{B}$ when $k \in \mathcal{Q}$;
- (e) $\bar{R}_{ko} = 0, k \in \mathcal{Q}$, if:
 - (a) all paths from w_o to w_k pass through a node in $w_{\mathcal{Q}\cup\mathcal{U}}$;
- (f) $\bar{R}_{k\ell} = 0, k \in \mathcal{Y}, \ell \in \mathcal{Z}$, if:
 - (a) all paths from w_ℓ to w_k pass through a node in $w_{\mathcal{Q}\cup\mathcal{U}}$; and
 - (b) there exist no direct or unmeasured paths from w_ℓ to any $w_m, m \in \mathcal{B}$.

The signals that are in $u_{\mathcal{P}}$ are characterized by a binary matrix \bar{S} , and thus are determined by the entries in \bar{R} that are equal to 1. The combined statements under result (d), then lead to the specification of $u_{\mathcal{P}}$ in Proposition 3, see also Proposition 4.3 in Ramaswamy (2022).

In order to specify $u_{\mathcal{K}}$, we will follow a reasoning of exclusion. The above results (a)-(f) specify columns in $\bar{R}_{\mathcal{Y}\star}$ that are zero, i.e. they determine the nodes of which the excitation is not explicitly present in (3). All remaining nodes, for which the corresponding terms in $\bar{R}_{\mathcal{Y}\star}$ cannot be shown to be 0 or 1, will typically have a parametrized transfer to $u_{\mathcal{Y}}$ and therefore will constitute $u_{\mathcal{K}}$. With $\bar{\mathcal{P}}$ being defined as the set of node indices that correspond to 0-columns in $\bar{R}_{\mathcal{Y}\star}$, it follows that $\mathcal{L} = \mathcal{P} \cup \bar{\mathcal{P}} \cup \mathcal{K}$, leading to $\mathcal{K} = \mathcal{L} \setminus \{\mathcal{P} \cup \bar{\mathcal{P}}\}$.

For specification of the set $\bar{\mathcal{P}}$, it is observed that from result (c) it follows that $\mathcal{A} \in \bar{\mathcal{P}}$, and from results (a),(b) and (e) that $\mathcal{Y} \cap \bar{\mathcal{P}} = \emptyset$. Moreover result (f) specifies the subset ($\bar{\mathcal{Z}}$) of unmeasured nodes that corresponds to zero columns in $\bar{R}_{\mathcal{Y}\star}$ and thus belongs to $\bar{\mathcal{P}}$. This leads to $\mathcal{K} = \mathcal{L} \setminus \{\mathcal{P} \cup \mathcal{A} \cup \bar{\mathcal{Z}}\}$ which proves the second part of Proposition 3.

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