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Equation error versus output error methods

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For the identification of linear processes on the basis of ARX-models, equation error least squares (EELS) (often indicated as the one step ahead prediction error method) is frequently used rather than output error least squares (OELS). This is mainly because the minimum of the convex EE-criterion can easily be found, in contrast to the OE-criterion, which often displays multiple local minima. Both methods lead to the correct parameter values when the system is in the model set chosen. But in many practical situations, such as human behaviour, the real process under study will be of infinite order causing essentially different models to be found from either EE or OE criteria. Various aspects of these differences are analysed in this study. Much attention has been paid to the performance of a simulation based on a model estimated with an EELS. This simulation performance can be predicted and bounds can be given without executing the simulation itself. Furthermore the simulation performance is very poor for systems where the energy in the initial impulse response samples is very small compared with the energy in the remainder of the response. For these systems an equation error estimate cannot even provide a proper initial guess for an OELS minimization algorithm. Examples are presented that illustrate this effect.

1. Introduction

It is often very important to characterize human behaviour by means of mathematical representations. The expression of transfer functions is a common mathematical means of representing dynamic properties of systems. System identification is then meant to estimate the transfer functions based on input and output signals. In this paper two essentially different criteria are considered to determine the parameters of possible transfer functions together constituting the model set, namely equation error (EE-) criterion and output error (OE-) criterion. These two criteria and their effects are often confused in practice. An EELS will provide good prediction models, while OELS yields good simulation models. The former is restricted to a short horizon, the latter refers principally to an infinite horizon.

In this paper the performance of the estimated models for one step ahead prediction and for simulation will be analysed and compared. In a very familiar

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setting, as explained in section 2, we disregard any disturbing noise as we want to focus our attention on the crucial phenomenon, which is that the process under study is not in the model set chosen. The disturbing noise does not essentially disturb the presented analysis as shown in Damen 1986.

The least squares equation error criterion, having the nice property of being quadratic-in-the-parameters, has been extensively applied as a linear regression method in system identification problems (see, for example, Eykhoff 1974, Iserman 1974, Ljung and Söderström 1983). Its use is also extended to the model reduction setting, (for example, Obinata and Inooka 1976, Mullis and Roberts 1976, Inouye 1983, Sufleta 1984). Introduction of output error criteria both in system identification and model reduction has always suffered from the high non-linearity of the error criterion. Nevertheless, system identification techniques are proposed by Steigitz and McBride 1965, Landau 1976 and Dugard and Landau 1980. An algorithm for output error model reduction has been discussed by Porat and Friedlander 1984.

For both equation error and output error strategies we take the same model set for the process and, consequently, the two strategies will produce different models within this model set. A general measure of 'how different these models are' cannot be given easily as, again, it is application-dependent. Nevertheless, we would like to answer questions such as:

'We intend to identify a model by a one step ahead prediction strategy with a predefined performance. Is it justified to use the resulting model for simulation purposes too?'

and

'Can the results of least squares equation error estimation be used as a good initial guess for the minimization of an output error criterion?'

In order to answer these questions we will focus our attention on the validation of equation error models with an output error validation criterion.

After the setting of the problem has been stated in section 2, an analysis will follow in section 3. In section 4 some examples will be given and a class of processes will be defined where indeed the answer to both questions posed above must certainly be 'no'.

2. Setting the problem

The circumstances which we will define are taken very idealistically in order to be able to stress the essentials, while deviations from this ideal situation can be brought in later. We study the discrete time description of single input single output, linear time-invariant systems. The model set for the process behaviour is given by:

$$y(k)=\frac{\beta_0+\beta_1 z^{-1}+\ldots+\beta_\mu z^{-\mu}}{1+\alpha_1 z^{-1}+\ldots+\alpha_\nu z^{-\nu}}u(k)=\frac{B(z^{-1})}{A(z^{-1})}u(k)=\hat{H}(z^{-1})\,u(k),\ \mu\leq\nu \qquad (1)$$

where z^{-1} is the delay operator: $z^{-1}u(k)=u(k-1)$, $u(k)$ is the input sample and $y(k)$ the output sample at time instant k.

Moreover we suppose that there is no perturbation noise, the number of available input and output samples tends to infinity and the input signal has a constant power spectrum over all frequencies:

$$\Phi_u(\omega)=c \qquad -\pi<\omega<\pi \tag{2}$$

The fundamental difference between this setting and the one that appears in most papers in this field, is that we do not assume the real system to be in the model set. The estimated transfer function $\hat{H}(z^{-1})$ can therefore never be equal to the real transfer function $H(z^{-1})$.

If the input and output signals of the real system are given by $u(k)$ and $y(k)$, the equation error and output error criteria can be illustrated by means of the scheme in figure 1.

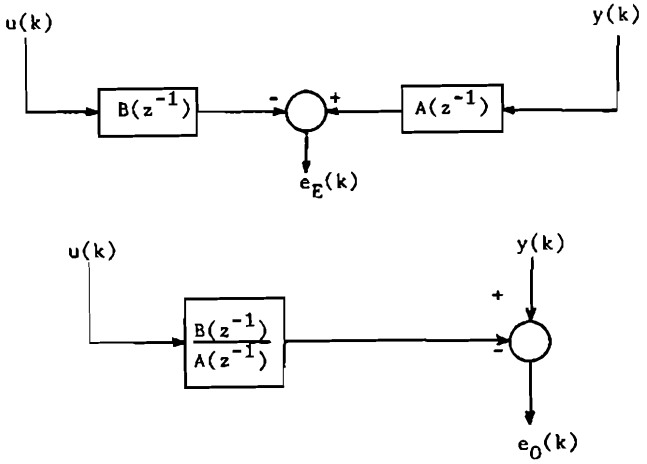

Figure 1. Equation error criterion (upper), output error criterion (lower).

The corresponding least squares criteria are defined by

$$V_E=\frac{1}{N}\sum_{k=0}^{N} e_E^2(k,\theta) \quad \text{for the equation error method (EELS)} \tag{3}$$

$$V_O=\frac{1}{N}\sum_{k=0}^{N} e_O^2(k,\theta) \quad \text{for the output error method (OELS)} \tag{4}$$

where N is the number of available input and output samples, and θ is the vector of parameters α_j, β_j. The respective model parameters are obtained as:

$$\hat{\theta}_E=\text{arg. min } V_E(\theta) \qquad \hat{\theta}_O=\text{arg. min. } V_O(\theta) \tag{5}$$

If $N\rightarrow\infty$ and under the condition of ergodicity we may also write:

$$V_E=E\{e_E^2(k,\theta)\} \tag{6}$$

$$V_O=E\{e_O^2(k,\theta)\} \tag{7}$$

where E stands for expectation.

The EELS model is easy to obtain. Since $e_E(k)$ is linearly dependent on the parameters an analytic solution can be obtained by only performing linear operations. It is well known that, in a prediction context, this method finds that model in the model set which gives the best (in a least squares sense) one step ahead *prediction* of the output, based on previous input and output samples, (see, for example, Ljung and Söderström 1983).

Obtaining the OELS model, on the other hand, is more complicated since $e_O(k)$ is very non-linear in the parameters. A solution can hardly be obtained without non-linear minimization techniques, and convergence to the global minimum of the criterion cannot be guaranteed in general. However, the OELS model is the model within the model set which gives the best simulation of the system based on previous input samples. In the present context with the chosen input signal this means a best fit of the impulse response of the original process. This latter criterion is very attractive in practical applications of simulation and control based on a long horizon.

Analysis of the two criteria in the frequency domain shows that the expressions (3) and (4) for $N \to \infty$ take the forms:

$$V_E(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{-j\omega}) - \hat{H}(e^{-j\omega},\theta)|^2 |A(e^{-j\omega},\theta)|^2 \Phi_u(\omega)\, d\omega \tag{8}$$

$$V_O(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{-j\omega}) - \hat{H}(e^{-j\omega},\theta)|^2 \Phi_u(\omega)\, d\omega \tag{9}$$

Where $H(e^{-j\omega})$ is the transfer function of the original process.

The extra component $|A(e^{-j\omega})|^2$ in V_E, weighting the errors between the original and modelled transfer function is dependent on the *estimated model*. This means that only *a posteriori*, when the estimation is performed, can an interpretation be given to the criterion that is minimized in the frequency domain.

Because models generally show a low pass character, the weighting in V_E by $|A(e^{-j\omega})|^2$ with respect to V_O will favour the estimation of the higher frequencies at the cost of an increasing bias in the lower frequencies (Ljung 1984). This property of EELS models has been noticed before (for example, by van Zee 1981). Numerical examples of this phenomenon are also given by Wahlberg and Ljung 1986.

Of course this effect can be translated to the different goals of V_E and V_O (one step ahead and infinite horizon). Nevertheless we have the impression that the goals are much more apparent in time domain. This is the reason for analysing the situation in time domain.

3. Equation error minimization, analysis and asymptotic results

The equation error for the model given by equation (1) is defined as:

$$e_E(k,\theta) = \sum_{i=0}^{r} \alpha_i y(k-i) - \sum_{j=0}^{p} \beta_j u(k-j) \tag{10}$$

where $\alpha_0 = 1$. In a simple setting, we suppose that the input signal $u(k)$ is sufficiently rich so that we may represent it as a white, zero mean noise sequence.

According to Mullis and Robers 1976, ARMA parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, which give the minimum equation error least squares, are given by:

$$\begin{bmatrix} \Psi & -H^T \\ -H & I \end{bmatrix} \begin{bmatrix} 1 \\ \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{bmatrix} = \begin{bmatrix} \delta_E \\ 0 \end{bmatrix} \tag{11}$$

Where T stands for 'transpose of',

$$\alpha=[\alpha_1\alpha_2 \ldots\ldots \alpha_\nu]^T; \qquad \beta=[\beta_0\beta_1 \ldots\ldots \beta_\mu]^T$$

$$\Psi_\nu=\begin{bmatrix} \Psi(0) & \Psi(1) & . & \Psi(\nu) \\ \Psi(1) & \Psi(0) & . & \Psi(\nu-1) \\ . & . & . & . \\ \Psi(\nu) & . & . & \Psi(0) \end{bmatrix} a[(\nu+1)\times(\nu+1)] \text{ Toeplitz matrix}$$

Where $\Psi(k)$ is the autocorrelation function of the output y with time shift

$$k=i-j:\Psi(i-j)=\sum_{\ell=0}^{\infty} h(\ell-i)h(\ell-j)$$

$$\mathbf{H}_{\mu,\nu}=\begin{bmatrix} h(0) & & & \\ h(1)h(0) & & \varphi & \\ h(2)h(1) & h(0) & & \\ . & . & . & . \\ h(\mu)h(\mu-1) & & . \quad . & h(\mu-\nu) \end{bmatrix} \text{having dimensions } [(\mu+1)\times(\nu+1)]$$

Where $h(k)$ is the impulse response of the process at the instant k. I=Identity $(\mu+1)\times(\mu+1)$. Where from now on α and β represent the values for the parameters for which V_E takes the minimum value δ_E. Of course $\delta_E=0$ if and only if the system is in the model set. Consequently the model parameters α, β constitute corresponding $\hat{\psi}$ and $\hat{H}$, such that

$$\begin{bmatrix} \hat{\Psi} & -\hat{H}^T \\ -\hat{H} & \underline{I} \end{bmatrix}\begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix}=\begin{bmatrix} \\ 0 \\ \\ \end{bmatrix} \tag{12}$$

With $\hat{\Psi}$, $\hat{H}$ similarly structured as Ψ, H, however now containing quantities related to the estimated model.

With respect to the uniqueness of the equation error model, the following theorem will be presented:

Theorem 1

The least squares equation error model defined by α and β is unique if and only if there does not exist an $(\mu-1,\nu-1)$ ordered model that describes the system data $\{h(i)\}_{i=0,\ldots,\mu}$, $\{\psi(j)\}_{j=0,\ldots,\nu}$ exactly. In other words: the model is unique if and only if it has not been over-parametrized.

Theorem 2

Given a stable linear system represented by its impulse response $\{h(i)\}_{i=0,\ldots,\infty}$. A (μ,ν) order equation error model fits the first $\mu+1$ samples of this impulse response exactly, i.e. $\hat{h}(i)=h(i)$ for $i=0,..,\mu$.

The above theorems are proved in Mullis and Roberts 1976 and in the appendix.

In fact, the β-parameters are adjusted in order to accomplish the fit, as mentioned in Theorem 2, whereas the α-parameters are used to give some fit to the tail of the impulse response h(i), i$>\mu$. The implication of this phenomenon for the performance of an equation error model is further commented on in the next section.

A more elaborate version of this last theorem for multivariable transfers is provided by Van den Hof 1987.

4. Output error results for equation error models

In this section, properties of an equation error model will be discussed based on a number of numerical examples. The equation error model will be validated in terms of its impulse response, comparing this response with the impulse response of the original system. It will be shown that a good prediction performance of the model, reflected in a small minimal value δ_E of the error criterion, does not guarantee a proper simulation performance of the model. Moreover, the occurrence of an equation error model with a bad fit in output error criterion is shown to be not dependent on the poles of the model alone, as sometimes suggested in the literature (Porat and Friedlander 1984, Ljung 1984).

A simulation example for a damped sine wave as an impulse response illustrates

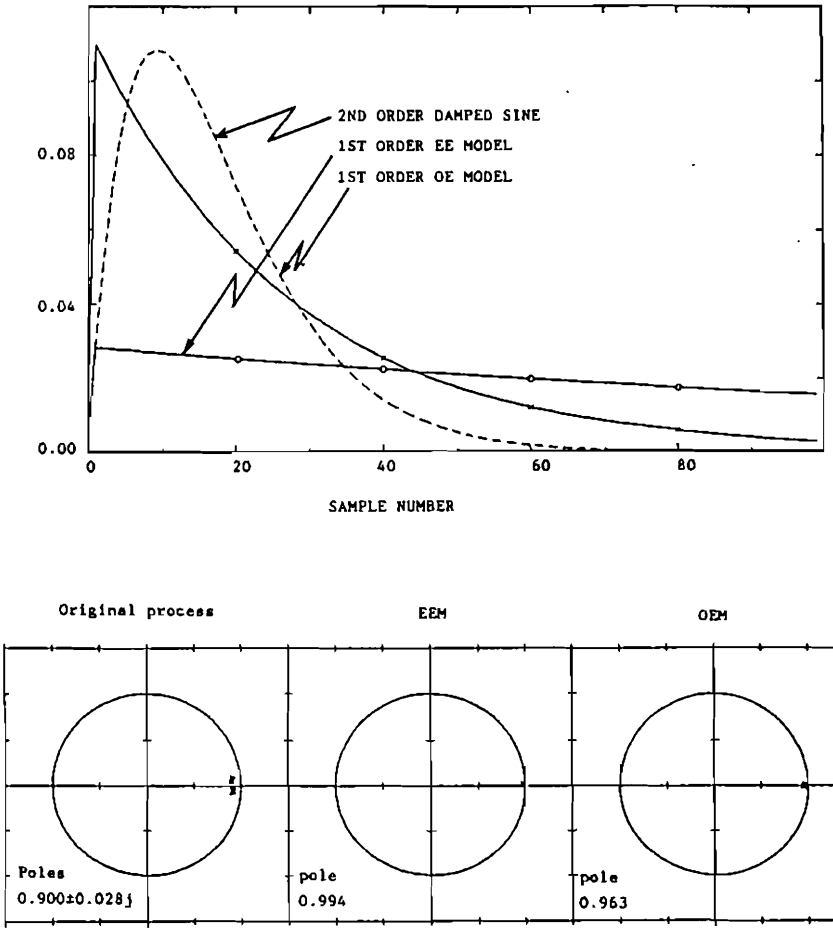

Figure 2. Impulse response and pole zero plot of a second order process with a damped sine character (h), and of its first order equation error (EE) and output error (OE) model, respectively denoted by $\hat{h}_E$ and $\hat{h}_O$. The process is defined by $h(k)=(0{\cdot}9)^k \cdot \sin(0{\cdot}01kx)$. The asymptotic EE model is calculated based on the relations as described in section 3. The OE model is the result of the pseudo-linear regression algorithm of Steiglitz and McBridge 1965, minimizing the sum of squared output errors over 240 samples. $\mu(0)=0{\cdot}19$; $\delta_E=0{\cdot}15 \cdot 10^{-2}$; $||h-\hat{h}_E||_2=0{\cdot}35$; $||h-\hat{h}_O||_2=0{\cdot}16$.

this, as given in figure 2. This is a second-order system which we try to model using a first-order model. According to Theorem 2 the equation error will show $\hat{h}(0)=h(0)=0$ and $\hat{h}(1)=h(1)$, being very small compared with the maximal value of $h(i)$. The tail of the model response will be a decreasing exponential (first order) defined by α_1, but it is clear that we can never obtain a good fit of the actual impulse response. This effect will even become stronger if the sampling rate is increased. As an illustration the response of a first-order output error model is also depicted in figure 2.

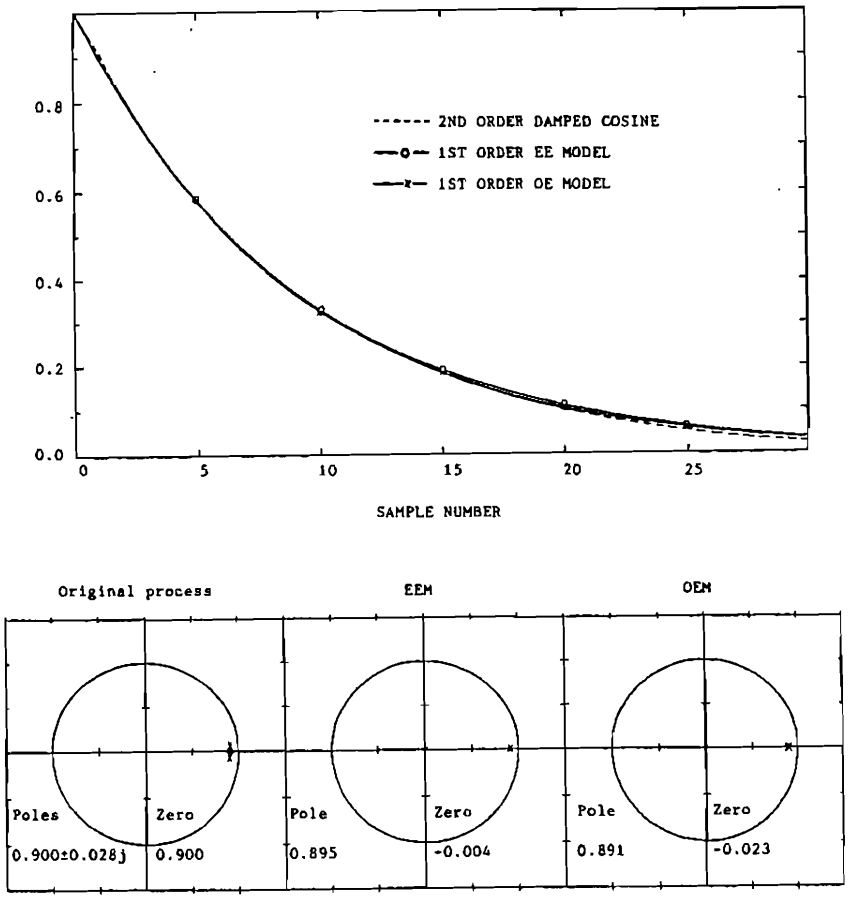

Figure 3. Impulse response and pole zero plot of a second order process with a damped cosine character (h), and of its first order equation error (EE) and output error (OE) model, resp. denoted by $\hat{h}_E$ and $\hat{h}_O$. The process is defined by $h(k)=(0{\cdot}9)^k\cdot\cos(0{\cdot}01k\pi)$. The asymptotic EE model is calculated based on the relations as described in ection 3. The OE model is the result of the pseudo-linear regression algorithm of Steiglitz and McBridge 1965, minimizing the sum of squared output errors over 240 samples. $\Psi(0)=5{\cdot}069$; $\delta_E=0{\cdot}73\cdot10^{-4}$; $||h-\hat{h}_E||_2=0{\cdot}058$; $||h-\hat{h}_O||_2=0{\cdot}044$.

In figure 3 a similar simulation example is shown for a dampted cosine wave. The poles of this system are the same as in the previous example, but now an excellent fit can be obtained. Since the estimated α_1 parameter of the equation error models in the examples are very close, this feature cannot be explained in frequency domain by

only considering the weighting function $|A(e^{-j\omega})|$ of the models. As a result this discrepancy can hardly be detected in frequency domain, but it is very apparent in time domain.

A third example illustrating the effect mentioned above is shown in figure 4 where it concerns a more or less artificial impulse response from a retina phenomenon. Detailed information on this data set can be found in Roufs and Blommaert 1981. The response $h(k)$ is considered to be zero for $k>76$.

For a fifth-order model ($\nu=\mu=5$) we observe that indeed the first six samples of the impulse response are fitted exactly, contrary to the tail where the fit is very poor. As in figure 2 we also see that the energy in the estimated impulse response ($\hat{\psi}(0)$) is substantially less than the energy in the actual response ($\psi(0)$). A relation between these two quantities is formalized in a theorem which is proved in Damen and Moses 1986:

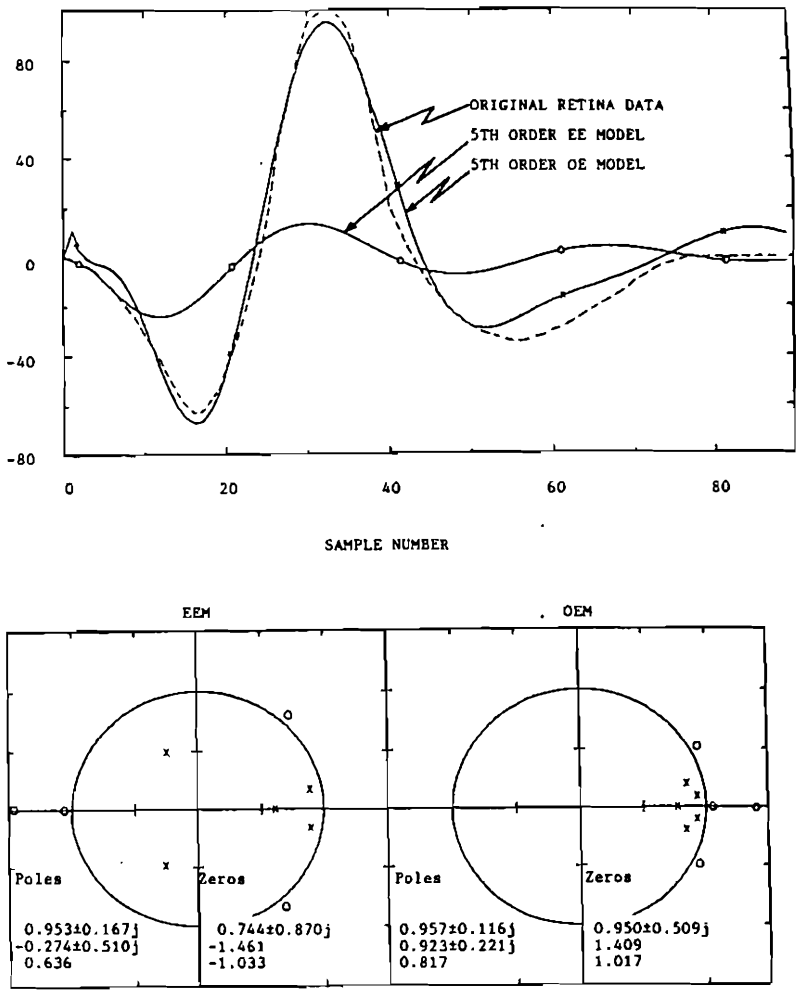

Figure 4. Original impulse response of a retina phenomenon, and the impulse responses and pole zero, plots of its fifth order EE and OE model. The OE model is obtained by a modified Newton minimization algorithm, taking an output error criterion over 240 samples. $\Psi(0)=141\ 123$; $\delta_E=262{\cdot}09$; $||h-\hat{h}_E||_2=318{\cdot}72$; $||h-h_O||_2=76{\cdot}67$.

Theorem 3
Given a stable equation error model with autoregressive polynomial $A(z^{-1})$ and corresponding reflection coefficients $K_1 \ldots K_*$ (Markel 1976), then

$$\psi(0)=\hat{\psi}(0)+\frac{\delta_E}{p}$$

where $\Psi(0)$, $\hat{\Psi}(0)$ respectively are the energy in the original and the model impulse response, δ_E is the minimal value of the equation error criterion and

$$p=\prod_{i=1}^{*}(1-K_i^2).$$

Remarks:
— Ψ and $\hat{\Psi}$ correspond to previous definitions.
— Since both $\delta_E \geq 0$ and $p>0$ it is clear that $\Psi(0) \geq \hat{\Psi}(0)$.
— If and only if the system is in the model set we obtain $\delta_E=0$ and consequently $\Psi(0)=\hat{\Psi}(0)$.
— If $\delta_E/\Psi(0)$ is very small, which implies that the relative prediction error is very small, it is not necessarily true that $\Psi(0) \approx \hat{\Psi}(0)$. Note that p might be very small too, particularly if we are dealing with poles close to the unit circle ($K_i \approx 1$, $p \ll 1$).

Based on the observation that a situation $\Psi(0) \gg \hat{\Psi}(0)$ will undoubtedly lead to a poor simulation performance of the model, we will work towards a lower bound for the output error criterion of an equation error model.

By using the Schwartz inequality in ℓ^2 infinite sequence Hilbert space it follows that:

$$\frac{||h-\hat{h}||_2}{\sqrt{\Psi(0)}} \geq 1-\sqrt{\frac{\hat{\Psi}(0)}{\Psi(0)}} \tag{13}$$

where

$$||h-\hat{h}||_2=\sum_{k=0}^{\infty}[h(k)-\hat{h}(k,\alpha,\beta)]^2$$

The drawback of the lower bound equation (13), however, is that we actually have to calculate $\hat{\psi}(0)$ in order to predict the simulation performance. Next we will show how this can be avoided.

In Damen and Moses 1984 it is shown that

$$\frac{\hat{\Psi}(0)}{\Psi(0)}=\frac{\delta_H}{\delta_E+\delta_H} \tag{14}$$

where δ_H is a quadratic function both in $h(i)$, $i=0,1,\ldots,\mu$ and in the reflection coefficients K_i determined by the parameter vector $\boldsymbol{\alpha}$. As soon as $\delta_H<\delta_E$ the simulation performance will be very poor as follows from equation (14). In order to calculate this δ_H we have to know the α-parameters again. However there is an upper

bound on this δ_H for all stable systems with ν poles. This leads to a lower bound for the simulation performance:

$$\frac{||h-\hat{h}||_2}{\sqrt{\Psi(0)}} \geq 1-\sqrt{\frac{\delta_{H\,max\,1}}{\delta_E+\delta_{H\,max\,1}}} \quad (=\text{L.B.1}) \tag{15}$$

where

$$\delta_{H\,max1}=\left(\frac{2\nu}{\nu}\right)\sum_{i=0}^{\mu}\min(\nu+1,\mu+1-i)\,h^2(i) \tag{16}$$

(Damen and Moses 1986).

This is quite a powerful result as it puts a bound irrespective of the $_\alpha$, which can be estimated afterwards. Of course the bound is not as tight as the one given in equation (13) as tightness is sacrificed for generality. We may also simplify the bound for the cost of extra loss of tightness, because obviously:

$$\delta_{H\,max1}<\delta_{H\,max2}=\min(\nu+1,\mu+1)\,||h^\mu||_2^2 \tag{17}$$

where $h^\mu=(h(0),h(1),\ldots,h(\mu))^T$.

Substitution of this simplified bound on δ_H, denoted by $\delta_{H\,max2}$, into equation (15) leads to a simplified lower bound on the relative simulation error, denoted by L.B.2. Both lower bounds L.B.1 and L.B.2 for the examples treated, are given in tables 1 and 2.

Table 1. Prediction performance, simulation performance and its lower bounds for the equation error models of the examples in figures 2 and 3.

	Damped sine (figure 2) $\underline{\mu}=\underline{\nu}=1$	Damped cosine (figure 3) $\underline{\mu}=\underline{\nu}=1$
Prediction performance		
$\sqrt{\frac{\underline{\delta}_E}{\underline{\Psi}(0)}}$	0·089	0·004
Simulation performance		
$\frac{\|\|\underline{h}-\underline{\hat{h}}\|\|^2}{\sqrt{\underline{\Psi}(0)}}$	0·789	0·026
Lower bound from equation (13)		
$1-\sqrt{\frac{\underline{\hat{\psi}}(0)}{\underline{\Psi}(0)}}$	0·430	$0{\cdot}36\times10^{-4}$
L.B. 1	0·285	$0{\cdot}65\times10^{-5}$
L.B. 2	0·177	$0{\cdot}50\times10^{-5}$

Now we can also use equation (15) in another way. Suppose that we have estimated $h(0)\ldots h(\mu)$, which is quite straightforward in the present setting. If we

require that the lower bound on the relative simulation error is less than say 0·01, this puts an upper bound on the δ_E to be achieved in the estimation:

$$\delta_E < \frac{1-(1-\varepsilon_0)^2}{(1-\varepsilon_0^2)}\delta_{H\ max} = \frac{0{\cdot}0199}{0{\cdot}9801}\delta_{H\ max}$$

In table 2, we are given these bounds for δ_E for various $\nu=\mu$ for the retina data, and for both $\delta_{H\ max1}$ and $\delta_{H\ max2}$:

Table 2. Prediction performance, simulation performance and its lower bounds for the equation error models of the retina example in figure 4. Model orders from 1 to 5.

$\mu=\nu=$	1	2	3	4	5
Prediction performance					
$\sqrt{\frac{\delta_E}{\Psi(0)}}$	0·167	0·045	0·045	0·044	0·043
Simulation performance					
$\frac{\|h-\hat{h}\|_2}{\sqrt{\Psi(0)}}$	0·999	0·939	0·973	0·924	0·848
Lower bound from equation (13)					
L.B. 1 $1-\sqrt{\frac{\hat{\Psi}(0)}{\Psi(0)}}$	0·984	0·878	0·932	0·867	0·763
L.B. 2	0·968	0·564	0·148	0·020	0·020
	0·968	0·376	0·119	0·006	0·005

Comment on table 2

As δ_E stabilizes at about 270, we observe that at least the order 4 or 5 will be necessary.

Note that if we should consider only the prediction performance of the estimated model, characterized by

$$\sqrt{\frac{\delta_E}{\Psi(0)}},$$

then the result of table 2 suggests that a second-order model would suffice and that an increase of the model order would hardly improve the model performance.

5. Conclusions

The performance of a simulation based upon an ARX model estimated with an equation error method was analysed. The process under study does not fit in the model set, and for the sake of simplicity, it is assumed that there is no disturbing noise.

Asymptotic equation error estimation results were employed and, first, it was shown how these were computed. Under the defined conditions, it was then shown and illustrated by various examples that:

The first $\mu+1$ samples of the impulse response fit exactly, where $\mu+1$ equals the number of moving average parameters of the model.

The energy $\Psi(0)$ in the estimated impulse response is always less than the

energy $\Psi(0)$ in the actual impulse response. The difference between both energies increases linearly with the equation error but also increases substantially when the poles of the estimated model are close to the unit circle. This dependence is represented by a very simple relation containing reflection coefficients.

The ratio $\hat{\Psi}(0)/\Psi(0)$ can be expressed as a function of the first $\mu+1$ actual impulse response samples. Furthermore, it follows that the upper bound on the ratio $\hat{\Psi}(0)/\Psi(0)$ decreases depending on the energy of the first $\mu+1$ impulse response samples, regardless of the dynamics of the process under study provided that it is stable. This implies that knowledge of the first $\mu+1$ impulse response samples leads, beforehand, to a limit on the simulation performance. This limit can be quantified prior to the estimation itself.

As impulse responses of many (if not all) physical processes start from zero it is clear that for a high sampling rate the first $\mu+1$ samples may be rather small, which implies, therefore, that a very bad simulation performance may be expected.

Quantification of this effect has been made possible, although disturbances were not taken into consideration. Extensions for this aspect, as well as the analysis for multivariable processes, will be necessary. In order to determine the simulation model which is best for the examples presented in the chosen model set, results of output error estimation have also been discussed and illustrated.

Table 3. Minimal value ($\underline{\delta}_E$) of the equation error criterion for retina equation error models with order 1, . . . ,5, and upperbounds for $\underline{\delta}_E$ when the relative simulation error is forced to have an upperbound 0·01. $\underline{\delta}_{E\ max1}$ is the upper bound for $\underline{\delta}_E$ using $\underline{\delta}_{H\ max1}$; $\underline{\delta}_{E\ max2}$ is the upper bound for $\underline{\delta}_{H\ max2}$.

Retina data (figure 4) $\mu=\nu=$	1	2	3	4	5
$\underline{\delta}_E$	3931	281	279	278	262
$\underline{\delta}_{E\ max1}$	13	39	259	1134	4899
$\underline{\delta}_{E\ max2}$	0·04	0·64	12.4	116·2	1315·44

Finally, it has become clear that equation error estimates as a result, may be poor 'initial guesses' for nonlinear minimization techniques in output error algorithms.

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Appendix

Proof of Theorem 1

Define

$$W_{\mu,\nu}=\begin{bmatrix} \Psi & -H^T \\ -H & I \end{bmatrix} \tag{17}$$

and consider the matrix $W^*_{\mu,\nu}$ being the submatrix of $w_{\mu,\nu}$ excluding its first row and column. Non-uniqueness of the model from equation (11) occurs if and only if $W^*_{\mu,\nu}$ is a singular matrix. By inspection of the structure of $W^*_{\mu,\nu}$ it can be verified that this matrix is singular if and only if $W_{\mu-1,\nu-1}$ is singular.

With equation (12) this last condition can easily be proved to be equivalent to the existence of an $(\mu-1, \nu-1)$ order model that generates the system data, $(\delta_E=0)$.

Proof of Theorem 2

In order to use equation (12) for the calculation of $\hat{\Psi}(i)$ and $\hat{h}(j)$ from α and β, it can be rearranged to an equivalent set of equations:

$$\begin{bmatrix} \mathbf{Q}_\nu - \mathbf{R}^T_{\nu,\omega} \\ \boldsymbol{\varphi} \quad \mathbf{S}_\mu \end{bmatrix} \begin{bmatrix} \hat{\Psi}^\nu \\ \hat{\mathfrak{h}}^\nu \end{bmatrix} = \begin{bmatrix} \theta \\ \beta \end{bmatrix} \tag{18}$$

where

$$\hat{\psi}^\nu = [\hat{\psi}(0)\hat{\psi}(1) \ldots . \hat{\psi}(\nu)]^T;\ \hat{\mathfrak{h}}^\mu = [\hat{h}(0)\hat{h}(1) \ldots . \hat{h}(\mu)]^T$$

$$\mathbf{Q}\nu = \begin{bmatrix} 1 & & & & \phi \\ \alpha_1 & 1 & & & \\ \alpha_2 & \alpha_1 & 1 & & \\ & & & & \\ \alpha_{\nu-1} & & & \alpha_1 & 1 \\ \alpha_\nu & \alpha_{\nu-1} & & \alpha_2 & \alpha_1 \quad 1 \end{bmatrix} + \begin{bmatrix} 0 & \alpha_1 & \alpha_2 & . & \alpha_{\nu-1} & \alpha_\nu \\ 0 & \alpha_2 & & & \alpha_\nu & \\ 0 & . & & . & & \\ . & \alpha_{\nu-1}\alpha_\nu & & & \phi & \\ 0 & \alpha_\nu & & & & \\ 0 & 0 & & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{R}_{\mu,\nu} = \begin{bmatrix} \beta_0 & & & \\ \beta_1 & \beta_0 & \phi & \\ . & . & . & \\ \beta_\mu & \beta_{\mu_{\omega 1}} & & \beta_{\theta-\nu} \end{bmatrix} \quad (\mu+1)\times(\nu+1) \text{ matrix with equivalent structure as } \mathbf{H}\mu,\nu$$

$$\mathbf{S}_\theta = \begin{bmatrix} 1 & & & \\ \alpha_1 & 1 & . & \phi \\ . & . & . & \\ \alpha_\mu & \alpha_{\mu-1} & . \; . \; \alpha_1 & 1 \end{bmatrix} \quad \mu+1 \text{ square matrix}$$

If the model α,β is stable, det ($\mathbf{Q}\nu$) will be unequal zero (Dugré, Beex and Sharf 1980). In that situation $\hat{\psi}$ and $\hat{\mathfrak{h}}\mu$ are uniquely determined by equation (18).

From equation 18 it follows that $\mathbf{S}_\mu \hat{\mathfrak{h}}^\mu = \beta$. Equation (11) can be written in a similar structure, leading to $\mathbf{S}_\mu \mathfrak{h}^\mu = \beta$. Since $\mathbf{S}_\mu$ is a non-singular matrix, subtraction of these equations shows that $\hat{\mathfrak{h}}^\mu = \mathfrak{h}^\mu$.

線形システムの伝達関数は出力誤差規範（ＯＥＬＳ）によって同定されるが、パラメータに関して非線形になるので、線形化した式誤差規範（ＥＥＬＳ）が頻繁に用いられる。これは線形化により最小値を求めることが非常に容易になるためである。システムの次数がモデルの次数と等しいときには、両規範は同じモデルパラメータを与えるのでＥＥＬＳが優れているが、人間の振舞のような実際の系では次数が非常に高く、低次数化されたモデルに対するパラメータ推定値は両規範によって大きく異なるので、目的によって規範を使い分ける必要がある。　シミュレーションには線形化されていないＯＥＬＳによって得られたパラメータを用いるべきであるが、推定が容易なことからＥＥＬＳによって得られたパラメータを用いて行うことが頻繁にある。このようなときに実際にシミュレーションを実行することなく、その成績を与える方法を開発した。この方法を用いて、種々の例について解析した結果、そのインパルス応答の初期部（モデルの次数程度のサンプル個数）のエネルギーが残りの部分のエネルギーと比べて大きいときに成績が悪く、このような系ではＥＥＬＳが適当でないことを例とともに示す。