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# Determining Identifiable Parameterizations for Large-scale Physical Models in Reservoir Engineering ⋆

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**Abstract:** In this paper identifiable parameterizations are determined for models of flow in porous media as applied in the field of petroleum reservoir engineering. Starting from a large-scale, physics-based model parameterization with an extensive parameter space, the best identifiable reduced dimensional parameterization is constructed. This is achieved through the development of an analytical expression for the (finite-time) information matrix of the problem. It is shown that the information matrix can be expressed in terms of controllability and observability properties of the model and the sensitivity of the state space matrices w.r.t. the parameter vector. A reduced dimensional subspace is then obtained after a singular value decomposition of the information matrix, leading to the use of basis functions (spatial patterns) in the original parameter space. The approach is applied to two reservoir models: a SISO model with 49 parameters and a MIMO model with 441 parameters.

Keywords: parameterization; structural identifiability; physical models; large-scale systems.

## 1. INTRODUCTION

This paper deals with determining identifiable parameterizations for large-scale, physical models in petroleum reservoir engineering. Petroleum reservoir engineering is concerned with maximizing the oil and gas production from subsurface reservoirs. A common way to increase the production is to inject water in the reservoir via injection wells to drive the oil via production wells towards the subsurface. However, due to strong heterogeneities in the porous reservoir rock the resulting oil-water front is not progressing uniformly and a large part of the oil is bypassed and not produced. This can be partly counteracted by manipulating the injection and production settings in the wells. The dynamic control strategy that maximizes the production is calculated based on a model of the reservoir. A reservoir model (Fig. 1) describes the flow of hydrocarbons through a porous medium in the subsurface. The model is non-linear due to the dependency of the system matrices on the states. A model typically contains $10^5$ to $10^6$ states, which are composed of the fluid pressure and fluid saturations in each grid block. The physical parameters in the model represent properties of the hydrocarbons, the heterogeneities in the porous medium and the interaction between the hydrocarbons and porous medium. The number of physical parameters is also in the order of $10^5$ to $10^6$. In the most simple model, the parameters represent the permeability in each grid block which determines how easily fluids flow through the porous medium. Although geological information might provide a rough idea about the permeability structures in the subsurface, the model parameters are basically unknown. For applying model-based control strategies, they are usually estimated together with the system states in procedures that are referred to as history matching. These basically consist of (extended) Kalman filter type of procedures, applied to measured data of e.g. pressures or production rates. The process of iteratively updating the parameters and states, and calculating an updated control strategy is called closed-loop reservoir management (Jansen et al. (2005)).

The parameter estimation problem consists of two parts. The first part is a property of the model structure itself:

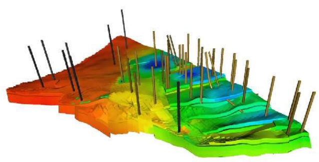

Fig. 1. Example of a reservoir model with wells.

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is it possible at all to distinguish two given parameters sets, provided the input is chosen the best possible way? This property is called structural identifiability of a model parameterization. The second part of the parameter estimation problem is to find out if the actual input is informative enough to allow this distinction. This leads to the requirement that the input is persistently exciting. In this paper the first part is investigated.
The notion of structural identifiability was first stated by Bellman and Åström (1970) and has been extensively studied in the field of compartmental modeling (Godfrey (1983), Norton (1980)). State-space model parameterizations are analyzed by Glover and Willems (1974), Grewal and Glover (1976) and Walter (1987). It is important to notice that observable and controllable systems can still be not structurally identifiable (DiStefano (1977)). A test for local structural identifiability of high-order state-space models is proposed in Dötsch and Van den Hof (1996). Lately there has been a renewed interest in structural identifiability analysis, with contributions from Bazanella et al. (2007) and Stigter and Peeters (2007), where the latter authors analyze non-linear systems.
For reservoir models it is shown in e.g. Tavassoli et al. (2004) that different parameter sets can lead to the same input-output behavior. Apparently, there is no unique relation between the input-output behavior of the model and the physical parameters. In other words, the model parameterization is not structurally identifiable. As a result, the parameter estimation problem is ill-posed. This is problematic, because an incorrect parameter estimate can lead to incorrect long-term predictions.
One way to overcome the ill-posedness in the parameter estimation problem is by constraining the solution space for the model parameters through the addition of regularization terms to the objective function. Another way is to restrict the parameter space to a low-dimensional space, guaranteeing (local) structural identifiability. In this paper we choose for the latter option and an identifiable parameterization is determined using local structural identifiability analysis.
First we will briefly review the local structural identifiability problem. Then we will present our approach, which is based on Dötsch and Van den Hof (1996). However, our approach is derived in a simplified manner, is fit for models with multiple inputs and outputs (MIMO), and the calculation is more efficient. In Section 4 the reservoir model is described, and two examples are presented in Section 5.

## 2. STRUCTURAL IDENTIFIABILITY

Consider a linear, time-invariant, discrete time, state-space model structure, parameterized in $\boldsymbol{\theta}$:

$$\mathbf{x}(k+1) = \mathbf{A}(\boldsymbol{\theta})\mathbf{x}(k) + \mathbf{B}(\boldsymbol{\theta})\mathbf{u}(k) \tag{1}$$

$$\mathbf{y}(k) = \mathbf{C}(\boldsymbol{\theta})\mathbf{x}(k), \tag{2}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$, $\mathbf{u}(k) \in \mathbb{R}^m$, $\mathbf{y}(k) \in \mathbb{R}^p$, and $\boldsymbol{\theta} \in \mathbb{R}^q$.
There are several approaches to evaluate the structural identifiability of a model. In this paper we use the local structural identifiability formulation of Glover and Willems (1974).

**Definition 1:** A model structure is said to be locally identifiable from the input-output behavior at $\hat{\boldsymbol{\theta}}$ if, for every $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$ in the neighborhood of $\hat{\boldsymbol{\theta}}$

$$G(q, \boldsymbol{\theta}_1) = G(q, \boldsymbol{\theta}_2) \rightarrow \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2,$$

with $G(q, \boldsymbol{\theta}) = \mathbf{C}(\boldsymbol{\theta})\left(q\mathbf{I} - \mathbf{A}(\boldsymbol{\theta})\right)^{-1}\mathbf{B}(\boldsymbol{\theta})$ and $q$ the forward shift operator.

In words: in the neighborhood of $\hat{\boldsymbol{\theta}}$ there are no two models with distinct parameters which have the same input-output behavior.
Note that $G(q, \boldsymbol{\theta})$ can be written as:

$$G(q, \boldsymbol{\theta}) = \sum_{k=1}^{\infty} \mathbf{M}(k, \boldsymbol{\theta}) q^{-k}, \tag{3}$$

where $\mathbf{M}(k, \boldsymbol{\theta}) = \mathbf{C}(\boldsymbol{\theta})\mathbf{A}^{k-1}(\boldsymbol{\theta})\mathbf{B}(\boldsymbol{\theta})$ are the Markov parameters. Based on (3) we can argue that equality of the models $G(q, \boldsymbol{\theta}_1)$ and $G(q, \boldsymbol{\theta}_2)$ is related to equality of the Markov parameters of $G(q, \boldsymbol{\theta}_1)$ and $G(q, \boldsymbol{\theta}_2)$. For reasons that become clear later the MIMO Markov parameters are decomposed into $p$ multi-input single-output (MISO) Markov parameters of dimension $1 \times m$, which are organized row-wise

$$\overrightarrow{\mathbf{M}}(k, \theta) := \left[\mathbf{M}_{1*}(k, \boldsymbol{\theta}), ..., \mathbf{M}_{p*}(k, \boldsymbol{\theta})\right], \tag{4}$$

where $\mathbf{M}_{j*}(k, \boldsymbol{\theta})$ denotes the $j$-th row of Markov parameter $\mathbf{M}(k, \boldsymbol{\theta})$.
We now present Lemma 1 on injective maps, which will lead together with Definition 1 to Proposition 2 (see also Glover and Willems (1974); Grewal and Glover (1976); Norton (1980)):
*Lemma 1.* Let $\Omega$ be an open set in $\mathbb{R}^n$ and $f : \Omega \rightarrow \mathbb{R}^m$ be a $k$-times continuously differentiable map with $k \geq 1$. If $\frac{\partial f(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$ has constant rank $l$ in a neighborhood of $\hat{\boldsymbol{\theta}}$, then $f$ is locally injective at $\hat{\boldsymbol{\theta}}$ if and only if $l = n$.
*Proposition 2.* Consider the map $\overrightarrow{S}_r : \Theta \subset \mathbb{R}^q \rightarrow \mathbb{R}^{pmr}$ defined by:

$$\overrightarrow{\mathbf{S}}_r(\boldsymbol{\theta}) := \left[\, \overrightarrow{\mathbf{M}}(1, \boldsymbol{\theta}) \; \overrightarrow{\mathbf{M}}(2, \boldsymbol{\theta}) \; \ldots \; \overrightarrow{\mathbf{M}}(r, \boldsymbol{\theta}) \,\right] \in \mathbb{R}^{1 \times pmr}. \tag{5}$$

Then the model structure is locally identifiable in $\boldsymbol{\theta}_0$ if, for sufficiently large $r$, rank $\left(\frac{\partial \overrightarrow{\mathbf{S}}_r(\theta)}{\partial \boldsymbol{\theta}}\right) = q$ in $\boldsymbol{\theta} = \boldsymbol{\theta}_0$.

This is equal to a rank test on the matrix:

$$\mathcal{I}_r := \left(\frac{\partial \overrightarrow{\mathbf{S}}_r(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right) \left(\frac{\partial \overrightarrow{\mathbf{S}}_r(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)^T \Bigg|_{\boldsymbol{\theta}_0}, \tag{6}$$

which has dimension $q \times q$. We will refer to (6) as an information matrix. It expresses the sensitivity of the parameters and combinations of parameters on the input-output behavior. If the model structure is not structurally identifiable, then the user of the model can decide to reparameterize the model such that the parameters that marginally contribute to the input-output behavior are eliminated.

## 3. IDENTIFIABILITY ANALYSIS BASED ON FINITE-TIME INFORMATION MATRIX

In this section we will first derive an expression for calculating the information matrix as given in (6). In subsequent subsections we explain how an identifiable parameterization is calculated, determine the minimum required number of Markov parameters $r$, and express the information matrix in terms of controllability, observability and matrix sensitivities.

### 3.1 Derivation

First recall that the chain rule for differentiating a matrix $\mathbf{A}^k$ w.r.t. $\theta_i \in \mathbb{R}$ gives us

$$\frac{\partial \mathbf{A}^k}{\partial \theta_i} = \frac{\partial \mathbf{A}\mathbf{A}^{k-1}}{\partial \theta_i} = \mathbf{A}\frac{\partial \mathbf{A}^{k-1}}{\partial \theta_i} + \frac{\partial \mathbf{A}}{\partial \theta_i}\mathbf{A}^{k-1}, \qquad (7)$$

where $\frac{\partial \mathbf{A}^k}{\partial \theta_i}$ has dimensions equal to that of $\mathbf{A}$. This can also be written as

$$\frac{\partial \mathbf{A}^k}{\partial \theta_i} = \sum_{l=1}^{k} \mathbf{A}^{l-1}\frac{\partial \mathbf{A}}{\partial \theta_i}\mathbf{A}^{k-l}. \qquad (8)$$

The Jacobian matrix of $\mathbf{A}^k \in \mathbb{R}^{n\times n}$ w.r.t. the parameter vector $\boldsymbol{\theta} \in \mathbb{R}^q$ is consequently given by:

$$\frac{\partial \mathbf{A}(\boldsymbol{\theta})^k}{\partial \boldsymbol{\theta}} = \sum_{l=1}^{k}\left(\left(\mathbf{I}_q \otimes \mathbf{A}^{l-1}\right)\frac{\partial \mathbf{A}}{\partial \boldsymbol{\theta}}\mathbf{A}^{k-l}\right), \qquad (9)$$

where $\mathbf{I}_q$ is the identity matrix with dimensions $q \times q$, $\otimes$ denotes the Kronecker product, and where $\frac{\partial \mathbf{A}}{\partial \boldsymbol{\theta}} \in \mathbb{R}^{qn\times n}$ consists of the partial derivatives $\frac{\partial \mathbf{A}}{\partial \theta_i}$ organized under each other.

Similarly, the Jacobian matrix of $\mathbf{M}_{\mathbf{j}*}(k,\boldsymbol{\theta}) \in \mathbb{R}^{1\times m}$ w.r.t. $\boldsymbol{\theta}$ is:

$$\begin{aligned}\frac{\partial \mathbf{M}_{j*}(k,\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} &= \frac{\partial \mathbf{C}_{j*}(\boldsymbol{\theta})\mathbf{A}^{k-1}(\boldsymbol{\theta})\mathbf{B}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \\ &\frac{\partial \mathbf{C}_{j*}}{\partial \boldsymbol{\theta}}\mathbf{A}^{k-1}\mathbf{B} + \left(\mathbf{I}_q \otimes \mathbf{C}_{j*}\mathbf{A}^{k-1}\right)\frac{\partial \mathbf{B}}{\partial \boldsymbol{\theta}} \\ &+ \sum_{l=1}^{k-1}\left(\mathbf{I}_q \otimes \mathbf{C}_{j*}\mathbf{A}^{l-1}\right)\frac{\partial \mathbf{A}}{\partial \boldsymbol{\theta}}\mathbf{A}^{k-1-l}\mathbf{B}.\end{aligned} \qquad (10)$$

Equation (10) shows that the Jacobian of each Markov parameter can be expressed using the system matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and the analytical partial derivatives of the state-space system matrices w.r.t. the parameter vector. As stated in (5) the Markov parameters are organized row-wise

$$\overrightarrow{\mathbf{S}}_r = \left[\overrightarrow{\mathbf{M}}(1), \ldots, \overrightarrow{\mathbf{M}}(r)\right] \in \mathbb{R}^{1\times pmr}, \qquad (11)$$

where we have omitted, for brevity, the dependence of $\mathbf{S}_r$ and $\mathbf{M}(k)$ on $\boldsymbol{\theta}$. The Jacobian of $\overrightarrow{\mathbf{S}}_r$ w.r.t. the parameter vector $\boldsymbol{\theta} \in \mathbb{R}^q$ is defined as:

$$\frac{\partial \overrightarrow{\mathbf{S}}_r}{\partial \boldsymbol{\theta}} = \left[\frac{\partial \overrightarrow{\mathbf{M}}(1)}{\partial \boldsymbol{\theta}}, \ldots, \frac{\partial \overrightarrow{\mathbf{M}}(r)}{\partial \boldsymbol{\theta}}\right] \in \mathbb{R}^{q\times pmr}, \qquad (12)$$

where $\frac{\partial \overrightarrow{\mathbf{M}}(k)}{\partial \boldsymbol{\theta}} \in \mathbb{R}^{q\times pm}$, for $k = (1, \ldots, r)$.

*Proposition 3.* Using the notational conventions stated before, the information matrix $\mathcal{I}_r$ for a multi-input multi-output system is defined as

$$\mathcal{I}_r := \left.\frac{\partial \overrightarrow{\mathbf{S}}_r}{\partial \boldsymbol{\theta}}\frac{\partial \overrightarrow{\mathbf{S}}_r^T}{\partial \boldsymbol{\theta}^T}\right|_{\boldsymbol{\theta}_0} = \left.\sum_{i=1}^{r}\sum_{j=1}^{p}\left(\frac{\partial \mathbf{M}_{j*}(i)}{\partial \boldsymbol{\theta}}\frac{\partial \mathbf{M}_{j*}^T(i)}{\partial \boldsymbol{\theta}^T}\right)\right|_{\boldsymbol{\theta}_0} \qquad (13)$$

with dimensions $q \times q$, and where $\frac{\partial \mathbf{M}_{j*}(i)}{\partial \boldsymbol{\theta}}$ is given by (10).

As in Dötsch and Van den Hof (1996) $\mathcal{I}_r$ is computed using the matrices $\mathbf{A}(\boldsymbol{\theta})$, $\mathbf{B}(\boldsymbol{\theta})$, $\mathbf{C}(\boldsymbol{\theta})$, $\frac{\partial \mathbf{A}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$, $\frac{\partial \mathbf{B}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$ and $\frac{\partial (\mathbf{C}\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$. Computation of the partial derivatives of $\mathbf{A}(\boldsymbol{\theta})$, $\mathbf{B}(\boldsymbol{\theta})$, $\mathbf{C}(\boldsymbol{\theta})$ w.r.t. $\theta_i$ $(i = 1, \ldots, q)$, for $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ is done analytically. However, the procedure given here is strongly simplified and much more direct compared to the one proposed by Dötsch and Van den Hof (1996), and has been extended to MIMO models. Also, computation of $\mathcal{I}_r$ with (13) is computationally more efficient because the matrix is calculated as a whole, instead of element by element.

### 3.2 Identifiable parameterization

After calculating $\mathcal{I}_r$ with (13) its rank is evaluated. The rank of $\mathcal{I}_r$ provides an estimate of the number of linearly independent rows or columns in $\mathcal{I}_r$. It is denoted as

$$l := \operatorname{rank}(\mathcal{I}_r). \qquad (14)$$

Here we use a singular value decomposition (SVD) to determine the numerical rank (Golub and Van Loan (1996)). Let $\mathcal{I}_r = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$ be the SVD of $\mathcal{I}_r$, where $\mathbf{U}$ and $\mathbf{V}$ are unitary and $\boldsymbol{\Sigma} = \operatorname{diag}(\sigma_1, \ldots, \sigma_q)$ with $\sigma_1 \geq \cdots \geq \sigma_l \gg \sigma_{l+1} \geq \ldots \geq \sigma_q$. The singular values $\sigma_{l+1}, \ldots \sigma_q$ are regarded as negligible. Numerical determination of the matrix rank $l$ requires a criterion for deciding when a singular value $\sigma_i$ should be treated as zero. In the example the choice is made that $\frac{\sigma_{l+1}}{\sigma_1} < 1 \times 10^{-5}$.

Accordingly, the SVD of $\mathcal{I}_r$ can be partitioned as follows

$$\mathcal{I}_r = \left[\, \mathbf{U}_1 \;\; \mathbf{U}_2 \,\right]\begin{bmatrix}\boldsymbol{\Sigma}_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{V}_1^T\\ \mathbf{V}_2^T\end{bmatrix}, \qquad (15)$$

where $\mathbf{U}_1 \in \mathbb{R}^{q\times l}$, $\mathbf{U}_2 \in \mathbb{R}^{q\times(q-l)}$, $\boldsymbol{\Sigma}_1 \in \mathbb{R}^{l\times l}$, $\mathbf{V}_1 \in \mathbb{R}^{q\times l}$ and $\mathbf{V}_2 \in \mathbb{R}^{q\times(q-l)}$. This means that the SVD of the information matrix can be utilized to determine an identifiable parameterization. From (15) we see that the columns of $\mathbf{U}_1$ provide an orthogonal basis of the column space of $\mathcal{I}_r$. The columns of $\mathbf{U}_1$ are regarded as directions in the parameter space that are structurally identifiable and serve as a mapping from high-dimensional parameter space $\boldsymbol{\theta}$ to a low-dimensional parameter space $\rho = \mathbf{U}_1^T\boldsymbol{\theta}$. It can be shown that $\frac{\partial \mathbf{S}_r}{\partial \rho}\frac{\partial \mathbf{S}_r}{\partial \rho}^T = \boldsymbol{\Sigma}_1$. In case we choose that the parameter to be estimated is the permeability in each grid block (see Section 4 for model description), each column of $\mathbf{U}_1$ with length $q$ can be projected on the $N$ grid blocks of the reservoir model. This can be done because each parameter value in $\boldsymbol{\theta}$ corresponds to one grid block $(q = N)$. Consequently, each column of $\mathbf{U}_1$ can also be interpreted as a spatial pattern, expressing the sensitivity of the Markov parameters w.r.t. the permeability vector.

The columns of $\mathbf{U}_2$ provide an orthogonal basis of the null space of $\mathcal{I}_r$. The columns of $\mathbf{U}_2$ are regarded as directions in the parameter space that are structurally *not* identifiable.

### 3.3 Number of Markov parameters

One of the questions is how many Markov parameters $r$ should be taken into account in order to arrive at correct expressions for the local identifiability analysis. In the SISO case it is well known that $2n$ Markov parameters uniquely determine a linear system with McMillan degree $n$. In the MIMO case the minimum number of Markov parameters that uniquely determine the underlying linear system is given by $\nu + \mu$, with $\nu$ the observability index and $\mu$ the controllability index. See e.g. Kailath (1980). It implies that $r$ is sufficient if $\operatorname{rank}(\mathbf{H}_{r-1}) = \operatorname{rank}(\mathbf{H}_r)$, where $\mathbf{H}_r$ denotes a block Hankel matrix containing $r$ Markov parameters (Damen et al. (1985)).

### 3.4 Observability, sensitivity and controllability

The information matrix in (13) can be expressed in terms of controllability and observability properties of the model and the sensitivity of the state matrices w.r.t. the parameter vector $\boldsymbol{\theta}$. In the following we assume that system matrices $\mathbf{B}$ and $\mathbf{C}$ are not dependent on $\boldsymbol{\theta}$, which leads to an expression without $\frac{\partial \mathbf{B}}{\partial \boldsymbol{\theta}}$ and $\frac{\partial \mathbf{C}}{\partial \boldsymbol{\theta}}$. This is also the case in the reservoir engineering example presented later.

For $q = 1$, $p = 1$ and $r = 4$ we can write $\frac{\partial \overrightarrow{\mathbf{S}}_4}{\partial \boldsymbol{\theta}} = [\, 0 \ \mathbf{X}]$, where

$$\mathbf{X} = \left[\, \mathbf{C} \ \mathbf{CA} \ \mathbf{CA}^2 \,\right] \begin{bmatrix} \frac{\partial \mathbf{A}}{\partial \boldsymbol{\theta}} & & \\ & \frac{\partial \mathbf{A}}{\partial \boldsymbol{\theta}} & \\ & & \frac{\partial \mathbf{A}}{\partial \boldsymbol{\theta}} \end{bmatrix} \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \\ & \mathbf{B} & \mathbf{AB} \\ & & \mathbf{B} \end{bmatrix}. \tag{16}$$

In this expression a block diagonal matrix of the sensitivity of the state space matrices w.r.t. the parameter vector is left multiplied with the observability matrix, and right multiplied with a block Toeplitz matrix containing the controllability matrix and shifted controllability matrices. For $q > 1$, $p = 1$ and $r = 4$, the expression becomes

$$\mathbf{X} = \left[\, (\mathbf{I}_q \otimes \mathbf{C}) \ (\mathbf{I}_q \otimes \mathbf{CA}) \ \left(\mathbf{I}_q \otimes \mathbf{CA^2}\right) \,\right] \times \begin{bmatrix} \frac{\partial \mathbf{A}}{\partial \boldsymbol{\theta}} & & \\ & \frac{\partial \mathbf{A}}{\partial \boldsymbol{\theta}} & \\ & & \frac{\partial \mathbf{A}}{\partial \boldsymbol{\theta}} \end{bmatrix} \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \\ & \mathbf{B} & \mathbf{AB} \\ & & \mathbf{B} \end{bmatrix},$$

which possesses a similar structure as (16).

## 4. RESERVOIR MODEL

The structural identifiability analysis is applied to a reservoir model. A reservoir model describes the fluid flow in a porous medium in time and space. The equations are based on a mass balance combined with Darcy's Law, which states that the fluid flow rate in a reservoir rock is proportional to the pressure gradient (Aziz and Settari (1986)). The model is non-linear, large-scale ($10^5$ - $10^6$ states), can contain several tens of wells that can have measurement and control capabilities, and is uncertain in the model parameters and initial state. See Jansen (2007) and references therein for more material about the physical model. However, as concluded in e.g. Heijn et al. (2004) the dynamics of the reservoir model is in most cases several orders less then the number of states. This is mainly due to the unobservable and uncontrollable part of the reservoir model.

In this analysis we use a two-dimensional model, that contains only one fluid. The resulting model is linear in the states, but non-linear in the parameters. A five-point finite difference discretization in space and an implicit discretization in time yield the following state-space ordinary differential equation in discrete time

$$\mathbf{p}(k+1) = \mathbf{A}(\boldsymbol{\theta})\,\mathbf{p}(k) + \mathbf{Bu}(k), \quad \mathbf{p}(0) = \mathbf{p}_0 \tag{17}$$

$$\mathbf{y}(k) = \mathbf{Cp}(k), \tag{18}$$

where $k \in \mathbb{Z}$ denotes discrete time. The state variables $\mathbf{p} \in \mathbb{R}^n_+$ denote (positive) fluid pressures in each grid block. The number of states is equal to the number of grid blocks. The input variables $\mathbf{u} \in \mathbb{R}^m$ denote control settings such as injection or production rates or pressures in grid blocks containing wells. The output variables $\mathbf{y} \in \mathbb{R}^p$ denote measured pressures in grid blocks containing wells. $\mathbf{A}(\theta) \in \mathbb{R}^{n\times n}$ is a penta-diagonal matrix with entries that are a function of grid block volume, fluid density, compressibility, fluid viscosity, porosity in each grid block, and permeability in each grid block. The latter parameter can be interpreted as the conductivity of the fluid through the porous medium. We choose not to use a well inflow model, and therefore $\mathbf{B} \in \mathbb{R}^{n\times m}$ is a sparse matrix containing ones in entries corresponding to a grid block containing a well. $\mathbf{C} \in \mathbb{R}^{p\times n}$ is also a sparse matrix containing ones in entries corresponding to a grid block containing a well. At the boundaries no-flow conditions are assumed.

The linear, time-invariant reservoir model of (17,18) can be seen as a state-space model that is parameterized by a parameter vector $\boldsymbol{\theta}$. Because the permeability in each grid block directly influences the flow, it is vital to estimate this parameter vector using the available measurements in order to obtain reliable model predictions and control strategies. Therefore, we propose to analyze the structural identifiability for this parameter vector and determine an identifiable subspace of the parameter space.

## 5. EXAMPLES AND APPLICATION

In this section the identifiability analysis is applied to two reservoir models. The first example is a SISO model with 25 parameters. It mainly serves to show the influence of the permeability values on the information matrix. The second example is a MIMO model with 441 parameters and five inputs and five outputs. It demonstrates the influence of the permeability values and also the position of the wells. The examples will show that the permeability estimation problem is ill-posed, as $\text{rank}(\mathcal{I}_r) < q$. More importantly, an identifiable parameterization is determined and visualized based on $\mathcal{I}_r$.

### 5.1 Single input / single output

In the first example the identifiability analysis is performed on a reservoir model with 49 states and an equal number of parameters. The compressibility is $c = 1 \times 10^{-10} Pa^{-1}$, the viscosity is $\mu = 1 \times 10^{-3} Pa \cdot s$, the porosity is $\phi = 0.2$ in each grid block, and each grid block has dimensions $10m \times 10m \times 10m$. The permeability is $10^{-13} m^2$ and equal in each grid block (Fig. 2, left picture, top view of a 2D representation). The model contains one input and one output, which are both located in the middle grid block. This can be regarded as one well that measures and controls the pressure in the middle of the reservoir. We first determined $r$ based on a rank evaluation of the two block Hankel matrices. The information matrix is calculated with $r = 4$, and $\text{rank}(\mathcal{I}_r)$ is evaluated. The singular values of $\mathcal{I}_r$ decrease rapidly and $\frac{\sigma_1}{\sigma_3} = 1 \times 10^{16}$. The rank of the information matrix is $l = 2$ with the cut-off value chosen as $\frac{\sigma_{l+1}}{\sigma_1} < 1 \times 10^{-5}$. This means that in this case 2 parameters can be identified using pressure measurements. $\mathbf{U}_1$ has consequently dimensions of $q \times 2$ and since each parameter is connected to a grid block each

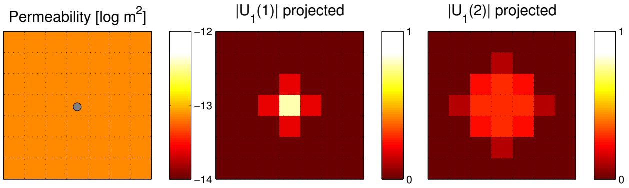


Fig. 2. Top view of a homogeneous permeability distribution with well location indicated by a grey dot (left) and corresponding dominant spatial patterns in parameter space (middle and right).

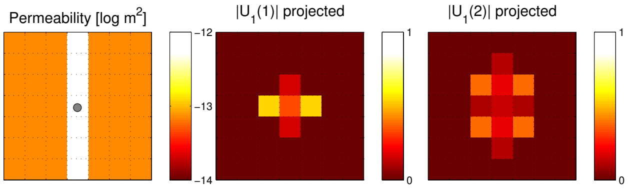


Fig. 3. Permeability distribution containing a streak with higher permeability values (left) and corresponding dominant spatial patterns in parameter space (middle and right).

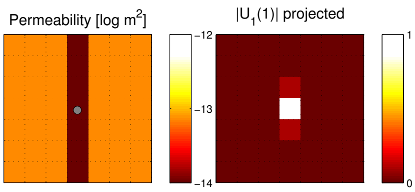


Fig. 4. Permeability distribution containing a streak with lower permeability values (left) and corresponding dominant spatial patterns in parameter space (right).

column of $\mathbf{U}_1$ can be projected onto the reservoir grid (Fig. 2, middle and left picture). These spatial patterns can be interpreted as the dominant direction in the parameter space to which the output is sensitive to changes in $\boldsymbol{\theta}$. We depicted the absolute values of $\mathbf{U}_1$ to show clearly which values are equal or close to 0. As can be seen in Fig. 2 the model parameters are mainly structurally identifiable around the well, because the values corresponding to these grid blocks are not equal or close to 0. Also, it is clear that a five-point spatial discretization scheme is used, because the parameter corresponding to the grid block containing a well and its four surrounding grid blocks are not close to 0. Finally, due to the symmetry in the model properties, the dominant directions in the parameter space also display symmetric patterns.

To show the influence of $\boldsymbol{\theta}_0$, we changed the uniform permeability distribution and added a high-permeable streak to the uniform permeability distribution (Fig. 3, left picture). The value of the permeability in the streak is 10 times higher and the fluids in this area flow easier. We calculated the information matrix with $r = 5$ and its rank is $l = 2$. However, the dominant directions in the parameter space that are sensitive to changes in the input-output behavior are different (Fig. 3, middle and right picture). Apparently, the input-output behavior is more sensitive to permeabilities with high values.

If the permeability in the streak is 10 times lower (Fig. 4, left picture), the rank of the information matrix is only $l = 1$. The corresponding dominant direction in parameter space is plotted in the right picture of Fig. 4, and

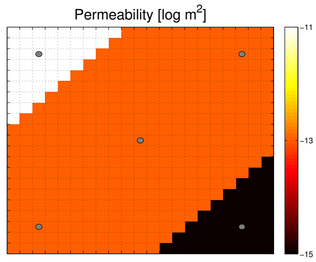


Fig. 5. Permeability distribution (top view) for MIMO example. Grey dots indicate well positions.

shows that the input-output behavior is less sensitive to permeabilities with low values.

### 5.2 Multiple input / multiple output

In the second example the identifiability analysis is performed on a reservoir model with 441 states and an equal number of parameters. The permeability distribution consists of three zones: the upper left corner has a high permeability, the lower right corner a low permeability, and the intermediate zone an intermediate permeability (Fig. 5). The other physical parameters are chosen the same as in the first example. The model contains 5 wells, all with measurement and control capabilities. This means that the model has 5 inputs and 5 outputs. The wells are distributed in a characteristic five-spot pattern, indicated in Fig. 5 by grey circular shapes. We computed the corresponding impulse response, which revealed that there is a negligible interaction between the wells in the case of a single-phase flow model. However, interaction between the wells is more likely in the case of a multi-phase flow model. For example, if water is injected via an injection well and displaces oil towards a production well, there is clearly interaction between the injection and production well when the front between the fluids reaches the production well.

It was found that in this case with 19 Markov parameters $\text{rank}(\mathbf{H}_r)$ was not increasing anymore and $\text{rank}(\mathbf{H}_{r-1}) = \text{rank}(\mathbf{H}_r)$. The information matrix is therefore calculated with $r = 19$. The singular values of $\mathcal{I}_r$ are plotted in the left part of Fig. 6. The 30 largest singular values are plotted again in the right part of Fig. 6. For this example the rank of $\mathcal{I}_r$ is more difficult to determine, because the difference between two subsequent singular values is less distinct than for the SISO example. However, we do see that at least $\sigma(18)$ to $\sigma(q)$ are close to machine precision. This means that the maximum number of parameters that can be structurally identified with perfect pressure measurements and finite machine precision is only 17 (out of a total of 441).

The absolute singular vectors, the columns of $|\mathbf{U}|$, corresponding to the 9 largest singular values are depicted in Fig. 7. The first singular vector, the most dominant direction in the parameter space, corresponds to a well in the part with a lower permeability. The next three singular vectors correspond to wells in the part with the intermediate permeability. The fifth to eighth singular vectors correspond to the well in the part with a higher permeability, where each subsequent vector covers a larger area. The last singular vector corresponds to a well in the part with the intermediate permeability. From this

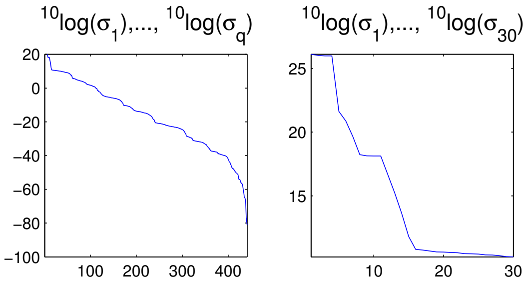


Fig. 6. Singular values of $\mathcal{I}_r$ for the MIMO case.

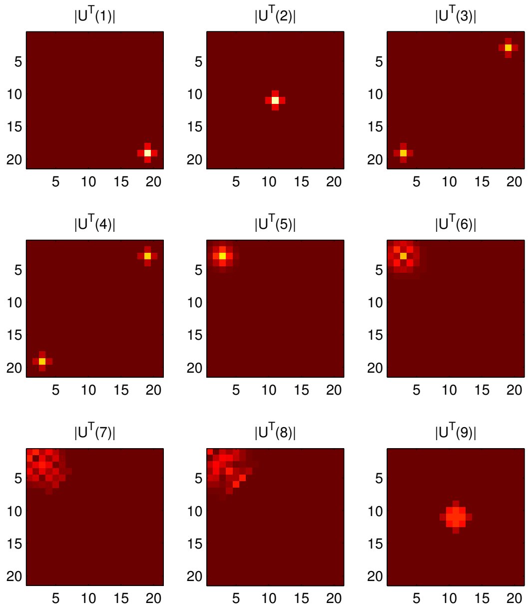


Fig. 7. For the MIMO case the first 9 spatial patterns in parameter space (top view).

example we conclude that only parameters in an area near a well are structurally identifiable. Furthermore, the structural identifiability is to a lesser extent affected by the permeability distribution.

## 6. CONCLUSION

In this paper a best identifiable, reduced-dimensional parameterization is constructed, which is applicable to large-scale, non-linearly parameterized and multi-input multi-output state-space models. This is achieved through the development of an analytical expression for the finite-time information matrix. It is shown that the information matrix can be expressed in terms of controllability and observability properties of the model and the sensitivity of the state matrices w.r.t. the parameter vector. A singular value decomposition of the information matrix is used to construct a reduced-dimensional subspace. In the original parameter space this leads to basis functions or spatial patterns. The approach was applied to two examples in petroleum reservoir engineering. It was demonstrated that the identifiable parameterization was mainly depending on the position of the actuators and sensors, and to a lesser extent on the permeability distribution.

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