The Behavioral Approach to Linear Parameter-Varying Systems

Roland Tóth, *Member, IEEE*, Jan C. Willems, *Life Fellow, IEEE*, Peter S. C. Heuberger, and Paul M. J. Van den Hof, *Fellow, IEEE*

***Abstract*—Linear parameter-varying (LPV) systems are usually described in either state-space or input–output form. When analyzing system equivalence between different representations it appears that the time-shifted versions of the scheduling signal (dynamic dependence) need to be taken into account. Therefore, representations used previously to define and specify LPV systems are not equal in terms of dynamics. In order to construct a parametrization-free description of LPV systems that overcomes these difficulties, a behavioral approach is introduced that serves as a basis for specifying system theoretic properties. LPV systems are defined as the collection of trajectories of system variables (like inputs and outputs) and scheduling variables. LPV kernel, input–output, and state-space system representations are introduced with appropriate equivalence transformations.**

***Index Terms*—Behavioral approach, dynamic dependence, equivalence, linear parameter-varying (LPV).**

I. Introduction

MANY physical/chemical processes encountered in practice have nonstationary or nonlinear behavior, and often their dynamics depend on external variables like space coordinates, temperature, etc. For such processes, the theory of *linear parameter-varying* (LPV) systems offers an attractive modeling framework [1]. This class of systems is particularly suited to deal with processes that operate in varying operating regimes. LPV systems can be seen as an extension of the class of *linear time-invariant* (LTI) systems. In LPV systems, the signal relations are considered to be linear, but the parameters in the description of these relations are assumed to be functions of a time-varying signal, the so-called *scheduling variable* p. As a result of the parameter variation concept, the LPV system class can describe both time-varying and nonlinear phenomena. Practical use of this framework is stimulated by the fact that LPV control design is well developed, extending results of optimal and robust LTI control theory to nonlinear, time-varying plants [1]–[9].

In a discrete-time setting, LPV systems are commonly described in a *state-space* (SS) form (see [1]–[9])

$$x(k) = A(p(k))x(k) + B(p(k))u(k) \tag{1a}$$
$$y(k) = C(p(k))x(k) + D(p(k))u(k) \tag{1b}$$

where $u : \mathbb{Z} \to \mathbb{R}^{n_\mathrm{U}}$ is the input, and $y : \mathbb{Z} \to \mathbb{R}^{n_\mathrm{Y}}$ is the output, $x : \mathbb{Z} \to \mathbb{R}^{n_\mathrm{X}}$ is the state vector and the system matrices $\{A, B, C, D\}$ are functions of the scheduling signal $p : \mathbb{Z} \to \mathbb{P}$, e.g., $A : \mathbb{P} \to \mathbb{R}^{\cdot \times \cdot}$, where the set $\mathbb{P} \subseteq \mathbb{R}^{n_\mathrm{P}}$ is the so called "*scheduling space*." It is assumed that p is an *external* signal of the system, i.e., p is not dependent on u or y. An exact definition of when this externality property holds for p will be given later.

In the identification literature, LPV systems are also described in the form of (filter-type) *input–output* (IO) representations [10]–[13]

$$y(k) = \sum_{i=1}^{n_a} a_i(p(k))y(k-i) + \sum_{j=0}^{n_b} b_j(p(k))u(k-j) \tag{2}$$

where $\{a_i, b_j\}$ are matrix functions of p. In equations (1a), (1b), and (2), the coefficients depend on the instantaneous time value of p, which is called *static-dependence*. In analogy with the LTI system theory, it is commonly assumed that representations (1a), (1b), and (2) define the same class of LPV systems and that conversion between these representations follows similar rules as in the LTI case (see [14]–[16]). However, it has been observed recently that this assumption is invalid if attention is restricted to static dependence [17].

Example 1: To illustrate the problem, consider the following second-order SS representation:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & a_2(p(k)) \\ 1 & a_1(p(k)) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} b_2(p(k)) \\ b_1(p(k)) \end{bmatrix} u(k)$$
$$y(k) = x_2(k).$$

With simple manipulations, this system can be written in an equivalent IO form

$$\begin{aligned} y(k) = {} & a_1(p(k-1))y(k-1) + a_2(p(k-2))y(k-2) \\ & + b_1(p(k-1))u(k-1) + b_2(p(k-2))u(k-2) \end{aligned}$$

which can clearly not be formulated as (2). □

Manuscript received October 19, 2009; revised June 16, 2010; October 25, 2010; December 09, 2010; and December 13, 2010; accepted January 22, 2011. Date of publication January 31, 2011; date of current version November 02, 2011. The SISTA research program of the K. U. Leuven is supported by the Research Council KUL under projects: GOA AMBioRICS, CoE EF/05/006 Optimization in Engineering (OPTEC), IOF-SCORES4CHEM; by the Flemish Government: FWO: projects G.0452.04 (new quantum algorithms), G.0499.04 (Statistics), G.0211.05 (Nonlinear), G.0226.06 (cooperative systems and optimization), G.0321.06 (Tensors), G.0302.07 (SVM/Kernel), G.0320.08 (convex MPC), G.0558.08 (Robust MHE), G.0557.08 (Glycemia2), G.0588.09 (Brain-machine) research communities (ICCoS, ANMMM, MLDM); G.0377.09 (Mechatronics MPC); by IWT: McKnow-E, Eureka-Flite+, SBO LeCoPro, SBO Climaqs; by the Belgian Federal Science Policy Office: IUAP P6/04 (DYSCO, Dynamical systems, control and optimization, 2007–2011); by the EU: ERNSI; FP7-HD-MPC (INFSO-ICT-223854); and by several contract research projects. Recommended by Associate Editor F. Wu

R. Tóth, P. S. C. Heuberger, and P. M. J. Van den Hof are with the Delft Center for Systems and Control (DCSC), Delft University of Technology, 2628 CD Delft, The Netherlands (e-mail: r.toth@tudelft.nl; p.s.c.heuberger@tudelft.nl; p.m.j.vandenhof@tudelft.nl).

J. C. Willems is with the Department of Electrical Engineering, Katholieke Universiteit Leuven, B-3001 Leuven-Heverlee, Belgium (e-mail: Jan.Willems@esat.kleuven.be).

Digital Object Identifier 10.1109/TAC.2011.2109439

In order to obtain equivalence between the SS and IO representations, it is necessary to allow for a dynamic mapping between p and the coefficients, i.e., $\{A, B, C, D\}$ and $\{a_i, b_j\}$ should be allowed to depend on (finitely many) time-shifted instances of $p(k)$, i.e., $\{\ldots, p(k-1), p(k), p(k+1), \ldots\}$ [17]. We call such a dependence *dynamic* in the sequel. Dynamic dependence has also been encountered and analyzed in terms of LPV control synthesis (see [18] and [19]), and its need is supported as well by LPV modeling of nonlinear/time-varying systems (see Example 2 and [20]). Currently, it is not well understood how to handle such dependencies in general and how to formulate algorithms that provide transformations between the representation forms (an intermediate solution for the SISO case is given in [17]).

The necessity of dynamic dependence clearly indicates that representations (1a), (1b), and (2) used previously to define and specify LPV systems are not equal in terms of dynamics. Furthermore, the lack of realization/transformation theory associated with these representations hinders the use of many identification methods based on IO models, like the extension of successful prediction error methods of the LTI case, e.g., [10] and [11], to provide state-space models for control synthesis. The lack of understanding of similarity transformation for (1a) and (1b) is also a source of many pitfalls both for identification and control synthesis in general [17]. Furthermore, the collection of transfer functions of (1a), (1b), and (2) for each value of $p(k)$, the so-called *frozen transfer functions*, does not specify the behavior of the system for nonconstant trajectories of p, which is often overlooked in the literature; see [21]–[23]. As no global transfer-function theory exists in the LPV case, definitions of input–output behavior of (1a), (1b), and (2) are relevant to be considered in terms of solutions of these difference equations in the time domain. These arguments indicate that the classical definitions of LPV systems and the "assumed" similarity transformation connected to them are inadequate, showing that the current LPV system theory is incomplete.

A parametrization-free definition of LPV systems and an algebraic framework where the previously considered representations and concepts of LPV systems are reestablished can be found by considering a behavioral approach to the problem. In this paper, the behavioral framework, originally developed for LTI systems [24], is extended to discrete-time LPV systems. In this framework, systems are described in terms of behaviors that corresponds to the collection of all valid signal trajectories. Our aim is to use the behavioral concept to establish well-defined LPV system representations as well as their interrelationships. Our further intention is to develop a unified LPV system theory that establishes connections between the available results.

The paper is organized as follows. In Section II, LPV systems are defined from the behavioral point of view. In Section III, an algebraic structure of polynomials is introduced to define parameter-varying difference equations as representations of the system behavior. This is followed, in Section IV, by developing kernel, IO, and SS representations of LPV systems, together with the basic notions of IO partitions and state variables. In Section V, it is explored when two kernel, IO, or SS representations are equivalent. In Section VI, equivalence transformations between SS and IO representations are worked out. Finally, in Section VII, the main conclusions are summarized. We only consider discrete-time systems, however analog results for the continuous-time case follow in a similar way (see [20]).

II. LPV Systems and Behaviors

The reason why the LPV framework has become popular in practical applications is that it represents an attractive intermediate case between LTI and nonlinear/time-varying descriptions. Driven by the need to address the control of complicated plant dynamics in a linear framework, LPV systems were invented to "embed" nonlinear behaviors into a linear structure enabling the use of convex control synthesis and simple stability analysis as extensions of well-worked-out LTI results. However, what makes all this possible is a particular concept behind the scheduling variable p. In order to give a formal definition of LPV systems, we first need to clarify the role of p and its so called *externality property*.

Assume that we are given a discrete-time system $\mathcal{G}$, depicted in Fig. 1(a), which describes the (possibly nonlinear) dynamical relation between the signals $w : \mathbb{Z} \to \mathbb{W}$, where $\mathbb{W}$ is a given set. Let $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{Z}}$ ($\mathbb{W}^{\mathbb{Z}}$ stands for all maps from $\mathbb{Z}$ to $\mathbb{W}$) containing all trajectories of w that are compatible with $\mathcal{G}$. Then, we call $\mathfrak{B}$ the behavior of the system $\mathcal{G}$. A common practice in LPV modeling is to introduce an auxiliary variable p, with range $\mathbb{P}$, and reformulate $\mathcal{G}$ as shown in Fig. 1(b), where it holds true that if the loop is disconnected and p is assumed to be a known signal, then the "remaining" relations of w are linear. Applying this reformulation with a disconnected p and assuming that all trajectories of p are allowed, i.e., p is a free variable with $p \in \mathbb{P}^{\mathbb{Z}}$, the possible trajectories of this reformulated system will form a behavior $\mathfrak{B}'$, which will contain $\mathfrak{B}$ as visualized in Fig. 1(c). This concept of formulating a linear but p-dependent description of $\mathcal{G}$ enables the use of simple stability analysis and convex controller synthesis, which will always be conservative w.r.t. $\mathcal{G}$, but computationally more attractive and robust than other approaches directly addressing $\mathfrak{B}$. The scheduling variable p can appear in many different relations w.r.t. the original variables w. If p is a free variable w.r.t. $\mathcal{G}$, then we can speak about a *true parameter-varying system* without conservativeness. However, it often happens that p depends on other signals. In the latter case, the resulting system is often referred as a *quasi parameter-varying system*. To decrease conservativeness of LPV controller synthesis or modeling w.r.t. such situations, very often the possible trajectories of p are restricted, for instance by supposing (boundary) restrictions on first- and higher-order derivatives/differences of p or by excluding specific trajectories due to physical constraints. In this way, p appears to be a free variable of the system, but with certain "external" restrictions, hence to express this property, we will call p an *external variable* in the sequel. Based on these concepts, the class of *parameter-varying* (PV) systems can be defined as follows.

Definition 1 (Parameter-Varying Dynamical System): A parameter-varying system $\mathcal{S}$ is defined as a quadruple $\mathcal{S} = (\mathbb{T}, \mathbb{P}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{T}$ is called the time axis, $\mathbb{P}$ denotes the scheduling set (i.e., $p(k) \in \mathbb{P}$), $\mathbb{W}$ is the signal space, and $\mathfrak{B} \subseteq (\mathbb{W} \times \mathbb{P})^{\mathbb{T}}$ is the *behavior*. Furthermore, the set of allowed scheduling trajectories

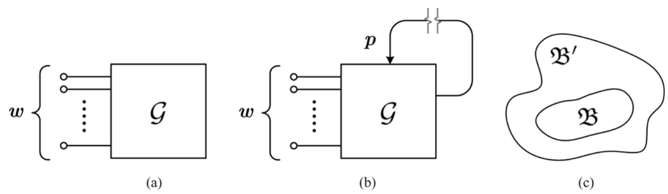

Fig. 1. Concept of LPV modeling. (a) Original plant. (b) Characterization of p. (c) Relation of the resulting behaviors.

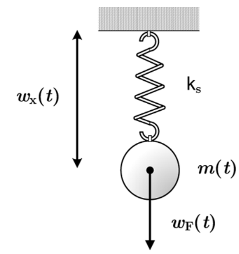

Fig. 2. Varying-mass connected to a spring.

$\pi_p\mathfrak{B} = \{p \in \mathbb{P}^{\mathbb{T}} \mid \exists w \in \mathbb{W}^{\mathbb{T}} \text{ s.t. } (w,p) \in \mathfrak{B}\}$ satisfies the externality property in the sense that there exists a behavior $\mathfrak{B}' \subseteq (\mathbb{W}\times\mathbb{P})^{\mathbb{T}}$ with p being a free variable, i.e., $\pi_p\mathfrak{B}' = \mathbb{P}^{\mathbb{T}}$, and $\mathfrak{B} \subseteq \mathfrak{B}'$ such that for each $p \in \pi_p\mathfrak{B}$ it holds that $(w,p)\in\mathfrak{B}' \Rightarrow (w,p)\in\mathfrak{B}$. In other words $(w,p)\in\mathfrak{B}'\setminus\mathfrak{B}$ implies that $p\notin\pi_p\mathfrak{B}$. □

The set $\mathbb{T}$ defines the time-axis of the system, describing *continuous-time* (CT), $\mathbb{T}=\mathbb{R}$, and *discrete-time* (DT), $\mathbb{T}=\mathbb{Z}$, systems alike, while $\mathbb{W}$ gives the range of the system signals w. The behavior $\mathfrak{B}\subseteq(\mathbb{W}\times\mathbb{P})^{\mathbb{T}}$ is the set of all signal and scheduling trajectories that are compatible with the system. Note that there is no prior distinction between inputs and outputs in this setting.

The scheduling set $\mathbb{P}$ is usually a closed subset of a vector space. The set of admissible scheduling trajectories of p, defined as the *projected scheduling behavior*

$$\mathfrak{B}_{\mathbb{P}} = \pi_p\mathfrak{B} \subseteq \mathbb{P}^{\mathbb{Z}} \tag{3}$$

describes all possible scheduling trajectories of $\mathcal{S}$. $\mathfrak{B}_{\mathbb{P}}$ in terms of Definition 1 implies that the scheduling variable $p\in\mathfrak{B}_{\mathbb{P}}$ is a "structurally free" variable of $\mathcal{S}$, but not literally as the trajectories of p can be restricted in $\mathfrak{B}$, i.e., $\pi_p\mathfrak{B}$ is not necessary equal to $\mathbb{P}^{\mathbb{Z}}$. A variable with such a property is called *external* or *semi-free*. Note that this definition of the behavior allows to include additional restrictions on the possible trajectories of p, but keeps the independence of p from the signal variables w, which is in line with the current concepts of the LPV literature (see Example 2)

For a given scheduling trajectory, $p\in\mathfrak{B}_{\mathbb{P}}$, we define the *projected signal behavior* as

$$\mathfrak{B}_p = \{w\in\mathbb{W}^{\mathbb{T}} \mid (w,p)\in\mathfrak{B}\}. \tag{4}$$

$\mathfrak{B}_p$ describes all possible signal trajectories compatible with p. In case of a constant scheduling trajectory, $p\in\mathfrak{B}_{\mathbb{P}}$ with $p(t)=\bar{\mathrm{p}}$ for all $t\in\mathbb{T}$ where $\bar{\mathrm{p}}\in\mathbb{P}$, the projected behavior $\mathfrak{B}_p$ is called a *frozen behavior* and denoted as

$$\mathfrak{B}_{\bar{\mathrm{p}}} = \{w\in\mathbb{W}^{\mathbb{T}} \mid (w,p)\in\mathfrak{B} \text{ with } p(t)=\bar{\mathrm{p}}, \forall t\in\mathbb{T}\}. \tag{5}$$

Definition 2 (Frozen System): Let $\mathcal{S}=(\mathbb{T},\mathbb{P},\mathbb{W},\mathfrak{B})$ be a PV system and consider $\mathfrak{B}_{\bar{\mathrm{p}}}$ for a $p(t)\equiv\bar{\mathrm{p}}$ in $\mathfrak{B}_{\mathbb{P}}$. The dynamical system $\mathcal{F}_{\bar{\mathrm{p}}}=(\mathbb{T},\mathbb{W},\mathfrak{B}_{\bar{\mathrm{p}}})$ is called a frozen system of $\mathcal{S}$. □

Define q as the unit forward time-shift operator, e.g., $qw(t)=w(t+1)$. With the previously introduced concepts, we can define discrete-time LPV systems as follows.

Definition 3 (DT-LPV System): Let $\mathbb{T}=\mathbb{Z}$. The parameter-varying system $\mathcal{S}$ is called LPV, if the following conditions apply.

- $\mathbb{W}$ is a vector space, and $\mathfrak{B}_p$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$ for all $p\in\mathfrak{B}_{\mathbb{P}}$ (linearity).
- For any $(w,p)\in\mathfrak{B}$ and any $\tau\in\mathbb{T}$, it holds that $(w(\cdot+\tau),p(\cdot+\tau))\in\mathfrak{B}$, in other words $q^{\tau}\mathfrak{B}=\mathfrak{B}$ (time-invariance). □

In terms of Definition 3, for a constant scheduling trajectory $p(k)\equiv\bar{\mathrm{p}}$, time-invariance of $\mathcal{S}$ implies time-invariance of $\mathcal{F}_{\bar{\mathrm{p}}}$. Based on this and the linearity condition of $\mathfrak{B}_p$, it holds for an LPV system that for each $\bar{\mathrm{p}}\in\mathbb{P}$ with $p(k)\equiv\bar{\mathrm{p}}$ in $\mathfrak{B}_{\mathbb{P}}$ the associated frozen system $\mathcal{F}_{\bar{\mathrm{p}}}$ is an LTI system, which is in accordance with previous definitions of LPV systems [1]. In this way, the projected behaviors of a given $\mathcal{S}$ w.r.t. constant scheduling trajectories define a set of LTI systems.

Definition 4 (Frozen System Set): Let $\mathcal{S}=(\mathbb{T},\mathbb{P},\mathbb{W},\mathfrak{B})$ be an LPV system. The set of LTI systems

$$\mathscr{F}_{\mathcal{S}} = \{\mathcal{F}=(\mathbb{T},\mathbb{W},\mathfrak{B}')| \\ \exists p\in\mathfrak{B}_{\bar{\mathrm{p}}} \text{ with } p(k)\equiv\bar{\mathrm{p}}\in\mathbb{P} \text{ s.t. } \mathfrak{B}'=\mathfrak{B}_{\bar{\mathrm{p}}}\} \tag{6}$$

is called the frozen system set of $\mathcal{S}$. □

Naturally, the LPV system concept is advantageous compared to general nonlinear systems, as the relation of the signals is linear. Definition 3 also reveals the advantage of this system class over LTV systems: The variation of the system dynamics is not associated directly with time, but with the variation of an external (semi-free) signal. Thus, the LPV modeling concept, compared to LTV systems, is more suitable for nonstationary/coordinate-dependent physical systems as it describes the underlying phenomena directly.

Example 2: To emphasize the advantage of LPV systems, we investigate the modeling of the motion of a varying mass connected to a spring (see Fig. 2). This problem is one of the typical phenomena occurring in systems with time-varying masses like in motion control (robotics, rotating crankshafts, rockets, etc.). Denote by w_x the position of the varying mass m. Let $\mathtt{k}_s>0$ be the spring constant, introduce w_F as the force acting on the mass, and assume that there is no damping. By Newton's second law of motion, the following equation holds:

$$\frac{d}{dt}\left(m\frac{d}{dt}w_x\right) = w_F - \mathtt{k}_s w_x. \tag{7}$$

Using an Euler type of discretization with step size $\mathtt{T}_d>0$, a DT approximation of (7) is

$$\left(\mathtt{T}_d^2\mathtt{k}_s + m(k)\right)w_x(k) - (m(k+1)+m(k))w_x(k+1) \\ +m(k+1)w_x(k+2) = \mathtt{T}_d^2 w_F(k) \tag{8}$$

It is immediate that by taking m as a scheduling variable, the behavior of this process can be described as an LPV system, preserving the physical insight of Newton's second law. Note that m is a free variable in (7), hence the resulting LPV system with $p = m$ describes the behavior of (7) without conservativeness. On the other hand, viewing m as a time-varying parameter, whose trajectory is fixed and known in time, results in an LTV system. Such a system would explain the behavior of the process for only a fixed trajectory of the mass. Furthermore, in an application it might be advantageous to restrict the possible trajectories of m to a subset of $\mathbb{R}^{\mathbb{Z}}$, as for example during operation of the system it is known that $|m(k+1) - m(k)| < \delta_m$. This restriction of the behavior can be exploited to decrease the conservativeness of the LPV description and focus the control synthesis on the interesting operating regime later on. However, with such a restriction, $p = m$ would not be a free variable anymore, but it would still be external. □

In the sequel, we restrict our attention to DT systems with $\mathbb{W} = \mathbb{R}^{n_{\mathbb{W}}}$ and with $\mathbb{P}$ a subset of $\mathbb{R}^{n_{\mathbb{P}}}$. In fact, we consider LPV systems described by finite-order linear difference equations with parameter-varying effects in the coefficients.

III. Algebraic Preliminaries

In order to reestablish the concept of LPV-IO and SS representations, we introduce difference equations with varying coefficients as the representation of the behavior $\mathfrak{B}$. These difference equations are described by polynomials of an algebraic ring where equivalence of representations and other system theoretic concepts can be characterized by simple algebraic manipulations.

A. Coefficient Functions

First, we define the set of functional dependencies considered in the sequel.

Definition 5 (Real-Meromorphic Function [25]): A real-meromorphic function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, is a function $f = g/h$, where $g, h : \mathbb{R}^n \rightarrow \mathbb{R}$ are holomorphic (analytic) functions and $h \neq 0$. □

Meromorphic functions consist of all rational, polynomial, trigonometric expressions, rational exponential functions, etc. Thus, this class contains the common functional dependencies that result during LPV modeling of physical systems. Next, we establish an algebraic field $\mathcal{R}$ of a wide class of multivariable real-meromorphic functions from which the p-dependent coefficients of the representations will follow. Variables of these functions will be associated with the elements of the scheduling variable and their time-shifts in order to represent dynamic dependencies. However, to uniquely define these dependencies (to establish a field), it must be ensured that in terms of an ordering, the "last" variable have a role in the considered functions. For instance, $f(\mathtt{x}_1, \mathtt{x}_2) = \mathtt{x}_1$ should be excluded from the considered set as only $\hat{f}(\mathtt{x}_1) = \mathtt{x}_1$ is need to express this functional dependence. To ensure this property, we introduce operators $\mho_j$ and $\mho_*$ to exclude nonunique functional dependencies in the construction of $\mathcal{R}$.

Let $\mathcal{R}_n$ denote the field of real-meromorphic functions with n variables. Denote the variables of a $r \in \mathcal{R}_n$ as $\zeta_1, \ldots \zeta_n$. Also define an operator $\mho_j$ on $\mathcal{R}_n$ with $1 \leq j \leq n$ such that

$$\mho_j(r(\zeta_1, \ldots, \zeta_n)) := r(\zeta_1, \ldots, \zeta_j, 0, \ldots, 0). \tag{9}$$

Note that $\mho_j$ projects a meromorphic function to a lower dimensional domain. Introduce

$$\bar{\mathcal{R}}_n = \{r \in \mathcal{R}_n \mid \mho_{n-1}(r) \neq r\}. \tag{10}$$

It is clear that $\bar{\mathcal{R}}_n$ consist of all functions $\mathcal{R}_n$ in which the variable ζ_n has a nonzero contribution, i.e., it plays a role in the function. Also define the operator $\mho_* : (\cup_{i\geq 0}\mathcal{R}_i) \rightarrow (\cup_{i\geq 0}\bar{\mathcal{R}}_i)$, which associates a given $r \in \mathcal{R}_n$ with a $r' \in \bar{\mathcal{R}}_{n'}$, $n \geq n'$, i.e., $\mho_*(r) = r'$, such that $r'(\zeta_1, \ldots, \zeta_{n'}) = r(\zeta_1, \ldots, \zeta_{n'}, 0, \ldots, 0)$ for all $\zeta_1, \ldots, \zeta_{n'} \in \mathbb{R}$, $\mho_{n'}(r) = r$ and n' is minimal. In this way, $\mho_*$ reduces the variables of a function till $\zeta_{n'}$ cannot be left out from the expression because it has a nonzero contribution to the value of the function. Now define the collection of all real-meromorphic functions with finite many variables as follows:

$$\mathcal{R} = \bigcup_{i\geq 0} \bar{\mathcal{R}}_i, \qquad \text{with } \bar{\mathcal{R}}_0 = \mathbb{R}. \tag{11}$$

The function class $\mathcal{R}$ will be used as the collection of coefficient functions [like $\{A, \ldots, D\}$ and $\{a_i, b_j\}$ in (1a), (1b), and (2)] for the representations, giving the basic building block of PV difference equations. These functions are not only used to express dependence over multidimensional p but also to enable a distinction between dynamic scheduling dependence of the coefficients and the dynamic relation between the signals of the system. The following lemma is important.

Lemma 1 (Field Property of $\mathcal{R}$): The set $\mathcal{R}$ is a field. □

To prove Lemma 1, the addition and multiplication operators on $\mathcal{R}$ are defined as follows.

Definition 6 (Addition/Multiplication Operator on $\mathcal{R}$): Let $r_1, r_2 \in \mathcal{R}$ such that $r_1 \in \bar{\mathcal{R}}_i$ and $r_2 \in \bar{\mathcal{R}}_j$ with $i, j \geq 0$. If $i \geq j$, there exists a unique function $r_2' \in \mathcal{R}_i$ such that $\mho_*(r_2') = r_2$. Let $r_1' = r_1$. In case $i < j$, r_1' and r_2' are defined respectively on $\mathcal{R}_j$. Then

$$r_1 \boxplus r_2 := \mho_*\left(r_1' + r_2'\right) \quad r_1 \boxdot r_2 := \mho_*\left(r_1' \cdot r_2'\right) \tag{12}$$

where $+$ and $\cdot$ are the Euclidean addition and multiplication operators of $\mathcal{R}_i$ (or $\mathcal{R}_j$). □

Based upon $\boxplus$ and $\boxdot$, the proof of Lemma 1 is straightforward and can be found in [20]. In the following, if it is not necessary to emphasize the difference between the Euclidian addition and $\boxplus$, we use $+$ to denote both operators in order to improve readability. The same abuse of notation is introduced for $\boxdot$.

B. Representing Scheduling Dependence

The next step is to associate the variables of the coefficient functions with elements of p and its time-shifts, which will provide the characterization of dynamic dependencies in the representations. Naturally, this association is dependent on the dimension of the scheduling space considered.

In case of a scalar p, i.e., $n_{\mathbb{P}} = 1$, we can associate each variable $\{\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_2, \ldots\}$ of a given $r \in \mathcal{R}$ with $\{p, qp, q^{-1}p, q^2p, \ldots\}$ in order to express a given dynamic coefficient dependency. For example, the dependence $2p \cdot \sin(q^{-1}p)$

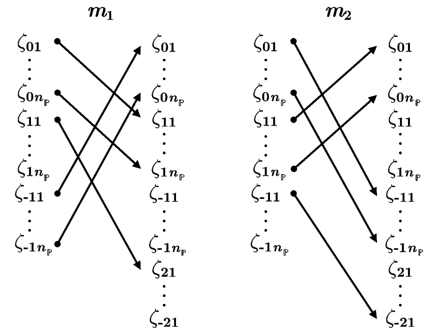

Fig. 3. Variable assignment by the functions m_1 and m_2 in Definition 7.

can be expressed in this way by a unique $r \in \mathcal{R}$ given as $r(\mathrm{x}_1\mathrm{x}_2, \mathrm{x}_3) = 2\mathrm{x}_1 \sin(\mathrm{x}_3)$.

Now, we can consider the general case. For a given $\mathbb{P}$ with dimension $n_\mathbb{P}$ and $r \in \bar{\mathcal{R}}_n$, label the variables of r according to the following ordering:

$$r(\zeta_{0,1}, \ldots, \zeta_{0,n_\mathbb{P}}, \zeta_{1,1}, \ldots, \zeta_{1,n_\mathbb{P}}, \zeta_{-1,1}, \ldots, \zeta_{-1,n_\mathbb{P}}, \zeta_{2,1}, \ldots).$$

For a given scheduling signal p, associate the variable $\zeta_{i,j}$ with $q^i p_j$. For this association we introduce the operator

$$\diamond : (\mathcal{R} \times \mathbb{P}^\mathbb{Z}) \to \mathbb{R}^\mathbb{Z}, \quad \text{defined by } r \diamond p = r(p, qp, q^{-1}p, \ldots).$$

The value of a (p-dependent) coefficient in an LPV system representation is now given by an operation $(r \diamond p)(k)$.

Example 3 (Coefficient Function): Let $\mathbb{P} = \mathbb{R}^{n_\mathbb{P}}$ with $n_\mathbb{P} = 2$. Consider the real-meromorphic coefficient function $r : \mathbb{R}^3 \to \mathbb{R}$, defined as $r(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) = \frac{1+\mathrm{x}_3}{1-\mathrm{x}_2}$. Then, for a scheduling signal $p : \mathbb{Z} \to \mathbb{R}^2$, $(r \diamond p)(k) = r(p_1, p_2, qp_1)(k) = \frac{1+p_1(k+1)}{1-p_2(k)}$. On the other hand, if $n_\mathbb{P} = 3$, then $(r \diamond p)(k) = r(p_1, p_2, p_3)(k) = \frac{1+p_3(k)}{1-p_2(k)}$, showing that the operator $\diamond$ implicitly depends on $n_\mathbb{P}$. □

In the sequel, the (time-varying) coefficient sequence $(r \diamond p)$ will be used to operate on a signal w [like $a_i(p)$ in (2)], giving the varying coefficient sequence of the representations. In this respect, an important property is that multiplication of the $\diamond$ operation with the shift operator q is not commutative—in other words, $q(r \diamond p) \neq (r \diamond p)q$. To handle this multiplication, for $r \in \mathcal{R}$ we define the shift operations $\overrightarrow{r}$, $\overleftarrow{r}$.

Definition 7 (Shift Operators): Let $r \in \bar{\mathcal{R}}_n$. For a given scheduling dimension $n_\mathbb{P}$, denote the variables of r as $\{\zeta_{i,j}\}$ based on the previously introduced labeling. The forward-shift and backward-shift operators on $\mathcal{R}$ are defined as

$$\overrightarrow{r} := \mho_*(r \circ m_1) \quad \overleftarrow{r} := \mho_*(r \circ m_2) \tag{13}$$

where $\circ$ denotes function composition, $m_1, m_2 \in (\mathcal{R}_{n+2n_\mathbb{P}})^n$, and m_1 assigns each variable $\zeta_{i,j}$ to $\zeta_{(i+1),j}$, while m_2 assigns each $\zeta_{i,j}$ to $\zeta_{(i-1),j}$ as depicted in Fig. 3. □

In other words, if $r \diamond p$ is dependent on p and qp, then $\overrightarrow{r}$ is the "same" function (disregarding the number of variables) except $\overrightarrow{r} \diamond p$ is dependent on qp and q^2p. With these notions, we can write $qr = \overrightarrow{r} q$ and $q^{-1}r = \overleftarrow{r} q^{-1}$, corresponding to

$$q(r \diamond p)w = (\overrightarrow{r} \diamond p)qw$$
$$q^{-1}(r \diamond p)w = (\overleftarrow{r} \diamond p)q^{-1}w$$

on the signal level.

Example 4: Consider the coefficient function r given in Example 3 with $n_\mathbb{P} = 2$. Then, $\overrightarrow{r}$ is a function $\mathbb{R}^5 \to \mathbb{R}$, given by $\overrightarrow{r}(\zeta_{0,1}, \zeta_{0,2}, \zeta_{1,1}, \zeta_{1,2}, \zeta_{-1,1}, \zeta_{-1,2}, \zeta_{2,1}) = \frac{1+\zeta_{2,1}}{1-\zeta_{1,2}}$. For a scheduling trajectory $p : \mathbb{Z} \to \mathbb{R}^2$, it holds that $(\overrightarrow{r} \diamond p)(k) = (r \diamond (qp))(k) = \frac{1+p_1(k+2)}{1-p_2(k+1)}$. □

The considered operator $\diamond$ can straightforwardly be extended to matrix functions $r \in \mathcal{R}^{n_r \times n_\mathbb{W}}$ where the operation $\diamond$ is applied to each scalar entry of the matrix.

C. Polynomials Over $\mathcal{R}$

Next, we define the algebraic structure of the representations we use to describe LPV systems. Introduce $\mathcal{R}[\xi]$ as all polynomials in the indeterminate ξ and with coefficients in $\mathcal{R}$. $\mathcal{R}[\xi]$ is a ring as it is a general property of polynomial spaces over a field, that they define a ring. Also introduce $\mathcal{R}[\xi]^{\cdot \times \cdot}$, the set of matrix polynomial functions with elements in $\mathcal{R}[\xi]$. Using $\mathcal{R}[\xi]$ and the operator $\diamond$, we are now able to define a PV difference equation.

Definition 8 (PV Difference Equation): Consider $R(\xi) = \sum_{i=0}^{n_\xi} r_i \xi^i \in \mathcal{R}[\xi]^{n_r \times n_\mathbb{W}}$ and $(w, p) \in (\mathbb{R}^{n_\mathbb{W}} \times \mathbb{R}^{n_\mathbb{P}})^\mathbb{Z}$.

$$(R(q) \diamond p)w := \sum_{i=0}^{n_\xi} (r_i \diamond p) q^i w = 0 \tag{14}$$

is called a PV difference equation with order $n_\xi = \deg(R)$. □

In this notation, the shift operator q operates on the signal w, while the operation $\diamond$ takes care of the time/scheduling-dependent coefficient sequence. Since the indeterminate ξ is associated with q, multiplication with ξ is noncommutative on $\mathcal{R}[\xi]^{n_r \times n_\mathbb{W}}$, i.e., $\xi r = \overrightarrow{r} \xi$ and $r\xi = \xi \overleftarrow{r}$.

In the following, we only consider scheduling trajectories for which the coefficients of $R(\xi) \diamond p$ are bounded, so the set of solutions associated with $R(\xi)$ is well defined. PV difference equations in the form of (14) are used to define the class of DT-LPV systems we consider in this paper. It will be shown that this class contains all the popular definitions of LPV-SS and IO models.

Example 5 (PV Difference Equation): Consider Example 2. Let $p = m$ with scheduling space $\mathbb{P} = [1, 2]$ and let $w = [w_x \quad w_F]^\top$. Then, the difference (8), which defines the possible signal trajectories of the DT approximation of the mass-spring system, can be written in the form of (14) with $n_\mathbb{W} = 2$, $n_\xi = 1$, $n_\mathbb{P} = 1$

$$(R(q) \diamond p)w = (r_0 \diamond p)w + (r_1 \diamond p)qw + (r_2 \diamond p)q^2 w = 0 \tag{15}$$

where $r_0 \diamond p = [\mathrm{T}_d^2 \mathrm{k}_s + p \quad -\mathrm{T}_d^2]$, $r_1 \diamond p = [-qp - p \quad 0]$, $r_2 \diamond p = [qp \quad 0]$. □

Due to its algebraic structure, it easily follows that $\mathcal{R}[\xi]$ is a domain, i.e., for all R_1, $R_2 \in \mathcal{R}[\xi]$ it holds that $R_1(\xi)R_2(\xi) = 0 \Rightarrow R_1(\xi) = 0$ or $R_2(\xi) = 0$. Then, with the above defined noncommutative multiplicative rules, $\mathcal{R}[\xi]$ defines an Ore algebra [26] and it is a left and right Euclidian domain [27]. The latter implies that there exists division by remainder. This means that if R_1, $R_2 \in \mathcal{R}[\xi]$ with $\deg(R_1) \geq \deg(R_2)$ and $R_2 \neq 0$, then there exist unique polynomials R', $R'' \in \mathcal{R}[\xi]$ such that $R_1(\xi) = R_2(\xi)R'(\xi) + R''(\xi)$ where $\deg(R_2) > \deg(R'')$. Due to the fact that $\mathcal{R}[\xi]$ is a domain, the rank of a polynomial $R \in \mathcal{R}[\xi]^{n_r \times n_\mathbb{W}}$ is well defined [28]. Denote by $\mathrm{span}_\mathcal{R}^{\mathrm{row}}(R)$

and $\mathrm{span}^{\mathrm{col}}_{\mathcal{R}}(R)$ the subspace spanned by the rows (columns) of $R \in \mathcal{R}[\xi]^{\cdot\times\cdot}$, viewed as a linear space of polynomial vector functions with coefficients in $\mathcal{R}^{\cdot\times\cdot}$. Then, it can be shown that

$$\mathrm{rank}(R) = \dim\left(\mathrm{span}^{\mathrm{row}}_{\mathcal{R}}(R)\right) = \dim\left(\mathrm{span}^{\mathrm{col}}_{\mathcal{R}}(R)\right). \quad (16)$$

The notion of unimodular matrices, essential to characterize equivalent representations, is also introduced.

Definition 9 (Unimodular Matrix): Let $M \in \mathcal{R}[\xi]^{n\times n}$. M is called unimodular if there exists a $M^\dagger \in \mathcal{R}[\xi]^{n\times n}$ such that $M^\dagger(\xi)M(\xi) = I$ and $M(\xi)M^\dagger(\xi) = I$. □

Any unimodular matrix operator in $\mathcal{R}[\xi]^{\cdot\times\cdot}$ is equivalent to the product of finite many elementary row and column operations [27].

1) Interchange row (column) i and row (column) j.
2) Multiply a row (column) i on the left (right) by a $r \in \mathcal{R}$, $r \neq 0$.
3) For $i \neq j$, add to row (column) i row (column) j multiplied by ξ^n, $n > 0$.

Example 6 (Unimodular Matrix): The matrix polynomials $M, M^\dagger \in \mathcal{R}[\xi]^{2\times 2}$, defined as

$$M(\xi) = \begin{bmatrix} r_2 & r_2\xi \\ r_1\xi & r_1\xi^2 + r_1 \end{bmatrix}$$
$$M^\dagger(\xi) = \begin{bmatrix} r_1 + \xi^2 r_1 & -\xi r_2 \\ -\xi r_1 & r_2 \end{bmatrix} \frac{1}{r_1 r_2}$$

are unimodular as $M(\xi)M^\dagger(\xi) = M^\dagger(\xi)M(\xi) = I$. Note that $\xi r_1 \neq r_1\xi$ due to the noncommutativity of the multiplication by ξ on $\mathcal{R}[\xi]$. □

Another important property of $\mathcal{R}[\xi]^{\cdot\times\cdot}$ is the existence of a Jacobson form (generalization of the Smith form).

Theorem 1 (Jacobson Form [27]): Let $R \in \mathcal{R}[\xi]^{n_r\times n_W}$ with $R \neq 0$ and $n = \mathrm{rank}(R)$. Then, there exist unimodular matrices $M_1 \in \mathcal{R}[\xi]^{n_r\times n_r}$ and $M_2 \in \mathcal{R}[\xi]^{n_W\times n_W}$ such that

$$M_1(\xi)R(\xi)M_2(\xi) = \begin{bmatrix} Q(\xi) & 0 \\ 0 & 0 \end{bmatrix} \quad (17)$$

where $Q = \mathrm{diag}(r'_1, \dots, r'_n) \in \mathcal{R}[\xi]^{n\times n}$ with monic nonzero $r'_i \in \mathcal{R}[\xi]$. Furthermore, there exist $g'_i \in \mathcal{R}[\xi]$ such that $r'_{i+1}(\xi) = g'_i(\xi)r'_i(\xi)$ for $i = 1, \dots, n-1$. □

Due to the algebraic structure of $\mathcal{R}[\xi]^{\cdot\times\cdot}$, the proof of Theorem 1 similarly follows as in [27].

Example 7 (Jacobson Form): Consider

$$R(\xi) = \begin{bmatrix} r + \xi & -1 & -1 \\ -r & 1 + \xi & -\overrightarrow{r} \end{bmatrix} \in \mathcal{R}[\xi]^{2\times 3}$$

where r is a meromorphic function and $\xi = q$. Then, the Jacobson form of R is

$$M_1(\xi)R(\xi)M_2(\xi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 + \overrightarrow{r} + \xi & 0 \end{bmatrix}$$
$$M_1(\xi) = \begin{bmatrix} 1 & 0 \\ -\overrightarrow{r} & 1 \end{bmatrix}$$
$$M_2(\xi) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & r \\ -1 & -1 & \xi \end{bmatrix}.$$

□

Now it is possible to show that there exists a duality between the solution spaces of PV difference equations and the polynomial modules in $\mathcal{R}[\xi]^{\cdot\times\cdot}$ associated with them, which is implied by a so-called *injective cogenerator* property. This property makes it possible to use the developed algebraic structure to characterize behaviors and manipulations on them. Originally, the injective cogenerator property has been shown for the solution spaces of the polynomial ring over $\mathcal{R}_1$ in [29]. In the Appendix, this proof is extended to $\mathcal{R}[\xi]$.

IV. System Representations

A. Kernel Representation

Using the developed concepts, we introduce *kernel representation* (KR) of an LPV system in the form of (14).

Definition 10 (DT-KR-LPV Representation): The parameter varying difference (14) is called a discrete-time kernel representation, denoted by $\mathfrak{R}_{\mathrm{K}}(\mathcal{S})$, of the LPV dynamical system $\mathcal{S} = (\mathbb{Z}, \mathbb{R}^{n_{\mathbb{P}}}, \mathbb{R}^{n_{\mathbb{W}}}, \mathfrak{B})$ with scheduling vector p and signals w, if

$$\mathfrak{B} = \{(w, p) \in (\mathbb{R}^{n_{\mathbb{W}}} \times \mathbb{R}^{n_{\mathbb{P}}})^{\mathbb{Z}} \mid (R(q) \diamond p)w = 0\}. \quad (18)$$

□

It is obvious that the behavior $\mathfrak{B}$ associated with (14) always corresponds to a LPV system in terms of Definition 3. It is also important, that the allowed trajectories of p in terms of (18) are not restricted by (14) (only those $p \in (\mathbb{R}^{n_{\mathbb{P}}})^{\mathbb{Z}}$ are excluded for which a coefficient $r_i \diamond p$ is unbounded). This is in accordance with the classical concept of p being an external variable of the system. One can also include further restrictions on $\mathfrak{B}_{\mathbb{P}} = \pi_p\mathfrak{B}$, like bounding the first- or higher-order differences of p, etc. However, to preserve the generality of the developed framework, we do not consider such restrictions in terms of representations.

Based on the concept of rank, the following theorem holds.

Theorem 2 (Full Row Rank KR Representation): Let $\mathfrak{B}$ be given with a KR representation (14). Then, $\mathfrak{B}$ can also be represented by a $R' \in \mathcal{R}[\xi]^{\cdot\times n_{\mathbb{W}}}$ with full row rank. □

The proof of this theorem is given in the Appendix.

B. IO Representation

Partitioning of the signals w into input signals $u \in (\mathbb{R}^{n_{\mathbb{U}}})^{\mathbb{Z}}$ and output signals $y \in (\mathbb{R}^{n_{\mathbb{Y}}})^{\mathbb{Z}}$, i.e., $w = \mathrm{col}(u, y)$, is often considered convenient. Such a partitioning is called an IO partition [24].

Definition 11 (IO Partition of an LPV System): Let $\mathcal{S} = (\mathbb{Z}, \mathbb{R}^{n_{\mathbb{P}}}, \mathbb{R}^{n_{\mathbb{W}}}, \mathfrak{B})$ be an LPV system. The partitioning of the signal space as $\mathbb{R}^{n_{\mathbb{W}}} = \mathbb{U} \times \mathbb{Y} = \mathbb{R}^{n_{\mathbb{U}}} \times \mathbb{R}^{n_{\mathbb{Y}}}$ and partitioning of $w \in (\mathbb{R}^{n_{\mathbb{W}}})^{\mathbb{Z}}$ correspondingly with $u \in (\mathbb{R}^{n_{\mathbb{U}}})^{\mathbb{Z}}$ and $y \in (\mathbb{R}^{n_{\mathbb{Y}}})^{\mathbb{Z}}$ is called an IO partition of $\mathcal{S}$, if the following conditions apply:

1) u is free, i.e., for all $u \in (\mathbb{R}^{n_{\mathbb{U}}})^{\mathbb{Z}}$ and $p \in \mathfrak{B}_{\mathbb{P}}$, there exists a $y \in (\mathbb{R}^{n_{\mathbb{Y}}})^{\mathbb{Z}}$ such that $(\mathrm{col}(u, y), p) \in \mathfrak{B}$;
2) y does not contain any further free component, i.e., given u, none of the components of y can be chosen freely for every $p \in \mathfrak{B}_{\mathbb{P}}$ (maximally free). □

An IO partition implies the existence of matrix-polynomial functions $R_y \in \mathcal{R}[\xi]^{n_\mathbb{Y} \times n_\mathbb{Y}}$ and $R_u \in \mathcal{R}[\xi]^{n_\mathbb{Y} \times n_\mathbb{U}}$ with R_y full row rank, such that (14) can be written as

$$(R_y(q) \diamond p)y = (R_u(q) \diamond p)u \tag{19}$$

with $n_\mathbb{W} = n_\mathbb{U} + n_\mathbb{Y}$ and the corresponding behavior $\mathfrak{B}$ is

$$\left\{(u, y, p) \in (\mathbb{U} \times \mathbb{Y} \times \mathbb{P})^\mathbb{Z} \mid (R_y(q) \diamond p)y = (R_u(q) \diamond p)u\right\}$$

with $\mathbb{U} = \mathbb{R}^{n_\mathbb{U}}$ and $\mathbb{Y} = \mathbb{R}^{n_\mathbb{Y}}$. An IO partition defines a causal mapping in case the solutions of (19) are restricted to have left compact support. Otherwise, initial conditions also matter [30]. Similar to the LTI case, LPV systems with no free variables are called autonomous.[1] Now, it is possible to introduce IO representations of DT-LPV systems.

Definition 12 (LPV-IO Representation): The discrete-time IO representation of an LPV system $\mathcal{S} = (\mathbb{Z}, \mathbb{P} \subseteq \mathbb{R}^{n_\mathbb{P}}, \mathbb{R}^{n_\mathbb{U}+n_\mathbb{Y}}, \mathfrak{B})$ with IO partition (u, y) and scheduling vector p is denoted by $\mathfrak{R}_{\mathrm{IO}}(\mathcal{S})$ and defined as a parameter-varying difference-equation system with order n_a

$$\sum_{i=0}^{n_a} (a_i \diamond p) q^i y = \sum_{j=0}^{n_b} (b_j \diamond p) q^j u \tag{20}$$

where $a_j \in \mathcal{R}^{n_\mathbb{Y} \times n_\mathbb{Y}}$ and $b_j \in \mathcal{R}^{n_\mathbb{Y} \times n_\mathbb{U}}$ with $a_{n_a} \neq 0$ and $b_{n_b} \neq 0$ are the meromorphic parameter-varying coefficients of the matrix polynomials $R_u(\xi) = \sum_{j=0}^{n_b} b_j \xi^j$ and full row rank $R_y(\xi) = \sum_{i=0}^{n_a} a_i \xi^i$ with $n_a \geq n_b \geq 0$. □

It is apparent that (20) is the "dynamic-dependent" counterpart of (2).

Example 8 (IO Partition and Representation): In Example 5, the sampled force variable w_x is a free variable as it represents the inhomogeneous part of difference equation (8). Thus, the choice of $w = [y \quad u]^\top = [w_x \quad w_F]^\top$ yields a valid IO partition. With m being the scheduling signal, the discrete-time PV behavior can be represented in the form of (20) with polynomials

$$R_y(\xi) = a_0 + a_1\xi + a_2\xi^2 \quad R_u(\xi) = b_0$$

which have coefficients: $a_0 \diamond p = \mathrm{T}_d^2 \mathrm{k}_s + p$, $a_1 \diamond p = -p - qp$, $a_2 \diamond p = qp$, $b_0 \diamond p = \mathrm{T}_d^2$. Obviously, $R_y(\xi)$ has full row rank. This implies that $R_y(\xi)$ and $R_u(\xi)$ define an IO representation of the model with coefficients as above. □

For LPV systems, the notion of transfer function or frequency response in the classical sense has no meaningful[2] interpretation. By using the approximative transfer-function calculus of LTV systems based on a formal series approach [31], some interpretation of these notions can be given for LPV systems. However, the direct extension of this approximative transfer function calculus to the class of systems considered here is not available yet.

[1]It is possible that the freedom of the components of w can change for specific scheduling trajectories. In this case, the autonomous part of the behavior is related to the scheduling dependent nature of the system.

[2]Some authors [21]–[23] introduce LPV transfer functions with varying parameters. As they commonly refer only to the collection of transfer functions associated with $\mathscr{F}_\mathcal{S}$, this notion of the LPV transfer function is misleading.

C. State-Space Representation

In the modeling of dynamical systems, auxiliary variables (often called *latent variables*) are commonly used [30]. The natural counterpart of (14) to cope with such variables is

$$(R_w(q) \diamond p)w = (R_L(q) \diamond p)w_L \tag{21}$$

where $w_L : \mathbb{Z} \to \mathbb{R}^{n_L}$ are the latent variables and $R_L \in \mathcal{R}[\xi]^{n_r \times n_L}$. The set of (21) is called a *latent variable representation* of the LPV *latent variable system* $(\mathbb{Z}, \mathbb{R}^{n_\mathbb{P}}, \mathbb{R}^{n_\mathbb{W}} \times \mathbb{R}^{n_L}, \mathfrak{B}_L)$, where the so-called *full behavior* $\mathfrak{B}_L$ of this system is defined as

$$\mathfrak{B}_L = \{(w, w_L, p) \in (\mathbb{R}^{n_\mathbb{W}} \times \mathbb{R}^{n_L} \times \mathbb{R}^{n_\mathbb{P}})^\mathbb{Z} \mid (21) \text{ holds}\}.$$

Additionally, $\mathfrak{B} = \pi_{(w,p)}\mathfrak{B}_L$ is introduced as the *manifest behavior* associated with $\mathfrak{B}_L$.

Example 9 (Latent Variable Representation): By considering the DT system in Example 5 with scheduling $p = m$ and $\mathbb{P} = [1, 2]$, the following latent variable representation of the model has the same manifest behavior:

$$\begin{bmatrix} \mathrm{T}_d^2 \mathrm{k}_s + p & -\mathrm{T}_d^2 \\ (-p - q^{-1}p) & 0 \\ (-q^{-1}p) & 0 \end{bmatrix} \begin{bmatrix} w_x \\ w_F \end{bmatrix} = \begin{bmatrix} q & 0 \\ -1 & q \\ 0 & 1 \end{bmatrix} w_L. \tag{22}$$

This can be proved by substituting the third row of (22) into the second row, giving

$$w_{L,1} = (p + q^{-1}p)w_x - pqw_x. \tag{23}$$

Substitution of (23) into the first row of (22) gives a PV difference equation in the variables w_x and w_F, which is equal to (8). □

Elimination of latent variables is always possible on $\mathcal{R}[\xi]^{\cdot \times \cdot}$.

Theorem 3 (Elimination Property): Given a LPV latent variable system $(\mathbb{Z}, \mathbb{R}^{n_\mathbb{P}}, \mathbb{R}^{n_\mathbb{W}} \times \mathbb{R}^{n_L}, \mathfrak{B}_L)$ with a signal variable w, a latent variable w_L, and scheduling variable p, there exists a $R' \in \mathcal{R}[\xi]^{\cdot \times n_\mathbb{W}}$ that defines a LPV-KR representation of $\mathfrak{B} = \pi_{(w,p)}\mathfrak{B}_L$. □

For a proof, see the Appendix. Now it is possible to define the concept of state for LPV systems.

Definition 13 (Property of State): Let $(\mathbb{Z}, \mathbb{R}^{n_\mathbb{P}}, \mathbb{R}^{n_\mathbb{W}} \times \mathbb{R}^{n_L}, \mathfrak{B}_L)$ be a LPV latent variable system. Then, the latent variable w_L is a state if for every $k_0 \in \mathbb{Z}$ and $(w_1, w_{L,1}, p)$, $(w_2, w_{L,2}, p) \in \mathfrak{B}_L$ with $w_{L,1}(k_0) = w_{L,2}(k_0)$ it follows that the concatenation of these signals at k_0 satisfies

$$(w_1, w_{L,1}, p) \underset{k_0}{\wedge} (w_2, w_{L,2}, p) \in \mathfrak{B}_L. \tag{24}$$

Then, $\mathfrak{B}_L$ is called a state-space behavior, and the latent variable w_L is called the state. □

To decide whether a latent variable is a state, the following theorem is important.

Theorem 4 (State-Kernel Form): The latent variable w_L is a state, iff there exist matrices $r_w \in \mathcal{R}^{n_r \times n_\mathbb{W}}$ and r_0,

$r_1 \in \mathcal{R}^{n_r \times n_L}$ such that the full behavior $\mathfrak{B}_L$ has the kernel representation

$$r_w w + r_0 w_L + r_1 q w_L = 0. \tag{25}$$

The proof of this theorem is given in the Appendix. Now we formulate the DT state-space representation, based on an IO partition (u, y), as a first-order PV difference equation system.

Definition 14 (DT-LPV-SS Representation): The discrete-time state-space representation of $\mathcal{S} = (\mathbb{Z}, \mathbb{P} \subseteq \mathbb{R}^{n_\mathbb{P}}, \mathbb{R}^{n_\mathbb{U}+n_\mathbb{Y}}, \mathfrak{B})$, with scheduling vector p is denoted by $\mathfrak{R}_{\mathrm{SS}}(\mathcal{S})$ and defined as a first-order parameter-varying difference equation system in the latent variable $x : \mathbb{Z} \to \mathbb{X}$

$$qx = (A \diamond p)x + (B \diamond p)u \tag{26a}$$
$$y = (C \diamond p)x + (D \diamond p)u \tag{26b}$$

where (u, y) is the IO partition of $\mathcal{S}$, x is the state vector, $\mathbb{X} = \mathbb{R}^{n_\mathbb{X}}$ is the state space

$$\mathfrak{B}_{\mathrm{SS}} = \{(u, x, y, p) \in (\mathbb{U} \times \mathbb{X} \times \mathbb{Y} \times \mathbb{P})^\mathbb{Z} \mid \text{(26a)–(26b) hold}\}$$

is the full behavior of (26a) and (26b), $\mathfrak{B}$ is equal to the manifest behavior of (26a) and (26b), i.e., $\mathfrak{B} = \pi_{u,y,p}\mathfrak{B}_{\mathrm{SS}}$, and

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \left[\begin{array}{c|c} \mathcal{R}^{n_\mathbb{X} \times n_\mathbb{X}} & \mathcal{R}^{n_\mathbb{X} \times n_\mathbb{U}} \\ \hline \mathcal{R}^{n_\mathbb{Y} \times n_\mathbb{X}} & \mathcal{R}^{n_\mathbb{Y} \times n_\mathbb{U}} \end{array}\right].$$

□

Note that in $\mathfrak{B}_{\mathrm{SS}}$, the latent variable x trivially fulfills the state property. It is apparent that (26a) and (26b) are the "dynamic-dependent" counterparts of (1a) and (1b).

Example 10 (SS Representation): Continuing Example 9, the LPV-SS representation of the model follows by taking $[y \quad u]^\top = [w_x \quad w_F]^\top$ as the IO partition and $x = w_L$ as the state

$$qx = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} \mathrm{T}_d^2 \mathrm{k}_s + p & -\mathrm{T}_d^2 \\ -p - q^{-1}p & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}$$
$$y = \begin{bmatrix} 0 & \frac{1}{-q^{-1}p} \end{bmatrix} x.$$

By substitution of the second equation into the first one, the state equation in the form of (26a) results, while the second equation gives the output equation in the form of (26b). Thus, the corresponding SS representation is

$$\left[\begin{array}{c|c} A \diamond p & B \diamond p \\ \hline C \diamond p & D \diamond p \end{array}\right] = \left[\begin{array}{cc|c} 0 & -\frac{p+\mathrm{T}_d^2\mathrm{k}_s}{q^{-1}p} & -\mathrm{T}_d^2 \\ 1 & 1 + \frac{p}{q^{-1}p} & 0 \\ \hline 0 & \frac{-1}{q^{-1}p} & 0 \end{array}\right].$$

□

V. Equivalence Relations

Using the behavioral framework, it is possible to consider equivalence of kernel representations, IO representations and state-space forms via equality of the represented behaviors.

A. Equivalent Kernel Forms

In the LTI case, two DT kernel representations are equivalent, i.e., they define the same system, if their associated behaviors are equal. Similar to the LTI framework, $R_1, R_2 \in \mathcal{R}[\xi]$ are expected to define an equal behavior if they are equivalent up to multiplication by a $r \in \mathcal{R}$, $r \neq 0$. However, r can be a rational function for which $(r \diamond p)(k) = \infty$ for some $p \in \mathbb{P}^\mathbb{T}$ and $k \in \mathbb{Z}$. The associated behavior of a kernel representation in terms of (18) is defined to contain only those trajectories of p for which a solution exists. The latter is guaranteed by the boundedness of $r \diamond p$. In this way, the behavior of R_1 is equal to the behavior of $R_2(\xi) = rR_1(\xi)$ except for those trajectories for which $r \diamond p$ is unbounded.

To consider equality of LPV-KR representations with this phenomenon of singularity in mind, we define the restriction of $\mathfrak{B}$ to $\bar{\mathfrak{B}}_\mathbb{P} \subseteq \mathfrak{B}_\mathbb{P}$ as

$$\mathfrak{B}|_{\bar{\mathfrak{B}}_\mathbb{P}} = \left\{(w, p) \in \mathfrak{B} \mid p \in \bar{\mathfrak{B}}_\mathbb{P}\right\}. \tag{27}$$

The equivalence of LPV-KR representations can now be introduced in an *almost everywhere* sense.

Definition 15 (Equivalent KR Representations): Two kernel representations with polynomials $R, R' \in \mathcal{R}[\xi]^{\cdot \times n_\mathbb{W}}$, $\mathbb{P} = \mathbb{R}^{n_\mathbb{P}}$ and behaviors $\mathfrak{B}, \mathfrak{B}' \subseteq (\mathbb{R}^{n_\mathbb{W}} \times \mathbb{R}^{n_\mathbb{P}})^\mathbb{Z}$ are called equivalent if $\mathfrak{B}|_{\mathfrak{B}_\mathbb{P} \cap \mathfrak{B}'_\mathbb{P}} = \mathfrak{B}'|_{\mathfrak{B}_\mathbb{P} \cap \mathfrak{B}'_\mathbb{P}}$, i.e., their behaviors are equal for all mutually valid trajectories of p. □

Example 11 (Almost Everywhere Equivalence): By continuing Example 5

$$\left(\frac{\mathrm{T}_d^2\mathrm{k}_s}{p} + 1\right) w_1 - \left(\frac{qp}{p} + 1\right) q w_1 + \left(\frac{qp}{p}\right) q^2 w_1 - \frac{\mathrm{T}_d^2}{p} w_2 = 0$$

has the same solutions as (15) except for those trajectories of $p = m$, where $m(k) = 0$ for some $k \in \mathbb{Z}$. Thus, this KR representation and (15) are equivalent in the almost everywhere sense. □

To characterize equivalence algebraically, we introduce unimodular transformations just as in the LTI case [24].

Theorem 5 (Unimodular Transformation): Consider $R \in \mathcal{R}[\xi]^{n_r \times n_\mathbb{W}}$ and $M' \in \mathcal{R}[\xi]^{n_r \times n_r}$, $M'' \in \mathcal{R}[\xi]^{n_\mathbb{W} \times n_\mathbb{W}}$ with M', M'' unimodular. For a given $n_\mathbb{P}$, define $R'(\xi) = M'(\xi)R(\xi)$ and $R''(\xi) = R(\xi)M''(\xi)$. Denote the behaviors corresponding to R, R' and R'' by $\mathfrak{B}$, $\mathfrak{B}'$ and $\mathfrak{B}''$ with scheduling space $\mathbb{P} \subseteq \mathbb{R}^{n_\mathbb{P}}$ and signal space $\mathbb{W} = \mathbb{R}^{n_\mathbb{W}}$. Then, $\mathfrak{B}|_{\mathfrak{B}_\mathbb{P} \cap \mathfrak{B}'_\mathbb{P}} = \mathfrak{B}'|_{\mathfrak{B}_\mathbb{P} \cap \mathfrak{B}'_\mathbb{P}}$ while $\mathfrak{B}|_{\mathfrak{B}_\mathbb{P} \cap \mathfrak{B}''_\mathbb{P}}$ and $\mathfrak{B}''|_{\mathfrak{B}_\mathbb{P} \cap \mathfrak{B}''_\mathbb{P}}$ are isomorphic. □

The proof of this theorem is given in the Appendix. Furthermore, if $R \in \mathcal{R}[\xi]^{n_r \times n_\mathbb{W}}$ is not full row rank, i.e., $\mathrm{rank}(R) = n < n_r$, then there exists a unimodular $M \in \mathcal{R}[\xi]^{n_r \times n_r}$ such that $M(\xi)R(\xi) = [(R'(\xi))^\top \quad 0]^\top$, where $R' \in \mathcal{R}[\xi]^{n \times n_\mathbb{W}}$ is full row rank and the corresponding behaviors are equivalent in terms of Theorem 5.

Definition 16 (Equivalence Relation): Introduce the symbol $\overset{n_\mathbb{P}}{\sim}$ to denote the equivalence relation on $\bigcup \mathcal{R}[\xi]^{\cdot \times \cdot}$ (all polynomial matrices with finite dimension) for an $n_\mathbb{P}$-dimensional scheduling space. $R_1 \in \mathcal{R}[\xi]^{n_1 \times n_\mathbb{W}}$ and $R_2 \in \mathcal{R}[\xi]^{n_2 \times n_\mathbb{W}}$ with

$i = \arg\max_{i\in\{1,2\}}(n_i)$ and $j = \{1,2\}\setminus i$ are called equivalent, i.e., $R_1 \overset{n_{\mathbb{P}}}{\approx} R_2$, if there exists a unimodular matrix function $M \in \mathcal{R}[\xi]^{n_i\times n_i}$ such that

$$M(\xi)R_i(\xi) = \begin{bmatrix} R_j(\xi) \\ 0 \end{bmatrix} \begin{matrix} \updownarrow & n_j \\ \updownarrow & n_i - n_j \end{matrix}. \tag{28}$$

This implies that if $R_1 \overset{n_{\mathbb{P}}}{\approx} R_2$, then the corresponding behaviors with $\mathbb{P} \subseteq \mathbb{R}^{n_{\mathbb{P}}}$ and $\mathbb{W} = \mathbb{R}^{n_{\mathbb{W}}}$ are equal (almost everywhere). Using $\overset{n_{\mathbb{P}}}{\approx}$ we can define equivalence classes as follows.

Definition 17 (Equivalence Class): For a given $n_{\mathbb{P}}$, the set $\mathcal{E}^{n_{\mathbb{P}}} \subseteq \bigcup \mathcal{R}[\xi]^{\cdot\times\cdot}$ is called an equivalence class if it is a maximal subset of $\bigcup \mathcal{R}[\xi]^{\cdot\times\cdot}$ such that for all $R_1, R_2 \in \mathcal{E}^{n_{\mathbb{P}}}$ it holds that $R_1 \overset{n_{\mathbb{P}}}{\approx} R_2$. □

An equivalence class defines the set of all KR representations that have equal behavior. Furthermore, it is an obvious consequence, that all R in a given $\mathcal{E}^{n_{\mathbb{P}}}$ have the same Jacobson form. An important subset of an equivalence class contains the so-called minimal representations.

Definition 18 (Minimality): Let $R \in \mathcal{R}[\xi]^{n_r\times n_{\mathbb{W}}}$. Then, R is called minimal if it has full row rank, i.e., $\operatorname{rank}(R) = n_r$. □

Consider a minimal $\mathfrak{R}_{\mathrm{K}}(\mathcal{S})$ described by a full row rank $R \in \mathcal{R}[\xi]^{n_r\times n_{\mathbb{W}}}$. Let $R(\xi) = [\,R'(\xi) \quad R''(\xi)\,]$ where $R' \in \mathcal{R}[\xi]^{n_r\times n_r}$ has full column rank. Note that such form can always be obtained by the permutation of the signal variables and it is not unique. Consider $n_{\deg} = \deg(r'_n)$ where r'_n results from the Jacobson form (see Theorem 1) of R'. Assume that R' is chosen w.r.t. R such that $n_{\deg}$ is maximal. It follows from Theorem 5 that all KR representations in the equivalence class of $\mathfrak{R}_{\mathrm{K}}(\mathcal{S})$ have the same $n_{\deg}$, hence $n_{\deg}$ can be called the degree of these representations. It can be also shown that this degree is equal to the required minimal number of state variables in a SS realization of $\mathfrak{R}_{\mathrm{K}}(\mathcal{S})$, hence $n_{\deg}$ can be considered as the order, i.e., *McMillan degree* of $\mathcal{S}$.

Example 12 (LPV Equivalence Relation and Minimality): Let the KR representation $\mathfrak{R}_{\mathrm{K}}(\mathcal{S})$ of an DT-LPV system $\mathcal{S}$ with $\mathbb{P} \subseteq \mathbb{R}$ be given by

$$R(\xi)\diamond p = \begin{bmatrix} qp & -qp \\ p & -p \end{bmatrix} + \begin{bmatrix} 0 & p(qp) \\ p^2 & 0 \end{bmatrix}\xi + \begin{bmatrix} -p(qp^2) & 0 \\ 0 & 0 \end{bmatrix}\xi^2.$$

Then, there exists a unimodular matrix $M \in \mathcal{R}[\xi]^{2\times 2}$

$$M(\xi)\diamond p = \begin{bmatrix} 0 & 1 \\ 1 & p\xi - \frac{qp}{p} \end{bmatrix}$$

such that

$$(M(\xi)R(\xi))\diamond p = \begin{bmatrix} p + p^2\xi & -p \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R'(\xi) \\ 0 \end{bmatrix}.$$

From Theorem 5, it follows that $R \overset{1}{\sim} R'$. Furthermore, $\operatorname{rank}(R') = 1$ implies that $\operatorname{rank}(R) = 1$, hence R' is minimal while R is not. By computing $n_{\deg}$ of R', the McMillan degree of $\mathcal{S}$ is 1. □

B. Equivalent IO Forms

The introduced equivalence concept generalizes to LPV-IO representations:

Definition 19 (Equivalence Relation, LPV-IO): Let (R_y, R_u) and (R'_y, R'_u) be LPV-IO representations with the same input and output dimensions $(n_{\mathbb{Y}}, n_{\mathbb{U}})$. For a given scheduling dimension $n_{\mathbb{P}}$, we call (R_y, R_u) and (R'_y, R'_u) equivalent, i.e., $(R_y, R_u) \overset{n_{\mathbb{P}}}{\approx} (R'_y, R'_u)$, if there exists a unimodular matrix $M \in \mathcal{R}[\xi]^{n_{\mathbb{Y}}\times n_{\mathbb{Y}}}$ such that

$$R'_y(\xi) = M(\xi)R_y(\xi) \quad \text{and} \quad R'_u(\xi) = M(\xi)R_u(\xi). \tag{29}$$

□

This implies the following minimality concept of LPV-IO representations.

Definition 20 (Minimal LPV-IO Representation): An IO representation defined through $R_y \in \mathcal{R}[\xi]^{n_{\mathbb{Y}}\times n_{\mathbb{Y}}}$ and $R_u \in \mathcal{R}[\xi]^{n_{\mathbb{Y}}\times n_{\mathbb{U}}}$ is called minimal for a given scheduling dimension $n_{\mathbb{P}}$, if there are no polynomials $R'_y \in \mathcal{R}[\xi]^{n_{\mathbb{Y}}\times n_{\mathbb{Y}}}$ and $R'_u \in \mathcal{R}[\xi]^{n_{\mathbb{Y}}\times n_{\mathbb{U}}}$ with $\deg(R_y) < \deg(R'_y)$ such that

$$(R_y, R_u) \overset{n_{\mathbb{P}}}{\approx} (R'_y, R'_u). \tag{30}$$

□

Using the IO equivalence relation and minimality, the definition of IO equivalence classes follows naturally.

Example 13 (LPV-IO Equivalence and Minimality): Let the IO representation $\mathfrak{R}_{\mathrm{IO}}(\mathcal{S})$ of an DT-LPV system $\mathcal{S}$ with $\mathbb{P} \subseteq \mathbb{R}$ be given by

$$R_y(\xi)\diamond p = \begin{bmatrix} p\xi & p^2 \\ p\xi^2 & p(qp)\xi \end{bmatrix} \quad R_u(\xi)\diamond p = \begin{bmatrix} p \\ p(\xi - 1) \end{bmatrix}.$$

Consider the unimodular matrix $M \in \mathcal{R}[\xi]^{2\times 2}$ given by

$$M(\xi)\diamond p = \begin{bmatrix} \frac{1}{p} & 0 \\ \frac{p}{qp}\xi & -1 \end{bmatrix}$$

then

$$(M(\xi)R_y(\xi))\diamond p = \begin{bmatrix} \xi & p \\ 0 & \xi \end{bmatrix} \quad (M(\xi)R_u(\xi))\diamond p = \begin{bmatrix} 1 \\ p \end{bmatrix}.$$

This implies that $(R'_y, R'_u) = (MR_y, MR_u)$ and (R_y, R_u) are equivalent for $n_{\mathbb{P}} = 1$ in terms of Theorem 5. From Definition 20, it follows that $\mathfrak{R}_{\mathrm{IO}}(\mathcal{S})$ is not minimal as $\deg(R_y) = 2$ is larger than $\deg(R'_y) = 1$. On the other hand, it is trivial that (R'_y, R'_u) defines a minimal IO representation of $\mathcal{S}$. By computing the Jacobson form of R'_y, the McMillan degree of $\mathcal{S}$ is 1. □

C. Equivalent State-Space Forms

We can also generalize the equivalence concept to LPV-SS representations. To do so, we first have to clarify state transformations in the LPV case.

By definition, the full behavior of LPV-SS representation is represented by a matrix $R_w \in \mathcal{R}^{n_r\times(n_{\mathbb{Y}}+n_{\mathbb{U}})}$ and a first-order polynomial $R_L \in \mathcal{R}[\xi]^{n_r\times n_{\mathbb{X}}}$ in the form

$$(R_w \diamond p)\operatorname{col}(u, y) = (R_L(q)\diamond p)x. \tag{31}$$

Similar to the LTI case, left and right side multiplication of R_w and R_L with unimodular $M_1 \in \mathcal{R}[\xi]^{n_r\times n_r}$ and $M_2 \in \mathcal{R}[\xi]^{n_{\mathbb{X}}\times n_{\mathbb{X}}}$ leads to $R'_w(\xi) = M_1(\xi)R_w$, $R'_L(\xi) = M_1(\xi)R_L(\xi)M_2(\xi)$. In terms of Theorem 5, the resulting polynomials R'_w and R'_L define an equivalent latent variable representation of $\mathcal{S}$, where the new latent variable is given as $x' = \left(M_2^{\dagger}(q) \diamond p\right) x$. To guarantee that the resulting latent variable representation qualifies as a SS representation, R'_L needs to be monic and $\deg(R'_L) = 1$ with $\deg(R'_w) = 0$ must be satisfied. This implies that the unimodular matrices must have zero order, i.e., $M_1 \in \mathcal{R}^{n_r \times n_r}$ and $M_2 \in \mathcal{R}^{n_\mathbb{X} \times n_\mathbb{X}}$, and M_1 must have a special structure in order to guarantee that R'_w and R'_L correspond to an equivalent SS representation. In that case, $x' = \left(M_2^{\dagger}(q) \diamond p\right) x$ is called a *state transformation* and $T = M_2^{\dagger}$ is called the *state transformation matrix* resulting in

$$x' = (T \diamond p)x. \tag{32}$$

A major difference w.r.t. LTI state transformations is that, in the LPV case, T is inherently dependent on p and this dependence is dynamic, i.e., $T \in \mathcal{R}^{n_\mathbb{X} \times n_\mathbb{X}}$. Additionally, it can be shown that an invertible $T \in \mathcal{R}^{n_\mathbb{X} \times n_\mathbb{X}}$ used as a state transformation is always equivalent with a right- and left-side multiplication by unimodular matrix functions yielding a valid SS representation of the LPV system. Based on this, two SS representations are equivalent if and only if their states can be related via an invertible state transformation (32).

Consider an LPV-SS representation (26a) and (26b). Let $T \in \mathcal{R}^{n_\mathbb{X} \times n_\mathbb{X}}$ be an invertible matrix function and consider x', given by (32), as a new state variable. Substitution of (32) into (26a) gives

$$q(T^{-1} \diamond p)x' = (A \diamond p)(T^{-1} \diamond p)x' + (B \diamond p)u. \tag{33}$$

Using that $qT^{-1} = \overrightarrow{(T^{-1})}q = \overrightarrow{T}^{-1}q$, (33) yields that the equivalent LPV-SS representation is

$$\left[\begin{array}{c|c} \overrightarrow{T}AT^{-1} & \overrightarrow{T}B \\ \hline CT^{-1} & D \end{array}\right]. \tag{34}$$

Definition 21 (Equivalence Relation, LPV-SS): Consider two LPV-SS representations with state-space matrices (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) in $\mathcal{R}^{\cdot\times\cdot}$ where $A_1 \in \mathcal{R}^{n_1 \times n_1}$ and $A_2 \in \mathcal{R}^{n_2 \times n_2}$ and $n_1 \geq n_2$. For a given scheduling dimension $n_\mathbb{P}$, these representations are called equivalent

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \overset{n_\mathbb{P}}{\approx} \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] \tag{35}$$

if there exists an invertible $T \in \mathcal{R}^{n_1 \times n_1}$ such that

$$\begin{aligned} \overrightarrow{T}A_1T^{-1} &= \begin{bmatrix} A_2 & 0 \\ * & * \end{bmatrix} \\ \overrightarrow{T}B_1 &= \begin{bmatrix} B_2 \\ * \end{bmatrix} \begin{matrix} \updownarrow n_2 \\ \updownarrow n_1 - n_2 \end{matrix} \\ C_1T^{-1} &= [\,C_2 \quad 0\,] \\ D_1 &= D_2. \end{aligned}$$

□

From the concept of LPV-SS equivalence, the concept of minimality directly follows.

Definition 22 (Minimal LPV-SS Representation): For a given $n_\mathbb{P}$, an SS representation, defined through the matrix functions (A, B, C, D), is called minimal if there exist no (A', B', C', D') with $n'_\mathbb{X} < n_\mathbb{X}$ such that

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \overset{n_\mathbb{P}}{\approx} \left[\begin{array}{c|c} A' & B' \\ \hline C' & D' \end{array}\right].$$

□

Again, using the concept of the SS equivalence relation and minimality, the definition of LPV-SS equivalence classes follows naturally. In addition, the state dimension $n_\mathbb{X}$ of a minimal $\mathfrak{R}_{\mathrm{SS}}(\mathcal{S})$ is equal to the McMillan degree of $\mathcal{S}$.

Example 14 (LPV-SS Equivalence and Minimality): Consider the LPV-SS representation derived in Example 10. Let $T \in \mathcal{R}^{2\times 2}$ be an invertible state transformation defined by

$$T \diamond p = \begin{bmatrix} -1 & -1 \\ 0 & -\frac{1}{q^{-1}p} \end{bmatrix}$$
$$\text{with} \quad T^{-1} \diamond p = \begin{bmatrix} -1 & q^{-1}p \\ 0 & -q^{-1}p \end{bmatrix} \quad \overrightarrow{T} \diamond p = \begin{bmatrix} -1 & -1 \\ 0 & -\frac{1}{p} \end{bmatrix}$$

giving

$$\left[\begin{array}{c|c} \overrightarrow{T}AT^{-1} & \overrightarrow{T}B \\ \hline CT^{-1} & D \end{array}\right] \diamond p = \left[\begin{array}{cc|c} 1 & -\mathtt{T}_d^2\mathtt{k}_s & \mathtt{T}_d^2 \\ \frac{1}{p} & 1 & 0 \\ \hline 0 & 1 & 0 \end{array}\right].$$

The obtained SS representation is an equivalent minimal SS representation of $\mathcal{S}$ as it is in an equivalence relation with $\mathfrak{R}_{\mathrm{SS}}(\mathcal{S})$ and its state dimension is the same. Note that this realization has only static dependence. □

Based on the developed state transformations and the concepts of state-observability and -reachability matrices, the classical canonical forms can also be defined (see [17] and [20]). Furthermore, Definition 21 highlights that applying p-dependent state transformation or system transposition according to the rules of the LTI theory deforms the dynamic relation. This "common practice" leads to inequivalent system representations with arbitrary large difference in terms of manifest behavior (see [17] and [20] for illustrative examples).

VI. Equivalence Transformations

Next, we introduce equivalence transformations between the SS and IO representation domains. These provide algorithms to obtain an IO (SS) realization of a given LPV-SS (IO) representation, solving the core problem of the existing LPV system theory, motivated in Example 1.

A. State Space to IO

As a consequence of Theorem 3, the following corollary holds.

Corollary 1 (Latent Variable Elimination): For any latent variable representation (31) with manifest behavior $\mathfrak{B}$ and polynomial matrices $R_w \in \mathcal{R}[\xi]^{n_r \times n_\mathbb{W}}$ and $R_L \in \mathcal{R}[\xi]^{n_r \times n_L}$, there exists a unimodular matrix $M \in \mathcal{R}[\xi]^{n_r \times n_r}$ such that

$$M(\xi)R_w(\xi) = \begin{bmatrix} R'_w(\xi) \\ R''_w(\xi) \end{bmatrix} \quad M(\xi)R_L(\xi) = \begin{bmatrix} R'_L(\xi) \\ 0 \end{bmatrix} \tag{36}$$

with R'_L of full row rank. The behavior defined by $(R''_w(\xi) \diamond p)\, w = 0$ is equal (almost everywhere) with $\mathfrak{B}$. □

Due to the latent nature of the variable w_L, such a transformation is always possible and does not change the manifest behavior, hence it is called an *equivalence transformation*. We can use this result to establish an IO realization of a given SS representation (26a) and (26b) by writing it in the latent form

$$R_w(q) = \begin{bmatrix} 0 & B \\ -I & D \end{bmatrix} \quad R_L(q) = \begin{bmatrix} Iq - A \\ -C \end{bmatrix}$$

with $w = \mathrm{col}(u, y)$, $w_L = x$, $R_w \in \mathcal{R}[\xi]^{(n_\mathbb{X}+n_\mathbb{Y})\times(n_\mathbb{X}+n_\mathbb{U})}$, and $R_L \in \mathcal{R}[\xi]^{(n_\mathbb{X}+n_\mathbb{Y})\times n_\mathbb{X}}$. According to Corollary 1, there exists a unimodular matrix

$$M(\xi) = \begin{bmatrix} M_{11}(\xi) & M_{12}(\xi) \\ M_{21}(\xi) & M_{22}(\xi) \end{bmatrix} \in \mathcal{R}[\xi]^{(n_\mathbb{X}+n_\mathbb{Y})\times(n_\mathbb{X}+n_\mathbb{Y})} \tag{37}$$

which in terms of $M(\xi)R_L(\xi) = [* \quad 0]^\top$ in (36) satisfies $M_{21}(\xi)(I\xi - A) - M_{22}(\xi)C = 0$. This yields that

$$\underbrace{\begin{bmatrix} * & * \\ -M_{21}(\xi) & M_{21}(\xi)B + M_{22}(\xi)D \end{bmatrix}}_{M(\xi)R_w(\xi)} = \underbrace{\begin{bmatrix} * \\ 0 \end{bmatrix}}_{M(\xi)R_L(\xi)}$$

and $R''_w(\xi) = [-M_{21}(\xi) \quad M_{21}(\xi)B + M_{22}(\xi)D]$ is in the form of an output side polynomial $R_y(\xi) = M_{21}(\xi)$ and an input side polynomial $R_u(\xi) = M_{21}(\xi)B + M_{22}(\xi)D$.

Corollary 2 (IO Equivalence Transformation): Let $\mathfrak{R}_{\mathrm{SS}}(\mathcal{S})$ be a state-space representation with manifest behavior $\mathfrak{B}$ and system matrices (A, B, C, D) where $A \in \mathcal{R}^{n_\mathbb{X}\times n_\mathbb{X}}$. Then, there exists a monic polynomial $\bar{R}_y \in \mathcal{R}[\xi]^{n_\mathbb{Y}\times n_\mathbb{Y}}$ with $\deg(\bar{R}_y) = n_\mathbb{X}$ and a $\bar{R}_u \in \mathcal{R}[\xi]^{n_\mathbb{Y}\times n_\mathbb{X}}$ with $\deg(\bar{R}_u) \leq n_\mathbb{X} - 1$ such that

$$\bar{R}_y(\xi)C = \bar{R}_u(\xi)(I\xi - A). \tag{38}$$

Let $R_c \in \mathcal{R}[\xi]^{n_\mathbb{Y}\times n_\mathbb{Y}}$ be the greatest common left-divisor of $\bar{R}_y$ and $\bar{R}_u B$ such that there exist $R_y, R_u \in \mathcal{R}[\xi]$ satisfying

$$R_c(\xi)R_y(\xi) = \bar{R}_y(\xi) \tag{39a}$$

$$R_c(\xi)R_u(\xi) = \bar{R}_u(\xi)B + \bar{R}_y(\xi)D. \tag{39b}$$

Then, the IO representation, given by $(R_y(q) \diamond p)y = (R_u(q) \diamond p)u$, defines a behavior equal to the manifest behavior of (26a) and (26b), thus it is an IO representation of $\mathcal{S}$. □

The algorithm defined by (38), (39a), and (39b) is structurally similar to the LTI case (see [32] and [33]), but it is more complicated as it involves multiplication with the time operators on the coefficients. Thus, this transformation can result in an increased complexity (like dynamic dependence) of the coefficient functions in the equivalent IO representation.

Example 15 (IO Equivalence Transformation): Consider the LPV-SS representation derived in Example 14. Let r be the identity function so $r \diamond p = p$. In terms of (38), we are looking for a $\bar{R}_u \in \mathcal{R}[\xi]^{1\times 2}$ with $\deg(\bar{R}_u) = 1$ and a monic polynomial $\bar{R}_y \in \mathcal{R}[\xi]$ with $\deg(\bar{R}_y) = 2$. Parameterize these polynomials as

$$\begin{aligned} \bar{R}_y(\xi) &= \xi^2 + a_1\xi + a_0 \\ \bar{R}_u(\xi) &= [\, b_{11}\xi + b_{12} \quad b_{21}\xi + b_{22} \,]. \end{aligned}$$

Then, in terms of (38)

$$\begin{aligned} &(\xi^2 + a_1\xi + a_0)\,[0 \quad 1] \\ &\quad = [\, b_{11}\xi + b_{12} \quad b_{21}\xi + b_{22} \,] \underbrace{\begin{bmatrix} \xi - 1 & \mathtt{T}_d^2\mathtt{k}_s \\ -\frac{1}{r} & \xi - 1 \end{bmatrix}}_{I\xi - A}. \end{aligned}$$

Solving this equation system, it follows that

$$\begin{aligned} a_0 &= \frac{\mathtt{T}_d^2\mathtt{k}_s + r}{\overrightarrow{r}} \\ a_1 &= -\frac{r}{\overrightarrow{r}} - 1 \\ b_{11} &= 0 \\ b_{12} &= \frac{1}{\overrightarrow{r}} \\ b_{21} &= 1 \\ b_{22} &= -\frac{r}{\overrightarrow{r}}. \end{aligned}$$

The resulting polynomials $\bar{R}_u$ and $\bar{R}_y$ are left coprime, hence

$$\begin{aligned} R_y(\xi) &= \bar{R}_y(\xi) = \xi^2 + a_1\xi + a_0 \\ R_u(\xi) &= \bar{R}_u(\xi)B + \bar{R}_y(\xi)D = \frac{\mathtt{T}_d^2}{\overrightarrow{r}}. \end{aligned}$$

After left-multiplying these polynomials with $\overrightarrow{r}$, the IO representation in the form of (20) with $n_a = 2$ and $n_b = 0$ has the coefficients

$$\begin{aligned} a_2 \diamond p &= qp \\ a_1 \diamond p &= -qp - p \\ a_0 \diamond p &= \mathtt{T}_d^2\mathtt{k}_s + p \\ b_0 \diamond p &= \mathtt{T}_d^2. \end{aligned}$$

In terms of $w = \mathrm{col}(y, u)$, the resulting LPV-IO representation is equal to (15), which shows its equivalence with the LPV-SS representation in Example 14. □

B. IO to State Space

Finding an equivalent SS representation of a given IO representation is accomplished by constructing a state mapping. This construction can be seen as the counterpart of the latent variable elimination. The aim is to introduce a latent variable into (19) such that it satisfies the state property, i.e., it defines a SS representation (Theorem 4). Similar to the LTI case (see [32] and [33]), the central idea of such a state construction is the *cut-and-shift-map* $\varrho_- : \mathcal{R}[\xi]^{\cdot\times\cdot} \to \mathcal{R}[\xi]^{\cdot\times\cdot}$ that acts on polynomial matrices as

$$\varrho_-\big(\underbrace{r_0 + r_1\xi + \cdots + r_n\xi^n}_{R(\xi)}\big) = \overleftarrow{r_1} + \cdots + \overleftarrow{r_n}\xi^{n-1}.$$

This operator can be seen as an intuitive way to introduce state variables for a kernel representation associated with R, as $w_L = \varrho_-(R(q)\diamond p)w$ implies that $(R(q)\diamond p)w = (r_0\diamond p)w + qw_L$. Repeated use of ϱ_- and stacking the resulting polynomial matrices gives

$$\underbrace{\begin{bmatrix} \varrho_-(R) \\ \varrho_-^2(R) \\ \vdots \\ \varrho_-^{n-2}(R) \\ \varrho_-^{n-1}(R) \end{bmatrix}}_{\Sigma_-(R)}(\xi) = \begin{bmatrix} r_1^{[1]} + \cdots + r_{n-1}^{[1]}\xi^{n-2} + r_n^{[1]}\xi^{n-1} \\ r_2^{[2]} + \cdots + r_{n-1}^{[2]}\xi^{n-3} + r_n^{[2]}\xi^{n-2} \\ \vdots \\ r_{n-1}^{[n-1]} + r_n^{[n-1]}\xi \\ r_n^{[n]} \end{bmatrix}.$$

where $r_i^{[j]}$ denotes the backward shift operation $\overleftarrow{\cdot}$ applied on r_i for j times. In case $R \in \mathcal{R}[\xi]^{n_r\times n_W}$ with $n_r = 1$, the rows of Σ_- are independent, thus it can be shown that $X = \Sigma_-(R)$ defines a minimal state map in the form of

$$x = (X(q)\diamond p)w. \tag{40}$$

In other cases (MIMO case), independent rows of $\Sigma_-(R)$ are selected to define a minimal X, but this selection is generally not unique. Later it is shown that a given state map implies a unique SS representation. Before that, we characterize all possible minimal state maps that lead to an equivalent SS representation.

Denote the left-side multiplication of $R(\xi)$ by ξ as ϱ_+ and introduce $\mathrm{module}_{\mathcal{R}[\xi]}(R)$ as the left module in $\mathcal{R}[\xi]^{n_r\times n_W}$ spanned by the rows of $R \in \mathcal{R}[\xi]^{n_r\times n_W}$, i.e.,

$$\mathrm{module}_{\mathcal{R}[\xi]}(R) = \mathrm{span}_{\mathcal{R}}^{\mathrm{row}}\left(\begin{bmatrix} R^\top & \varrho_+(R)^\top & \ldots \end{bmatrix}^\top\right).$$

This module represents the set of equivalence classes on $\mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(\Sigma_-(R))$. Let $X \in \mathcal{R}[\xi]^{\cdot\times n_W}$ be a polynomial matrix with independent rows (full row-rank) and such that

$$\begin{aligned} &\mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(X) \oplus \mathrm{module}_{\mathcal{R}[\xi]}(R) \\ &\quad = \mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(\Sigma_-(R)) + \mathrm{module}_{\mathcal{R}[\xi]}(R) \end{aligned} \tag{41}$$

where $\oplus$ denotes direct sum. Then, similar to the LTI case (see [32] and [33]), it is possible to show that X is a minimal state map of the LPV system $\mathcal{S}$, and it defines a state variable by (40) [20]. This way, it is possible to obtain all minimal, equivalent SS realizations of $\mathcal{S}$ which have a kernel representation associated with R.

The next step is to characterize these SS representations w.r.t. an IO partition. For a given kernel representation associated with the polynomial $R \in \mathcal{R}[\xi]^{n_r\times n_W}$, a valid input–output partition (u, y) of the representation is characterized by choosing a selector matrix $S_u \in \mathbb{R}^{\cdot\times n_W}$ giving $u = S_u w$ and a complementary matrix $S_y \in \mathbb{R}^{\cdot\times n_W}$ giving $y = S_y w$.

Assume that a full row rank $X \in \mathcal{R}[\xi]^{\cdot\times n_W}$ is given, which satisfies (41). Then, X and S_u jointly lead to

$$\begin{aligned} &\mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(\varrho_+(X)) \\ &\quad \subseteq \mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(X) \oplus \mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(S_u) \oplus \mathrm{module}_{\mathcal{R}[\xi]}(R). \end{aligned} \tag{42}$$

On the other hand, S_y gives

$$\begin{aligned} &\mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(S_y) \\ &\quad \subseteq \mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(X) \oplus \mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(S_u) \oplus \mathrm{module}_{\mathcal{R}[\xi]}(R). \end{aligned} \tag{43}$$

These inclusions imply that there exist unique matrix functions $\{A, B, C, D\}$ in $\mathcal{R}^{\cdot\times\cdot}$ and polynomial matrix functions X_u, $X_y \in \mathcal{R}[\xi]^{\cdot\times\cdot}$ with appropriate dimensions such that

$$\xi X(\xi) = AX(\xi) + BS_u + X_u(\xi)R(\xi) \tag{44a}$$

$$S_y = CX(\xi) + DS_u + X_y(\xi)R(\xi). \tag{44b}$$

Then, the resulting matrix function $\{A, B, C, D\}$ defines a minimal state representation of the LPV system $\mathcal{S}$. This algorithm provides an SS realization of both LPV-IO and LPV-KR representations. Specific choices of X leads to specific canonical forms. Note that a similar algorithm can be deduced for a realization in an image type of representation, i.e., latent variable representation (31), where $R_w(q) = I$.

Example 16 (SS Equivalence Transformation): Consider the LPV-IO representation derived in Example 15:

$$R_y(\xi) = \xi^2 - \left(1 + \frac{r}{\overrightarrow{r}}\right)\xi + \frac{\mathtt{T}_d^2\mathtt{k}_s + r}{\overrightarrow{r}} \quad R_u(\xi) = \frac{\mathtt{T}_d^2}{\overrightarrow{r}}.$$

Denote $R(\xi) = [\,R_y(\xi) \quad -R_u(\xi)\,]$, and generate the state map

$$X(\xi) = \Sigma_-(R(\xi)) = \begin{bmatrix} \xi - \left(1 + \frac{\overleftarrow{r}}{r}\right) & 0 \\ 1 & 0 \end{bmatrix}.$$

Now with $S_y = [1 \quad 0]$ and $S_u = [0 \quad 1]$, equations (44a) and (44b) read as

$$\underbrace{\begin{bmatrix} \xi^2 - \left(1 + \frac{r}{\overrightarrow{r}}\right)\xi & 0 \\ \xi & 0 \end{bmatrix}}_{\xi X(\xi)} = \underbrace{\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} \xi - \left(1 + \frac{\overleftarrow{r}}{r}\right) & 0 \\ 1 & 0 \end{bmatrix}}_{X(\xi)} + \underbrace{\begin{bmatrix} 0 & \beta_1 \\ 0 & \beta_2 \end{bmatrix}}_{BS_u} + \begin{bmatrix} X_{u1}(\xi) \\ X_{u2}(\xi) \end{bmatrix} R(\xi),$$

$$\underbrace{[1 \quad 0]}_{S_y} = \underbrace{[c_1 \quad c_2]}_{C} \cdot \underbrace{\begin{bmatrix} \xi - \left(1 + \frac{\overleftarrow{r}}{r}\right) & 0 \\ 1 & 0 \end{bmatrix}}_{X(\xi)} + \underbrace{[0 \quad d_1]}_{DS_u} + X_y(\xi)R(\xi).$$

By solving these equations, it follows that

$$\begin{aligned} \alpha_{11} &= 0 \\ \alpha_{12} &= -\frac{\mathtt{T}_d^2\mathtt{k}_s + r}{\overrightarrow{r}} \\ \alpha_{21} &= 1 \\ \alpha_{22} &= 1 + \frac{\overleftarrow{r}}{r} \end{aligned}$$

$$\begin{aligned}\beta_1 &= \frac{\mathtt{T}_d^2}{\overrightarrow{r}}\\ \beta_2 &= 0\\ c_1 &= 0\\ c_2 &= 1\\ d_1 &= 0\\ X_{u1}(\xi) &= 1\\ X_{u2}(\xi) &= 0\\ X_y(\xi) &= 0.\end{aligned}$$

Then, the obtained LPV-SS representation is

$$\mathfrak{R}_{\mathrm{SS}}(\mathcal{S}) = \left[\begin{array}{cc|c} 0 & -\frac{\mathtt{T}_d^2\mathtt{k}_s+p}{qp} & \frac{\mathtt{T}_d^2}{qp}\\ 1 & 1+\frac{q^{-1}p}{p} & 0\\ \hline 0 & 1 & 0\end{array}\right]$$

which through

$$T\diamond p = \begin{bmatrix} p & q^{-1}p\\ 0 & 1\end{bmatrix}$$

is in equivalence relation with the LPV-SS representation of Example 14. The latter proves that the IO representation given by R_y and R_u has the same manifest behavior as $\mathfrak{R}_{\mathrm{SS}}(\mathcal{S})$. □

VII. Conclusion

In this paper, we have extended the behavioral approach to LPV systems in order to lay the foundations of an LPV system theory that provides a clear understanding of this system class and the relations of its representations. We have defined LPV systems as the collection of signal and scheduling trajectories, and it has been shown that representations of these systems need dynamic dependence on the scheduling variable. By the use of such system descriptions, it has been proven that equivalence relations and transformations between these descriptions can be developed, giving a common ground where model structures of LPV system identification and concepts of LPV control can be compared, analyzed, and further developed.

Appendix

A. Proof of the Injective Cogenerator Property

The concept of the proof is based on [29]. Let $\mathbb{R}_\infty = \mathbb{R}\cup\{-\infty,\infty\}$ and denote by $\mathcal{Q}_n$ all maps w from $\mathbb{Z}\times\mathbb{R}^n$ to $\mathbb{R}_\infty$ which are essentially bounded w.r.t. $\mathbb{R}^n$, i.e., $\|\mathsf{w}(k,\mathtt{x})\| < \infty$ with $(k,\mathtt{x})\in\mathbb{Z}\times\mathbb{R}^n$ except for $\mathtt{x}\in\mathbb{S}(\mathsf{w})\subset\mathbb{R}^n$ where the set $\mathbb{S}(\mathsf{w})$ has measure 0. The set $\mathcal{Q}_n$ is a real vector space for each $n\in\mathbb{N}$. Denote $\bar{\mathcal{Q}}_n\subset\mathcal{Q}_n$ all $\mathsf{w}\in\mathcal{Q}_n$ for which there exist a $k\in\mathbb{Z}$ and $\mathtt{x}_1,\ldots\mathtt{x}_n\in\mathbb{R}$ such that $\mathsf{w}(k,\mathtt{x}_1,\ldots,\mathtt{x}_n)\neq\mathsf{w}(k,\mathtt{x}_1,\ldots,\mathtt{x}_{n-1},0)$. Denote $\mathcal{Q}=\bigcup_{n\in\mathbb{N}}\bar{\mathcal{Q}}_n$. $\mathcal{Q}$ is an (additive) Abelian group.

Consider a $R\in\mathcal{R}[\xi]^{n_r\times n_\mathrm{W}}$ with $\mathbb{P}=\mathbb{R}^{n_\mathbb{P}}$. For a $\mathsf{w}\in\mathcal{Q}$, $R\circledast\mathsf{w}=0$ means that any $(w,p)\in(\mathbb{R}_\infty^{n_\mathrm{W}}\times\mathbb{R}^{n_\mathbb{P}})^\mathbb{Z}$ satisfying

$$w(k) = \mathsf{w}(k,[\,p(k)\quad p(k+1)\quad p(k-1)\quad \ldots]) \tag{45}$$

for all $k\in\mathbb{Z}$, also satisfies $(R(q)\diamond p)(k)w(k)=0$ for all $k\in\mathbb{Z}\setminus\mathbb{J}(\mathsf{w},p)$, where $\mathbb{J}(\mathsf{w},p)=\{k\in\mathbb{Z}\mid [\,p(k)\quad p(k+1)\quad p(k-1)\quad\ldots]\in\mathbb{S}(\mathsf{w})\}$. As $\mathbb{S}(\mathsf{w})$ has zero measure, this means that there exists also a (bounded solution) $(w,p)\in(\mathbb{R}^{n_\mathrm{W}}\times\mathbb{R}^{n_\mathbb{P}})^\mathbb{Z}$ satisfying (45) such that $(R(q)\diamond p)(k)w(k)=0$ holds for all $k\in\mathbb{Z}$. The set $\mathfrak{B}_*$ given as $\mathfrak{B}_*=\{\mathsf{w}\in\mathcal{Q}^{n_\mathrm{W}}\mid R\circledast\mathsf{w}=0\}$, is called the complete solution space of the linear system of PV difference equations (KR-representation) $(R(q)\diamond p)w=0$. Note that the behavior $\mathfrak{B}$ of R defined by (18), contains the set of trajectories (w,p) that satisfy $\mathsf{w}\in\mathfrak{B}_*$ and are bounded, while $\mathfrak{B}_*$ describes the relationship of the trajectories containing the descriptions of possible solutions that are excluded from $\mathfrak{B}$ due to the singularity of the coefficients in R.

Let $M_1\in\mathcal{R}[\xi]^{n_r\times n_r}$ and $M_2\in\mathcal{R}[\xi]^{n_\mathrm{W}\times n_\mathrm{W}}$ be unimodular matrices such that (17) is the Jacobson form of R with $Q=\mathrm{diag}(r_1,\ldots,r_n)\in\mathcal{R}[\xi]^{n\times n}$. It can be shown (see [27]), that $(R(q)\diamond p)w=0$ has the same solutions as

$$(M_1(q)R(q)\diamond p)w = \left(Q(q)M_2^\dagger(q)\diamond p\right)w = 0 \tag{46}$$

so there is an isomorphism of solution spaces

$$\mathfrak{B}_*\cong\tilde{\mathfrak{B}}_* := \{\tilde{\mathsf{w}}\in\mathcal{Q}^{n_\mathrm{W}}\mid [\,Q\quad 0\,]\circledast\tilde{\mathsf{w}}=0\}, \tag{47a}$$
$$\mathsf{w}\rightarrow\tilde{\mathsf{w}} := M_2^\dagger(q)\mathsf{w} \tag{47b}$$

where $r_i\circledast\tilde{\mathsf{w}}_i=0$ for $i\in\{1,\ldots,n\}$. Introduce $\mathcal{M}_R=\mathrm{module}_{\mathcal{R}[\xi]}(R)$ as the left module in $\mathcal{R}[\xi]^{n_r\times n_\mathrm{W}}$ generated by the rows of $R\in\mathcal{R}[\xi]^{n_r\times n_\mathrm{W}}$. Then

$$\mathfrak{B}_*\cong\hom_{\mathcal{R}[\xi]}(\mathcal{M}_R,\mathcal{Q}^{n_\mathrm{W}}) \tag{48}$$

which corresponds to the so-called Malgrange isomorphism. Explicitly, (48) assigns to each $\mathsf{w}\in\mathfrak{B}_*$ the linear map $\phi_\mathsf{w}:\mathcal{M}_R\rightarrow\mathcal{Q}$ defined by $\phi_\mathsf{w}([r]):=r(q)\mathsf{w}$ where $[r]$ denotes the residue class of $r\in\mathcal{R}[\xi]^{1\times n_\mathrm{W}}$ in $\mathcal{M}_R$, and the well definedness of ϕ_w follows from

$$[r_1]=[r_2]\rightarrow r_1-r_2\in\mathrm{span}_\mathcal{R}^{\mathrm{row}}(R)\rightarrow r_1(q)\mathsf{w}=r_2(q)\mathsf{w}$$

for all $\mathsf{w}\in\mathfrak{B}_*$ which also implies that $\mathcal{Q}^{n_\mathrm{W}}\cong\hom_{\mathcal{R}[\xi]}(\mathcal{R}[\xi]^{1\times n_\mathrm{W}},\mathcal{Q}^{n_\mathrm{W}})$. Conversely, for a linear map $\phi:\mathcal{M}_R\rightarrow\mathcal{Q}$ one defines $\mathsf{w}_i:=\phi([e_i])$, where e_i is the i-th natural basis vector of $\mathcal{R}[\xi]^{1\times n_\mathrm{W}}$. Then, we have

$$\begin{aligned}\phi([r]) &= \phi\left(\left[\sum_{i=1}^{n_\mathrm{W}} r_ie_i\right]\right) = \sum_{i=1}^{n_\mathrm{W}} r_i(q)\phi([e_i])\\ &= \sum_{i=1}^{n_\mathrm{W}} r_i(q)\mathsf{w}_i = r(q)\mathsf{w}.\end{aligned}$$

Due to (45), the above equation implies an isomorphism of left modules

$$\mathrm{module}_{\mathcal{R}[\xi]}(R)\cong\mathrm{module}_{\mathcal{R}[\xi]}([\,Q\quad 0\,]) \tag{49a}$$
$$[r]\rightarrow[rM_2]. \tag{49b}$$

Let $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ be left modules in $\mathcal{R}[\xi]^{n_r \times n_W}$ and let $\phi_{12} : \mathcal{M}_1 \to \mathcal{M}_2$ and $\phi_{23} : \mathcal{M}_2 \to \mathcal{M}_3$ be linear maps, i.e., left module homomorphisms. Then

$$\mathcal{M}_1 \overset{\phi_{12}}{\to} \mathcal{M}_2 \overset{\phi_{23}}{\to} \mathcal{M}_3 \tag{50}$$

is exact if $\operatorname{im}(\phi_{12}) = \ker(\phi_{23})$. The same notion can be used if $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ are Abelian groups and ϕ_{12}, ϕ_{23} are group homomorphisms. Then, $\mathcal{Q}$ is called an injective cogenerator if the sequence

$$\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \tag{51}$$

is exact iff the sequence

$$\hom_{\mathcal{R}[\xi]}(\mathcal{M}_1, \mathcal{Q}^{n_W}) \leftarrow \hom_{\mathcal{R}[\xi]}(\mathcal{M}_2, \mathcal{Q}^{n_W}) \leftarrow \hom_{\mathcal{R}[\xi]}(\mathcal{M}_3, \mathcal{Q}^{n_W})$$

of Abelian groups is exact.

For injectivity, one needs to prove according to [28, Corollary 3.17]: For every $0 \neq R \in \mathcal{R}[\xi]$ and every $\mathsf{w}_\mathrm{u} \in \mathcal{Q}$, there exists a $\mathsf{w}_\mathrm{y} \in \mathcal{Q}$ such that $R \circledast \mathsf{w}_\mathrm{y} = \mathsf{w}_\mathrm{u}$. Let $R(\xi) = \sum_{i=0}^{n_\xi} r_i \xi^i$ be given with $r_{n_\xi} \neq 0$. If $n_\xi = 0$, there is nothing to prove. Since $\mathcal{R}$ is a field, assume that $r_{n_\xi} = 1$. Then, $R \circledast \mathsf{w}_\mathrm{y} = \mathsf{w}_\mathrm{u}$ can be rewritten as a first-order system

$$(r_{\mathrm{x},1} q + r_{\mathrm{x},0}) \circledast \mathsf{w}_\mathrm{x} = r_\mathrm{u} \circledast \mathsf{w}_\mathrm{u} \tag{52}$$

where $\mathsf{w}_\mathrm{x} = [\mathsf{w}_\mathrm{y} \;\dots\; q^{-(n_\xi - 1)}\mathsf{w}_\mathrm{y}]^\top$, $r_{\mathrm{x},1} = I$, $r_\mathrm{u} = [0 \;\dots\; 0 \;\; 1]^\top \in \mathbb{R}^{n_\xi}$ and

$$r_{\mathrm{x},0} = \begin{bmatrix} 0 & -I \\ r_0 & r_* \end{bmatrix} \in \mathcal{R}^{n_\xi \times n_\xi} \tag{53}$$

with $r_* = [r_1 \;\dots\; r_n]$. Let $\mathbb{S}(R)$ denote the set of singularities of the meromorphic coefficients r_i in R. Note that $\mathbb{S}(R)$ has measure 0. Let $\mathbb{S}(\mathsf{w}_\mathrm{y}) := \mathbb{S}(\mathsf{w}_\mathrm{u}) \cup \mathbb{S}(R)$ which has still zero measure. Hence, $\mathbb{R}^{\cdot} \setminus \mathbb{S}(\mathsf{w}_\mathrm{y})$ is a countable union of open intervals $I_i \in \mathbb{R}^{\cdot}$ and on each I_i it holds that R_0 and w_u are bounded. Therefore, there exists a bounded solution $\mathsf{w}_\mathrm{x} : (\mathbb{Z} \times I_i) \to \mathbb{R}^{n_\xi}$ to (52) on each I_i. By concatenating them, one gets a solution $\mathsf{w}_\mathrm{x} \in \mathcal{Q}^{n_\xi}$ and thus $\mathsf{w}_\mathrm{y} \in \mathcal{Q}$.

For the cogenerator property, it has to be shown that if for some $R \in \mathcal{R}[\xi]$, $R \circledast \mathsf{w}_\mathrm{y} = 0$ has only the zero solution, then this implies that $R \in \mathcal{R}$ and $R \neq 0$. Assume the contrary and let $\deg(R) = n_\xi \geq 1$. Then, one can rewrite $R \circledast \mathsf{w}_\mathrm{y} = 0$ as $q\mathsf{w}_\mathrm{x} = -r_{\mathrm{x},0} \circledast \mathsf{w}_\mathrm{x}$ like in the previous part. Let $\mathbb{S}(\mathsf{w}_\mathrm{y}) = \mathbb{S}(R)$, then on each of the intervals I_i, the solution set of this homogenous equation is an n_ξ-dimensional subspace of $(\mathbb{R}^{n_\xi})^{\mathbb{R} \times I_i}$, in particular there exist nonzero solutions. By concatenating them, we get a nonzero solution $\mathsf{w}_\mathrm{x} \in \mathcal{Q}^n$. If $\mathsf{w}_\mathrm{y} = \mathsf{w}_{\mathrm{x},1}$ was identically zero, then $\mathsf{w}_\mathrm{x} = [\mathsf{w}_\mathrm{y} \;\dots\; q^{-(n_\xi - 1)}\mathsf{w}_\mathrm{y}]^\top$ would be identically zero which leads to a contradiction. ■

B. Proof of Theorem 2

Consider $\mathfrak{R}_\mathrm{K}(\mathcal{S})$ with $R \in \mathcal{R}[\xi]^{n_r \times n_W}$, $\mathbb{P} = \mathbb{R}^{n_\mathbb{P}}$, and behavior $\mathfrak{B}$ in terms of (18). Without loss of generality, let $R \neq 0$ as the behavior $\mathfrak{B} = (\mathbb{R}^{n_W} \times \mathbb{R}^{n_\mathbb{P}})^\mathbb{Z}$ can be represented by the empty matrix which is full rank by convention. Let $M_1 \in \mathcal{R}[\xi]^{n_r \times n_r}$ and $M_2 \in \mathcal{R}[\xi]^{n_W \times n_W}$ be unimodular matrices such that (17) is the Jacobson form of R in terms of Theorem 1 with $Q = \operatorname{diag}(r_1, \dots, r_n) \in \mathcal{R}[\xi]^{n \times n}$. Partition $M_2^\dagger = [W_1 \;\; W_2]^\top$ according to the partition of the Jacobson form. Since M_1 is unimodular, the solution space of $(R(q) \diamond p)w = 0$ is equal to the solution space of $(M_1(q)R(q) \diamond p)w = 0$ (see the previous proof). Thus, $R'(\xi) := Q(\xi)W_1(\xi)$ also represents $\mathfrak{B}$ in an almost everywhere sense, i.e., for all trajectories of $p \in \mathfrak{B}_\mathbb{P}$ for which the coefficients of R' are bounded, and $\operatorname{rank}(R') = n$. ■

C. Proof of Theorem 3

Based on the proof of the injective cogenerator property (Appendix-A), consider

$$\mathfrak{B}_* = \{\mathsf{w} \in \mathcal{Q}^{n_W} \mid \exists \mathsf{w}_L \in \mathcal{Q}^{n_L} : R_w \circledast \mathsf{w} = R_L \circledast \mathsf{w}_L\} \tag{54}$$

where $R_w \in \mathcal{R}[\xi]^{n_r \times n_W}$ and $R_L \in \mathcal{R}[\xi]^{n_r \times n_L}$ defines an LPV latent variable representation in the form of (21) with $\mathbb{P} = \mathbb{R}^{n_\mathbb{P}}$. Then, showing that $\mathfrak{B}_*$ has a kernel representation is equivalent with showing that the manifest behavior of (21) has a kernel representation in an almost everywhere sense. Define the left kernel of R_L as

$$\ker_{\mathcal{R}[\xi]}(R_L) = \{r \in \mathcal{R}[\xi]^{1 \times n_r} \mid r(\xi)R_L(\xi) = 0\} \tag{55}$$

which is a left submodule of $\mathcal{R}[\xi]^{1 \times n_r}$. Thus, it is finitely generated, i.e., there exists a $Q \in \mathcal{R}[\xi]^{n \times n_r}$ such that $\operatorname{img}_{\mathcal{R}[\xi]}(Q) = \{r(\xi)Q(\xi) \mid r \in \mathbb{R}[\xi]^{1 \times n_r}\}$ is equal to $\ker_{\mathcal{R}[\xi]}(R_L)$. Then, we have an exact sequence

$$\mathcal{R}[\xi]^{1 \times n} \overset{\cdot Q}{\to} \mathcal{R}[\xi]^{1 \times n_r} \overset{\cdot R_L}{\to} \mathcal{R}[\xi]^{1 \times n_L} \tag{56}$$

and therefore the sequence $\mathcal{Q}^n \overset{Q(q)}{\leftarrow} \mathcal{Q}^{n_r} \overset{R_L(q)}{\leftarrow} \mathcal{Q}^{n_L}$ is also exact. This signifies that $R_w(q)\mathsf{w} \in \operatorname{img}_\mathcal{Q}(R_L) := \{R_L(q)\mathsf{w}_L \mid \mathsf{w}_L \in \mathcal{Q}^{n_L}\}$ iff $R_w(q)\mathsf{w} \in \ker_\mathcal{Q}(Q)$, i.e., $\mathfrak{B}_* = \{\mathsf{w} \in \mathcal{Q}^{n_W} \mid QR_w \circledast \mathsf{w} = 0\}$. ■

D. Proof of Theorem 4

The concept of the proof is based on [32]. To simplify the discussion, we prove only the so-called *Markovian case* as the state case follows trivially from this concept due to the linearity and time-invariance of LPV systems. We call the discrete-time LPV system $\mathcal{S} = (\mathbb{Z}, \mathbb{P}, \mathbb{W}, \mathfrak{B})$ Markovian, if for all $p \in \mathfrak{B}_\mathbb{P}$

$$(w_1, w_2 \in \mathfrak{B}_p) \wedge (w_1(0) = w_2(0)) \to (w_1 \underset{0}{\wedge} w_2) \in \mathfrak{B}_p.$$

In the following, we prove that $\mathcal{S}$ is Markovian, iff there exist matrices $r_0, r_1 \in \mathcal{R}^{n_r \times n_W}$ such that $\mathfrak{B}$ has the kernel representation: $r_0 w + r_1 \xi w = 0$. where $\xi = q$. The "if" part is trivial. To show the "only if" case, assume that a KR representation of $\mathcal{S}$ is given with $R \in \mathcal{R}[\xi]^{n_r \times n_W}$ for which the solutions of (14) satisfy the above given connectability condition. Without loss of generality it can be assumed that R is full row rank. Also, there exists a unimodular $M \in \mathcal{R}[\xi]^{\cdot \times n_r}$ such that $R'(\xi) = M(\xi)R(\xi)$ is in a row reduced form, meaning that the matrix formed by the coefficient functions of the highest powers in ξ of the rows $R'(\xi)$ has full row rank. Due to the fact that M

is a left-side unimodular transformation, the behaviors of R and R' are equivalent.

We show now that $\deg(R') = 1$. Assume the contrary and write R' in the IO form

$$(R_1(q) \diamond p)w_1 = (R_2(q) \diamond p)w_2 \tag{57}$$

where $\mathrm{col}(w_1, w_2) = w$ corresponds to an IO partition and $\deg(R_1) \geq \deg(R_2)$. The assumption that $\deg(R') > 1$ implies that $\deg(R_1) > 1$. Similarly, the assumption of $(R'(q) \diamond p)w = 0$ is Markovian implies that $(R_1(q) \diamond p)w_1 = 0$ is Markovian.

Now, let w_1', w_1'' be the solutions of $(R_1(q) \diamond p)w_1 = 0$ for a $p \in \mathfrak{B}_{\mathbb{P}}$ with $w_1'(0) = w_1''(0)$. Since (w_1, w_2) is an IO partition of $\mathcal{S}$, thus $\mathrm{col}\,(w_1', 0)$ and $\mathrm{col}\,(w_1'', 0)$ are also solutions of $(R'(q) \diamond p)w = 0$ and in order to obtain contradiction it suffices to prove contradiction for autonomous systems. Let $n_\xi = \deg(R_1)$ and by assumption $n_\xi > 1$. Introduce auxiliary variables $\breve{w}_{ij}$ defined as

$$\breve{w}_{ij} := q^i w_j, \qquad (i, j) \in \mathbb{I}_0^{n_\xi} \times \mathbb{I}_1^{n_\mathrm{W}} \tag{58}$$

where $w = [\,w_1 \quad \ldots \quad w_{n_\mathrm{W}}\,]^\top$. Collect these variables in a column vector

$$\breve{w} = [\,\breve{w}_{01} \quad \breve{w}_{02} \quad \ldots \quad \breve{w}_{0n_\mathrm{W}} \quad \breve{w}_{11} \quad \ldots \quad \breve{w}_{n_\xi n_\mathrm{W}}\,]^\top. \tag{59}$$

Now consider the system with latent variable $\breve{w}$ as

$$q\breve{w} = (r \diamond p)\breve{w} \tag{60a}$$

$$w_j = \breve{w}_{0j}, \qquad \forall j \in \mathbb{I}_1^{n_\mathrm{W}} \tag{60b}$$

where the coefficient $r \in \mathcal{R}^{(n_\xi n_\mathrm{W}) \times (n_\xi n_\mathrm{W})}$ is determined from the coefficients of $R_1(\xi)$ and the definition (58). The manifest behavior of (60a) is equivalent with the manifest behavior of $R_1(\xi)$, which can be checked by elimination of the latent variables of (60a) and (60b). However, the manifest behavior cannot be Markovian as (60a) and (60b) have exactly one solution $(w, \breve{w})$ for each initial condition $\breve{w}(0)$ and scheduling trajectory $p \in \mathfrak{B}_{\mathbb{P}}$. This contradicts Markovianity since two solutions $(w, \breve{w})$ and $(w', \breve{w}')$ with $\breve{w}_{0j}(0) = \breve{w}_{0j}'(0)$, $\forall j \in \mathbb{I}_1^{n_\mathrm{W}}$ cannot be connected unless also $\breve{w}_{ij}(0) = \breve{w}_{ij}'(0)$, $\forall (i, j) \in \mathbb{I}_1^{n_\xi - 1} \times \mathbb{I}_1^{n_\mathrm{W}}$. ■

E. Proof of Theorem 5

First consider the left-side transformation. Let $R \in \mathcal{R}[\xi]^{n_r \times n_\mathrm{W}}$ and $R' \in \mathcal{R}[\xi]^{n \times n_r}$ and $\mathbb{P} = \mathbb{R}^{n_\mathbb{P}}$. Based on the proof of the injective cogenerator property, consider $\mathfrak{B}_*$ and $\mathfrak{B}_*'$ as the complete behaviors of R and R'. Then, the inclusion $\mathfrak{B}_*' \subseteq \mathfrak{B}_*$ can be expressed as an exact sequence

$$0 \rightarrow \mathfrak{B}_*' \rightarrow \mathfrak{B}_* \tag{61}$$

which is equivalent to the exact sequence

$$0 \leftarrow \mathrm{module}_{\mathcal{R}[\xi]}(R') \leftarrow \mathrm{module}_{\mathcal{R}[\xi]}(R). \tag{62}$$

Equivalently, we have $\mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(R') \supseteq \mathrm{span}_{\mathcal{R}}^{\mathrm{row}}(R)$ or $R'(\xi) = Q(\xi)R(\xi)$ for some $Q \in \mathcal{R}[\xi]^{n \times n_r}$. If $\mathfrak{B}_* = \mathfrak{B}_*'$, then $R'(\xi) = Q_1(\xi)R(\xi)$ and $R(\xi) = Q_2(\xi)R'(\xi)$, which shows that R and R' has the same rank. If additionally, R and R' are full rank, then this implies that $Q_1 = Q_2^\dagger$, ergo Q_1 and Q_2 are unimodular. As the complete behaviors are equal therefore this implies that the behaviors of R and R' for each commonly valid trajectories of p are equal.

Consider the right-side transformation. Based on the proof of the injective cogenerator property, there is a homomorphism between the the complete behaviors of $R(\xi)$ and $R'(\xi) = R(\xi)Q_1(\xi)$ and also between $R(\xi) = R'(\xi)Q_2(\xi)$ and $R'(\xi)$. This implies that if $Q_1 = Q_2^\dagger$, ergo Q_1 and Q_2 are unimodular, then there exists a isomorphism between the behaviors. ■

REFERENCES

[1] W. Rugh and J. S. Shamma, "Research on gain scheduling," *Automatica*, vol. 36, no. 10, pp. 1401–1425, 2000.

[2] A. Packard, "Gain scheduling via linear fractional transformations," *Syst. Control Lett.*, vol. 22, no. 2, pp. 79–92, 1994.

[3] P. Apkarian and P. Gahinet, "A convex characterization of gain-scheduled $\mathcal{H}_\infty$ controllers," *IEEE Trans. Autom. Control*, vol. 40, no. 5, pp. 853–864, May 1995.

[4] C. W. Scherer, "Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control for time-varying and linear parametrically-varying systems," *Int. J. Robust Nonlinear Control*, vol. 6, no. 9–10, pp. 929–952, 1996.

[5] K. Zhou and J. C. Doyle, *Essentials of Robust Control*. Upper Saddle River, NJ: Prentice-Hall, 1998.

[6] F. Wu, "A generalized LPV system analysis and control synthesis framework," *Int. J. Control*, vol. 74, no. 7, pp. 745–759, 2001.

[7] B. Lu and F. Wu, "Switching LPV control designs using multiple parameter-dependent Lyapunov functions," *Automatica*, vol. 40, no. 11, pp. 1973–1980, 2004.

[8] F. Wu and K. Dong, "Gain-scheduling control of LFT systems using parameter-dependent lyapunov functions," *Automatica*, vol. 42, no. 1, pp. 39–50, 2006.

[9] C. W. Scherer, "Robust controller synthesis is convex for systems without control channel uncertainties," in *Model-Based Control: Bridging Rigorous Theory and Advanced Technology*, P. M. J. Van den Hof, C. W. Scherer, and P. S. C. Heuberger, Eds. New York: Springer, 2009, pp. 13–31.

[10] B. Bamieh and L. Giarré, "Identification of linear parameter varying models," *Int. J. Robust Nonlinear Control*, vol. 12, pp. 841–853, 2002.

[11] M. Butcher, A. Karimi, and R. Longchamp, "On the consistency of certain identification methods for linear parameter varying systems," in *Proc. 17th IFAC World Congress*, Seoul, Korea, Jul. 2008, pp. 4018–4023.

[12] R. Tóth, P. S. C. Heuberger, and P. M. J. Van de Hof, "Asymptotically optimal orthonormal basis functions for LPV system identification," *Automatica*, vol. 45, no. 6, pp. 1359–1370, 2009.

[13] V. Laurain, M. Gilson, R. Tóth, and H. Garnier, "Refined instrumental variable methods for identification of LPV box-Jenkins models," *Automatica*, vol. 46, no. 6, pp. 959–967, 2010.

[14] M. G. Wassink, M. van de Wal, C. W. Scherer, and O. Bosgra, "LPV control for a wafer stage: Beyond the theoretical solution," *Control Eng. Practice*, vol. 13, no. 2, pp. 231–245, 2004.

[15] M. Steinbuch, R. van de Molengraft, and A. van de Voort, "Experimental modeling and LPV control of a motion system," in *Proc. Amer. Control Conf.*, Denver, CO, Jun. 2003, pp. 1374–1379.

[16] L. Giarré, D. Bauso, P. Falugi, and B. Bamieh, "LPV model identification for gain scheduling control: An application to rotating stall and surge control problem," *Control Eng. Practice*, vol. 14, no. 4, pp. 351–361, 2006.

[17] R. Tóth, F. Felici, P. S. C. Heuberger, and P. M. J. Van de Hof, "Discrete time LPV I/O and state space representations, differences of behavior and pitfalls of interpolation," in *Proc. Eur. Control Conf.*, Kos, Greece, Jul. 2007, pp. 5418–5425.

[18] J. W. Lee, "On uniform stabilization of discrete-time linear parameter-varying control systems," *IEEE Trans. Autom. Control*, vol. 51, no. 10, pp. 1714–1721, Oct. 2006.

[19] J. W. Lee and G. E. Dullerud, "Optimal disturbance attenuation for discrete-time switched and Markovian jump linear systems," *SIAM J. Control Optim.*, vol. 45, no. 4, pp. 1329–1358, 2006.

[20] R. Tóth, "Modeling and identification of linear parameter-varying systems, an orthonormal basis function approach," Ph.D. dissertation, Delft University of Technology, Delft, The Netherlands, 2008.
[21] X. Wei, "Advanced LPV techniques for diesel engines," Ph.D., Johannes Kepler University, Linz, Austria, 2006.
[22] B. Paijmans, W. Symens, H. Van Brussel, and J. Swevers, "A gain-scheduling-control technique for mechatronic systems with position dependent dynamics," in *Proc. Amer. Control Conf.*, Minneapolis, MN, Jun. 2006, pp. 2933–2938.
[23] R. A. Nichols, R. T. Reichert, and W. J. Rugh, "Gain scheduling for $\mathcal{H}$-infinity controllers: A flight control example," *IEEE Trans. Control Syst. Technol.*, vol. 1, no. 2, pp. 69–79, Jun. 1993.
[24] J. C. Willems, "Paradigms and puzzles in the theory of dynamical systems," *IEEE Trans. Autom. Control*, vol. 36, no. 3, pp. 259–294, Mar. 1991.
[25] S. G. Krantz*, Handbook of Complex Variables*. Cambridge, MA: Birkhäuser-Verlag, 1999.
[26] O. Ore, "Theory of non-commutative polynomials," *Ann. Math.*, vol. 34, no. 2, pp. 480–508, 1933.
[27] P. M. Cohn*, Free Rings and Their Relations*. New York: Academic, 1971.
[28] T. Y. Lam, "On the quality of row rank and column rank," *Expositiones Math.*, vol. 18, pp. 161–164, 2000.
[29] E. Zerz, "An algebraic analysis approach to linear time-varying systems," *IMA J. Math. Control Inf.*, vol. 23, pp. 113–126, 2006.
[30] J. C. Willems, "The behavioral approach to open and interconnected systems," *IEEE Control Syst.*, vol. 27, no. 6, pp. 46–99, Dec. 2007.
[31] E. W. Kamen, P. P. Khargonekar, and K. R. Poolla, "A transfer-function approach to linear time-varying discrete-time systems," *SIAM J. Control Optim.*, vol. 23, no. 4, pp. 550–565, 1985.
[32] P. Rapisarda and J. C. Willems, "State maps for linear systems," *SIAM J. Control Optim.*, vol. 35, no. 3, pp. 1053–1091, 1997.
[33] J. W. Polderman and J. C. Willems*, Introduction to Mathematical Systems Theory, A Behavioral Approach*. New York: Springer, 1991.

Roland Tóth (M'08) was born in 1979 in Miskolc, Hungary. He received the B.Sc. degree in electrical engineering and M.Sc. degree in information technology in parallel at the University of Pannonia, Veszprém, Hungary, in 2004, and the Ph.D. degree (*cum laude*) from the Delft Center for Systems and Control (DCSC), Delft University of Technology (TUDelft), Delft, The Netherlands, in 2008.

He was a Post-Doctoral Research Fellow with DCSC, TUDelft, in 2009 and with the Berkeley Center for Control and Identification, University of California, Berkeley, in 2010. Currently, he is an Assistant Professor at DCSC, TUDelft. His research interests are in issues of linear parameter-varying (LPV) and nonlinear system identification and modeling, machine learning, process modeling and control, and behavioral system theory.

Dr. Tóth received the TUDelft Young Researcher Fellowship Award in 2010.

Jan C. Willems (S'66–M'68–SM'78–F'80–LF'05) was born in Bruges, Belgium. He graduated in electromechanical engineering from the University of Gent, Gent, Belgium, in 1963, and received the M.Sc. degree in electrical engineering from the University of Rhode Island, Kingston, in 1965, and the Ph.D. degree in electrical engineering from the Massachusetts Institute of Technology (MIT), Cambridge, in 1968.

He was on the Electrical Engineering faculty of MIT from 1968 to 1973, including a one-year post-doctoral stay at Cambridge University, Cambridge, U.K.. From 1973 until 2003, he was a Professor of systems and control with the Mathematics Department, University of Groningen, Groningen, The Netherlands. In 2003, he became an Emeritus Professor. He is presently a Guest Professor with the Department of Electrical Engineering, University of the K. U. Leuven, Leuven, Belgium. More details, a description of his research activities, and a list of publications can be found on his Web site, http://homes.esat.kuleuven.be/~jwillems.

Peter S. C. Heuberger was born in Maastricht, The Netherlands, in 1957. He received the M.Sc. degree in mathematics from the University of Groningen, Groningen, The Netherlands, in 1983, and the Ph.D. degree in mechanical engineering from Delft University of Technology, Delft, The Netherlands, in 1991.

From 1991 to 2006, he was with the Dutch National Institute for Public Health and the Environment (RIVM), and from 2006 to 2007, with the Dutch Environmental Assessment Agency (MNP), both in Bilthoven, The Netherlands. From 1996 to 2007, he also held a position with Delft University of Technology in the Mechanical Engineering Systems and Control Group and the Signals, Systems and Control Group of the Department of Applied Physics. Since 2007, he has been an Assistant Professor with the Delft Center for Systems and Control, Delft, The Netherlands. His research interests are in issues of system identification, uncertainty, model reduction, linear parameter-varying systems, and in the theory and application of orthogonal basis functions.

Paul M. J. Van den Hof (S'85–M'88–SM'95–F'08) received the M.Sc. and Ph.D. degrees in electrical engineering from Eindhoven University of Technology, Eindhoven, The Netherlands, in 1982 and 1989, respectively.

Since 1986, he has been with Delft University of Technology, where he was appointed as Full Professor in 1999. Since 2003, he has been Founding Co-Director of the Delft Center for Systems and Control (DCSC), with appointments in the faculty of Mechanical, Maritime and Materials Engineering and the faculty of Applied Sciences. Since 2005, he has also been scientific Director of the national research and graduate school "Dutch Institute of Systems and Control" (DISC), and national representative of the Dutch NMO in IFAC. His research interests are in issues of system identification, identification for control, and model-based control and optimization, with applications in industrial process control systems, including petroleum reservoir engineering systems, mechanical servo systems, and physical measurement systems.

Prof. Van den Hof is an IFAC Fellow. He has been a member of the IFAC Council (1999–2005), elected member of the Board of Governors of IEEE Control Systems Society (2003–2005), and member of the Administrative Council of the European Union Control Association (EUCA). He was an Associate Editor of *Automatica* (1992–1999), *Automatica* Editor for Technical Communiques and Rapid Publications (1999–2005), and Guest Editor of the *Automatica* special issue "Data-Based Modeling and System Identification" in March 2005. He was General Chair of the 13th IFAC Symposium on System Identification, held in Rotterdam, The Netherlands, in 2003, and Editor of the symposium proceedings.