

# Excitation allocation for generic identifiability of a single module in dynamic networks: A graphic approach <sup>★</sup>

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**Abstract:** For identifiability of a single module in a dynamic network, excitation signals need to be allocated at particular nodes in the network. Current techniques provide analysis tools for verifying identifiability in a given situation, but hardly address the synthesis question: where to allocate the excitation signals to achieve generic identifiability. Starting from the graph topology of the considered network model set, a new analytic result for generic identifiability of a single module is derived based on the concept of *disconnecting sets*. For the situation that all node signals are measured, the vertices in a particular disconnecting set provide the potential locations to allocate the excitation signals. Synthesis approaches are then developed to allocate excitation signals to guarantee generic identifiability.

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*Keywords:* System identification, identifiability, dynamic networks, graph theory

## 1. INTRODUCTION

Due to the increasing complexity of current technological systems, the study of large-scale interconnected dynamic systems receives considerable attention. As a modeling framework for dynamic networks, we consider the network of transfer functions introduced in (Gonçalves and Warnick, 2008; Van den Hof et al., 2013), where vertices represent the internal signals, and directed edges denote transfer functions which are called modules. Identification problems in this setup involve multiple aspects, including estimation of one local module (Van den Hof et al., 2013; Dankers et al., 2015; Materassi and Salapaka, 2019), estimation of the topology (Materassi and Innocenti, 2010; Shi et al., 2019), estimation of the full network model (Weerts et al., 2018b) and identifiability of the models (Weerts et al., 2018a; Hendrickx et al., 2019; Cheng et al., 2019a).

The analysis of identifiability of dynamic networks aims for conditions under which network models in a model set can be distinguished based on data. This analysis provides guidelines for users to choose an appropriate model set such that the identification problem has a unique solution. Two notions of identifiability have been addressed in the literature, including *global identifiability* and *generic identifiability*. On the one hand, global identifiability requires models to be distinguishable from all other models in the model set (Weerts et al., 2018a; van Waarde et al., 2018). On the other hand, generic identifiability requires that almost all models can be distinguished from the other models in the set (Bazanella et al., 2017; Hendrickx et al., 2019; Weerts et al., 2018c). The advantage of considering

generic identifiability is that the theoretical results can typically be formulated into graphical conditions on the network structure. This formulation can lead to efficient graphical approaches to analyze generic identifiability.

In addition, two different practical settings are considered for the study of generic identifiability. In one setting, all node signals can be measured and a subset of nodes is excited (Weerts et al., 2018c; Cheng et al., 2019a), while also allowing excitation through noise signals; while in the other setting, a subset of node signals is measured and all nodes can be excited while excitation is provided through external excitation only (Bazanella et al., 2017; van Waarde et al., 2018; Hendrickx et al., 2019). A recent contribution (Bazanella et al., 2019) extends the later work to partial measurement and partial excitation.

For generic identifiability when all node signals are measured, analysis results for generic identifiability of a single module can be found in (Weerts et al., 2018c). However, the results do not provide a structured method to allocate excitation signals for generic identifiability. In (Cheng et al., 2019b), an approach to allocate signals is developed for generic identifiability of a full network based on the pseudotree covering; however, this approach is hardly applicable to a single module. In this work, the research question is how to allocate the excitation signals such that one single module becomes generically identifiable.

This research question is addressed by deriving a new analytic result for generic identifiability of a single module using the concept of disconnecting sets. Given the output of a target module, it is further found that the vertices in the disconnecting set from the input of the module to the other inputs of the output provide the potential locations for allocating the excitation signals. Then synthesis approaches are developed to allocate excitation signals such

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that generic identifiability of a single module is guaranteed. Due to the limited space, the proofs are presented in the longer version of this paper (Shi et al., 2020).

## 2. PRELIMINARIES AND PROBLEM FORMULATION

### 2.1 Dynamic networks and graphical representation

A dynamic network encompasses the relationship among scalar-valued *internal signals*  $w_j$ , deterministic excitation signals  $r_k$  which can be manipulated by the user, and unmeasured disturbances  $e_l$ , where  $j \in \{1, \dots, L\}$ ,  $k \in \{1, \dots, K\}$ ,  $l \in \{1, \dots, p\}$  and  $p \leq L$ . Following (Van den Hof et al., 2013), the interconnection among the signals is modeled using linear time-invariant systems, and for any  $j$ , the equation of  $w_j$  is written as

$$w_j(t) = \sum_{j \neq i} G_{ji}(q)w_i(t) + \sum_k R_{jk}(q)r_k(t) + \sum_l H_{jl}(q)e_l(t), \quad (1)$$

where  $q$  is the delay operator, i.e.  $q^{-1}w_j(t) = w_j(t-1)$ , and its matrix form is obtained as

$$w(t) = G(q)w(t) + R(q)r(t) + H(q)e(t),$$

where  $G(q)$  is hollow,  $w(t) = [w_1(t), \dots, w_L(t)]^\top$ ,  $r(t) = [r_1(t), \dots, r_K(t)]^\top$ , and  $e(t) = [e_1(t), \dots, e_p(t)]^\top$ . Furthermore, model (1) satisfies the following assumptions:

#### Assumption 1

- $G_{ji}(q)$  is a stable and proper rational transfer operator if nonzero, and  $R_{jk}(q)$  is a stable and proper rational transfer operator;
- $(I - G(q))^{-1}$  is stable and all principal minors of  $\lim_{z \rightarrow \infty} (I - G(z))$  are non-zero;
- $H(q)$  is monic, proper and minimum-phase when  $p = L$ ; When  $p < L$ , i.e., rank-reduced noises,  $H(q)$  is structured as  $H(q) = \begin{bmatrix} H_a \\ H_b \end{bmatrix}$ , with  $H_a$  square, proper, monic, stable and minimum phase, see (Weerts et al., 2018a) for more details;
- $e(t)$  is white noise vector with covariance matrix  $\Lambda > 0$ ;
- $w(t)$  and  $r(t)$  can be measured.

We further define  $X(q) \triangleq [R(q) \ H(q)]$  and call both excitation and noise signals *external signals*. We use *modules* to refer to the transfer functions in  $G(q)$ .

It can be found that a network model is completely specified by  $(G(q), R(q), H(q), \Lambda)$ . Following (Weerts et al., 2018a,c), a set of such models can be obtained by a rational parametrization of every entry in network matrices  $(G(q), R(q), H(q))$  and a parameterization of  $\Lambda$ :

*Definition 1.* A parameterized network model set for a network of  $L$  internal variables,  $K$  excitation signals and  $p$  noise process is defined as

$$\mathcal{M} = \{(G(q, \theta), R(q, \theta), H(q, \theta)), \Lambda(\theta) | \theta \in \Theta \subseteq \mathbb{R}^n\},$$

with all its elements satisfying Assumption 1. ■

There can also be prior knowledge or users' choices such that certain entries in the network matrices of  $\mathcal{M}$  are fixed to zero. Then these entries in the matrices of  $\mathcal{M}$  should be zero functions. This structural information of  $\mathcal{M}$  can

also be encoded by a directed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V} = \mathcal{W} \cup \mathcal{X}$  is a set of vertices representing the set of all internal signals  $\mathcal{W}$  and the set of all external signals  $\mathcal{X}$ , and  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  denotes a set of directed edges. A directed edge exists if and only if the corresponding entry in the matrices of  $\mathcal{M}$  is not fixed to zero. For example,  $G_{ji}(q, \theta)$  is not a zero function if and only if the directed edge from  $w_i$  to  $w_j$  exists in  $\mathcal{G}$ . Then this model set with its structural information is specified by the set  $\mathcal{M}$  with its graph  $\mathcal{G}$ . In this work, we typically assume that the above structural information of a model set is available.

As mentioned in (Weerts et al., 2018a,c), there can also be information that certain entries of the network matrices can be non-parameterized and known, while the other non-zero entries are unknown and parameterized.

For notation, we sometimes use  $G$  to denote  $G(q, \theta)$ , similarly for other functions of  $\theta$  and  $q$ . We denote  $T \triangleq (I - G)^{-1}X$ ,  $T_{wr} \triangleq (I - G)^{-1}R$  and  $T_{we} \triangleq (I - G)^{-1}H$ , while  $\Phi_{\bar{v}}(w)$  is the power spectrum of  $T_{we}(q)e(t)$ .

### 2.2 Generic identifiability of a single module

Generic identifiability is a property of a network model set. By combining the identifiability concept in (Weerts et al., 2018a) and the concept of genericity in (Bazanella et al., 2017), generic identifiability of module  $G_{ji}$  can be defined as follows.

*Definition 2.* Module  $G_{ji}$  in  $\mathcal{M}$  is generically identifiable if for almost all  $\theta_1 \in \Theta$ , it holds that

$$\begin{cases} T_{wr}(q, \theta_1) = T_{wr}(q, \theta_2) \\ \Phi_{\bar{v}}(w, \theta_1) = \Phi_{\bar{v}}(w, \theta_2) \end{cases} \implies G_{ji}(q, \theta_1) = G_{ji}(q, \theta_2),$$

for all  $\theta_2 \in \Theta$ .  $G_{ji}$  is globally identifiable if the above implication holds for all  $\theta_1 \in \Theta$ . ■

In the above definition, the notion of almost all means all elements in a set except the elements in a subset of Lebesgue measure zero.

Since  $T_{wr}$  and  $\Phi_{\bar{v}}$  can typically be identified based on the measured  $w(t)$  and  $r(t)$ , identifiability refers to the situation that a unique module can be found based on  $T_{wr}$  and  $\Phi_{\bar{v}}$ . Following directly from Proposition 1 and Definition 5 of (Weerts et al., 2018a), Definition 2 can be simplified under certain conditions.

*Lemma 1.* Let  $\mathcal{M}$  satisfy one of the following conditions:

- all modules in  $G(q, \theta)$  are strictly proper, or
- there is no algebraic loop<sup>1</sup> in  $G(q, \theta)$  and  $H^\infty(\theta)\Lambda(\theta)H^\infty(\theta)^T$  is diagonal for all  $\theta$ , with  $H^\infty(\theta) := \lim_{z \rightarrow \infty} H(z, \theta)$ ;

then module  $G_{ji}$  in  $\mathcal{M}$  is generically identifiable if for almost all  $\theta_1 \in \Theta$ , it holds that

$$T(q, \theta_1) = T(q, \theta_2) \implies G_{ji}(q, \theta_1) = G_{ji}(q, \theta_2), \quad (2)$$

for all  $\theta_2 \in \Theta$ .  $G_{ji}$  is globally identifiable if the above implication holds for all  $\theta_1 \in \Theta$ . ■

In this work, we consider  $\mathcal{M}$  that satisfies both the assumptions in Definition 1 and the conditions in Lemma 1.

<sup>1</sup> There exists an algebraic loop around node  $w_{n_1}$  if there exists a sequence of integers  $n_1, \dots, n_k$  such that  $G_{n_1 n_2}^\infty G_{n_2 n_3}^\infty \dots G_{n_k n_1}^\infty \neq 0$ , with  $G_{n_1 n_2}^\infty := \lim_{z \rightarrow \infty} G_{n_1 n_2}(z)$ .

In addition, the model set should also satisfies (Weerts et al., 2018a,c):

- Every parameterized entry in  $(G, R, H)$  covers all possible proper transfer functions<sup>2</sup>;
- All parameterized transfer functions are parameterized independently.

In this setting, generic identifiability of a single module is related to the maximum number of vertex disjoint paths as defined in Section 2.3 (Weerts et al., 2018c). Before introducing the results in the above paper, we first introduce the important signals used in the results:

- $\mathcal{W}_j$  denotes the set of all in-coming internal signals of  $w_j(t)$  through unknown modules;
- $\mathcal{X}_j$  denotes all external signals that do not have unknown directed edge to  $w_j$ .

Thus,  $\mathcal{X}_j$  also contains all external signals that do not have directed edges to  $w_j$  as the corresponding transfer functions are zero. Then the graphical conditions in (Weerts et al., 2018c) are collected here<sup>3</sup>:

*Theorem 2.* Given a network model set  $\mathcal{M}$  and its graph  $\mathcal{G}$ , the following conditions are equivalent:

- (1) module  $G_{ji}$  of  $\mathcal{M}$  is generically identifiable;
- (2)  $b_{\mathcal{X}_j \rightarrow w_j} = b_{\mathcal{X}_j \rightarrow \mathcal{W}_j \setminus \{w_i\}} + 1$ ;
- (3) there exists a set  $\mathcal{P}$  containing the maximum number of vertex disjoint paths from  $\mathcal{X}_j$  to  $\mathcal{W}_j \setminus \{w_i\}$  and also a directed path from  $\mathcal{X}_j$  to  $w_i$ , such that  $\mathcal{P}$  and the directed path are vertex disjoint. ■

Condition (2) does not appear explicitly in (Weerts et al., 2018c); however, condition (3) can be phrased equivalently as condition (2). Related graphic results can be found in (Hendrickx et al., 2019) for a different setting where all internal signals are excited and a subset of them is measured. For that same situation, a graphic simplification procedure is developed in (van Waarde et al., 2018) for testing global identifiability of a full network.

Theorem 2 is demonstrated in the following example.

*Example 1.* Given the network model set in Fig. 1(a), where the target module is  $G_{41}$  and all indicated transfer functions are unknown, we have  $\mathcal{W}_4 = \{w_1, w_2, w_3\}$  and  $\mathcal{X}_4 = \{r_1\}$  such that  $b_{\mathcal{X}_4 \rightarrow \mathcal{W}_4} = 1$  consisting of one path  $r_1 \rightarrow w_1$ . However, since  $b_{\mathcal{X}_4 \rightarrow \mathcal{W}_4 \setminus \{w_1\}}$  also consists of one path  $r_1 \rightarrow w_2$  and thus  $b_{\mathcal{X}_4 \rightarrow \mathcal{W}_4 \setminus \{w_1\}} = b_{\mathcal{X}_4 \rightarrow \mathcal{W}_4} = 1$ ,  $G_{41}$  is not generically identifiable based on Theorem 2.  $\triangle$

The above result allows to analyse generic identifiability for a given situation. However it does not address the synthesis problem: Given a network model set, how to allocate external signals such that module  $G_{ji}$  becomes generically identifiable? In this work, we assume that the user can allocate excitation signals. In the next sections, new analytic results are derived, which lead to several synthesis approaches.

<sup>2</sup> within the constraints of Assumption 1 and Lemma 1

<sup>3</sup> The following generic identifiability result holds under the assumption that all nonzero entries in  $G, R, H$  are fully parametrized, and that there are no fixed, non-parametrized modules unequal to zero. However the result also holds true when the non-parametrized modules are chosen “generically”, i.e. when they do not introduce any particular dependencies in the network model (Cheng et al., 2019b).

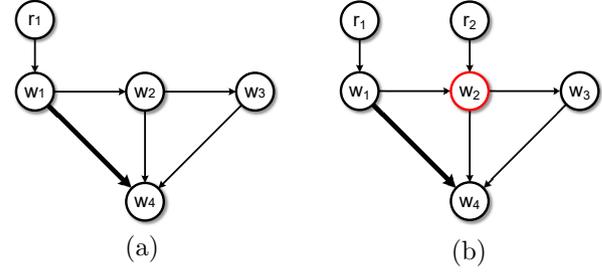


Fig. 1. Generic identifiability of  $G_{41}$  is considered (thick line).  $G_{41}$  is not generically identifiable in (a) but becomes generically identifiable in (b) if an extra signal  $w_2$  is allocated at  $w_2$ .

### 2.3 Notations and concepts

Given  $\bar{\mathcal{W}} \subseteq \mathcal{W}$  and  $\bar{\mathcal{X}} \subseteq \mathcal{X}$ ,  $T_{\bar{\mathcal{X}}\bar{\mathcal{X}}}$  denotes a submatrix of  $T$  with the rows and the columns corresponding to  $\bar{\mathcal{W}}$  and  $\bar{\mathcal{X}}$  respectively. Throughout the paper,  $w_i$  is used to denote both a signal and a vertex in  $\mathcal{G}$ . For any directed edge  $(w_i, w_j) \in \mathcal{E}$ ,  $w_i$  is called an in-neighbor of  $w_j$ , and  $w_j$  is called an out-neighbor of  $w_i$ . For any vertex in  $\mathcal{V}$ , e.g.  $w_i$ ,  $\mathcal{N}_{w_i}^+$  and  $\mathcal{N}_{w_i}^-$  denote the set of all out-neighbors and the set of all in-neighbors of  $w_i$ , respectively. A (directed) *walk* in  $\mathcal{G}$  from one vertex to another, e.g. from  $w_i$  to  $w_j$ , is a sequence of vertices and out-going edges starting from  $w_i$  to  $w_j$ , while a (directed) *path* is a walk without repeating any vertex. The *length* of a directed path is the number of edges in the path, and a single vertex is regarded as a directed path to itself with length zero. In addition, we use  $w_i \rightarrow w_j$  to denote a directed path from  $w_i$  to  $w_j$ , and  $\mathcal{V}_1 \rightarrow \mathcal{V}_2$  denotes paths from some vertices in  $\mathcal{V}_1$  to some vertices in  $\mathcal{V}_2$ . We refer to *internal vertices* as the vertices in a path excluding the starting and the ending vertices, and note that a directed path with length zero or length one does not have any internal vertex.

Two directed paths are *internally vertex disjoint* if they do not share any internal vertex, while they are called *vertex disjoint* if they do not share any vertex, including the starting and ending vertices. If two paths share any common vertex, we say that they *intersect*. Given two subsets of vertices  $\mathcal{V}_1$  and  $\mathcal{V}_2$ ,  $b_{\mathcal{V}_1 \rightarrow \mathcal{V}_2}$  denotes the maximum number of vertex disjoint paths from  $\mathcal{V}_1$  to  $\mathcal{V}_2$ .

## 3. DISCONNECTING SETS FOR GENERIC IDENTIFIABILITY

In this section, we derive a new analytic result based on the concept of disconnecting sets, which shows to be more suitable for synthesis. A vertex set  $\mathcal{D}$  is a disconnecting set from a vertex set  $\mathcal{V}_1$  to a set  $\mathcal{V}_2$  if there is no directed path from  $\mathcal{V}_1$  to  $\mathcal{V}_2$  when the vertices in  $\mathcal{D}$  are removed (Schrijver, 2003). Note that a disconnecting set from  $\mathcal{V}_1$  to  $\mathcal{V}_2$  may also include vertices in  $\mathcal{V}_1 \cup \mathcal{V}_2$ . We call a disconnecting set with the minimum cardinality a *minimum disconnecting set*. The duality between vertex disjoint paths and disconnecting sets is explained in Menger’s theorem, which is also explored in (Hendrickx et al., 2019).

*Theorem 3.* (Menger’s theorem (Schrijver, 2003)). Let  $\mathcal{V}_1, \mathcal{V}_2$  be two subsets of the vertices in a directed graph. The maximum number of vertex disjoint paths from  $\mathcal{V}_1$  to  $\mathcal{V}_2$

equals the cardinality of a minimum disconnecting set from  $\mathcal{V}_1$  to  $\mathcal{V}_2$ . ■

We illustrate the relevance of the concept of disconnecting sets to our synthesis problem in the following example.

*Example 2.* Given the network model in Fig. 1(a),  $\{w_2\}$  is a disconnecting set from  $\{w_1\}$  to the other in-neighbors of  $w_4$ , i.e.,  $\{w_2, w_3\}$ . Now, we allocate an extra excitation signal  $r_2$  at  $w_2$ , as shown in Fig. 1(b), and we find that  $b_{\mathcal{X}_4 \rightarrow \mathcal{W}_4} > b_{\mathcal{X}_4 \rightarrow \mathcal{W}_4 \setminus \{w_1\}}$ , which implies that  $G_{41}$  is generically identifiable according to Theorem 2. △

In Example 2, generic identifiability of  $G_{ji}$  is achieved when the vertices in a disconnecting set from  $\{w_i\}$  to the other in-neighbors of  $w_j$  are excited. In the following, we prove that this result holds for the general case. Before proceeding, a graphical result is provided.

*Lemma 4.* In a simple directed graph, given a set  $\mathcal{P}$  of vertex disjoint paths from vertex set  $\mathcal{V}_1$  to a vertex set  $\mathcal{V}_2$ , there exists a set  $\mathcal{P}_{new}$  of vertex disjoint paths from  $\mathcal{V}_1$  to  $\mathcal{V}_2$  such that  $|\mathcal{P}_{new}| = |\mathcal{P}|$ , and paths in  $\mathcal{P}_{new}$  are internally vertex disjoint with  $\mathcal{V}$ , where  $\mathcal{V}$  can be either  $\mathcal{V}_1$ ,  $\mathcal{V}_2$  or  $\mathcal{V}_1 \cup \mathcal{V}_2$ .

The above graphical result will later be used to derive the synthesis approach in Algorithm 1. Moreover, based on Lemma 4, a new analytic result for generic identifiability of a single module can be derived.

*Theorem 5.* Given atork model set  $\mathcal{M}$  and igrph $\mathcal{G}$ , module  $G_{ji}$  is generically identifiable in  $\mathcal{M}$  if and only if there exists a set of external signals  $\bar{\mathcal{X}}_j \subseteq \mathcal{X}_j$  and a disconnecting set  $\mathcal{D}$  from  $\{w_i\} \cup \bar{\mathcal{X}}_j$  to  $\mathcal{W}_j \setminus \{w_i\}$  such that

$$b_{\bar{\mathcal{X}}_j \rightarrow \mathcal{D} \cup \{w_i\}} = |\mathcal{D}| + 1. \quad (3)$$

In the above result, a disconnecting set  $\mathcal{D}$  from  $\{w_i\} \cup \bar{\mathcal{X}}_j$  to  $\mathcal{W}_j \setminus \{w_i\}$  always exists, and one simple example is  $\mathcal{W}_j \setminus \{w_i\}$ . However, generic identifiability of  $G_{ji}$  requires the existence of a special disconnecting set which satisfies equation (3). In addition, note that  $\mathcal{D}$  and  $\mathcal{W}_j \setminus \{w_i\}$  are not necessarily disjoint, while  $\{w_i\}$  and  $\mathcal{D}$  have to be *disjoint* to satisfy equation (3).

Theorem 5 can be visualized in Fig. 2, where the paths from  $w_i$  and  $\bar{\mathcal{X}}_j$  to  $\mathcal{W}_j \setminus \{w_i\}$  intersect with  $\mathcal{D}$ . Due to (3), a set of vertex disjoint paths exist that consists of a path  $\bar{\mathcal{X}}_j \rightarrow w_i$  and multiple paths  $\bar{\mathcal{X}}_j \rightarrow \mathcal{D}$ . As shown in Fig. 2, we can further concatenate  $\bar{\mathcal{X}}_j \rightarrow \mathcal{D}$  and  $\mathcal{D} \rightarrow \mathcal{W}_j \setminus \{w_i\}$ , which leads to a set of paths  $\mathcal{P}$  from  $\bar{\mathcal{X}}_j$  to  $\mathcal{W}_j \setminus \{w_i\}$ , that are vertex disjoint with the path  $\bar{\mathcal{X}}_j \rightarrow w_i$ . Then based on condition (3) of Theorem 2,  $G_{ji}$  is generically identifiable.

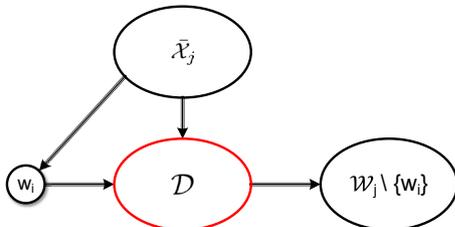


Fig. 2. A graphical visualization of Theorem 5.

Theorem 5 is also closely related to the existing identification method. The considered signals in the disconnecting

set can be shown to be the selected signals for the identification of  $G_{ji}$  using the direct method (Dankers et al., 2015). There, the internal signals selected in the predictor are the ones that intersect with all loops around the output  $w_j$  and all parallel paths from  $w_i$  to  $w_j$ , where the parallel paths mean all the paths except the edge from  $w_i$  to  $w_j$ .

*Proposition 6.* Given  $\mathcal{G}$  of a network model set where all non-zero modules are unknown, consider the module  $G_{ji}$  and a set of internal signals  $\mathcal{D}$  with  $w_i \notin \mathcal{D}$ . Then  $\mathcal{D}$  disconnects from  $w_i$  to  $\mathcal{W}_j \setminus \{w_i\}$  if and only if  $\mathcal{D}$  contains an internal vertex of every parallel path from  $w_i$  to  $w_j$  and a vertex of every loop around  $w_j$ .

The above result shows that the identifiability result in this work and the identification method in (Dankers et al., 2015) coincide from the aspect of signal selection.

*Remark 1.* A suggestion that results from Theorem 5 is that  $G_{ji}$  could be generically identifiable from measured internal signals  $\mathcal{D} \cup \{w_i, w_j\}$  only. This is indeed true, as it can be shown that with  $\mathcal{U} \triangleq \{w_i\} \cup \mathcal{D}$ , and in the case that  $\bar{\mathcal{X}}_j$  contains only excitation signals, a consistent estimate of  $G_{ji}$  can be obtained using the estimated submatrices  $\hat{T}_{j\mathcal{U}}$  and  $\hat{T}_{\mathcal{U}\bar{\mathcal{X}}_j}$  of  $T$ . This is a direct generalization of the indirect method presented in (Gevers et al., 2018) and extended in (Bazanella et al., 2019), where typically all the in-neighbors of  $w_j$  are used instead of  $\mathcal{U}$ . The formal proof of this result will be presented in future work. △

#### 4. SYNTHESIS APPROACH

Given a network model set with a set of initial external signals  $\mathcal{X}_j^0$  that do not have unknown directed edges to  $w_j$ , the synthesis problem aims to allocate a minimum number of additional excitation signals  $\mathcal{X}_j^a$  such that generic identifiability of a module  $G_{ji}$  is guaranteed. For synthesis, it is assumed that if  $r_k$  is allocated directly at  $w_j$ , its corresponding transfer function  $R_{jk}$  is known.

Based on Theorem 5, the main idea of the synthesis approaches is to first compute a disconnecting set  $\mathcal{D}$ , and then allocate external signals at suitable vertices such that (3) is satisfied.

To reduce the number of allocated excitation signals, a minimum disconnecting set should be used for synthesis; additionally, recall from Theorem 5 that  $w_i \notin \mathcal{D}$  is necessary for (3). This implies that a minimum disconnecting set  $\mathcal{D}$  subject to  $w_i \notin \mathcal{D}$  needs to be found. As standard graphical algorithms for computing minimum disconnecting sets do not take into account any constraint, we redefine the disconnecting set to make standard algorithms applicable.

*Lemma 7.* Given a network model set  $\mathcal{M}$  and its graph  $\mathcal{G}$ , for any subset  $\bar{\mathcal{X}}_j \subseteq \mathcal{X}_j$ , a minimum disconnecting set from  $\mathcal{N}_{w_i}^+ \cup \bar{\mathcal{X}}_j$  to  $\mathcal{W}_j \setminus \{w_i\}$  is a minimum disconnecting set  $\mathcal{D}$  from  $\{w_i\} \cup \bar{\mathcal{X}}_j$  to  $\mathcal{W}_j \setminus \{w_i\}$  subject to  $w_i \notin \mathcal{D}$ . ■

Based on the above approach, a minimum disconnecting set  $\mathcal{D}$  from  $\mathcal{N}_{w_i}^+ \cup \bar{\mathcal{X}}_j$  to  $\mathcal{W}_j \setminus \{w_i\}$  can be computed for the synthesis problem, which is an unconstrained problem and thus can be solved by standard graphic algorithms, e.g. the Ford-Fulkerson algorithm (Schrijver, 2003). Then based on the above lemma and Theorem 5, the following synthesis approach can be derived.

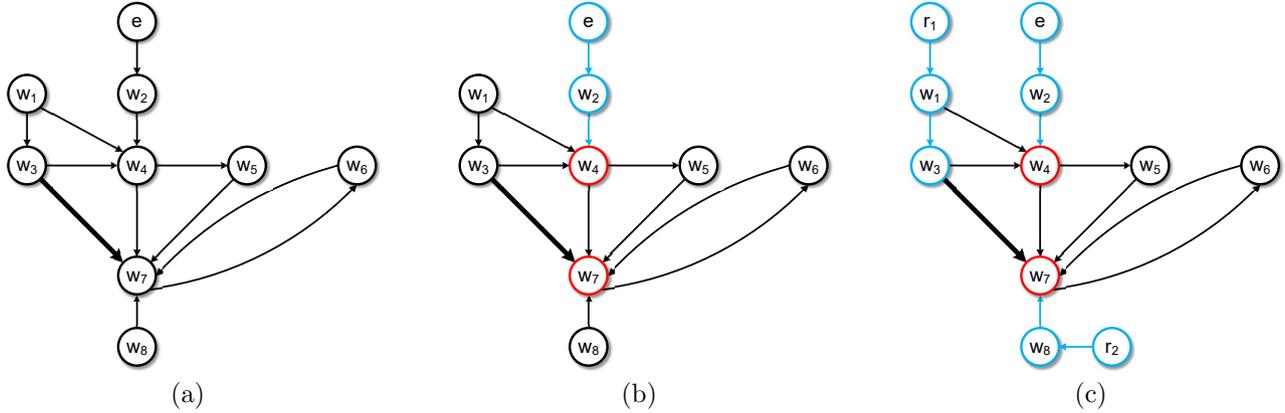


Fig. 3. An example of allocating signals for generic identifiability of  $G_{73}$  (thick line) using Algorithm 1. Starting from the network model in (a), a disconnecting set (red vertices) is computed in (b). Since there already exists an external signal  $e$ , we only need to add  $r_1$  and  $r_2$  as in (c), which achieves generic identifiability of  $G_{73}$ .

**Lemma 8.** In a network model set  $\mathcal{M}$  with its graph  $\mathcal{G}$ , given any minimum disconnecting set  $\mathcal{D}$  from  $\mathcal{N}_{w_i}^+$  to  $\mathcal{W}_j \setminus \{w_i\}$ , assigning distinct excitation signals to every vertex in  $\mathcal{D} \cup \{w_i\}$  leads to generic identifiability of  $G_{ji}$  in  $\mathcal{M}$ . ■

The above result leads to a synthesis approach with simple implementation. However, the method is strictly local in the sense that all excitation signals are directly allocated at  $w_i$  and  $\mathcal{D}$ . In addition, the method does not take into account the initially present external signals  $\mathcal{X}_j^0$ ; however, these signals can also be useful to satisfy equation (3) and thus reduce the number of required allocated signals. Therefore, we present a more comprehensive synthesis approach in Algorithm 1, which makes use of  $\mathcal{X}_j^0$  and explores the freedom of allocating additional excitation signals in the graph. Note that  $\mathcal{X}_j^0$  can contain both excitation signals and noise signals.

Given a network model set with a target module  $G_{ji}$  and a set of pre-existing signals  $\mathcal{X}_j^0$ , Algorithm 1 first computes a minimum disconnecting set  $\mathcal{D}$  from  $\{w_i\} \cup \mathcal{X}_j^0$  to  $\mathcal{W}_j \setminus \{w_i\}$ , and then removes the elements in  $\mathcal{D} \cup \{w_i\}$  that are already excited by the existing external signals through vertex disjoint paths. Then it allocates additional signals to excite the remaining vertices in  $\mathcal{D} \cup \{w_i\}$  through vertex disjoint paths. The validity of Algorithm 1 is shown as follows.

**Theorem 9.** In the returned  $\mathcal{M}_{out}$  with the graph  $\mathcal{G}_{out}$  from Algorithm 1, the target module  $G_{ji}$  is generically identifiable. ■

We demonstrate Algorithm 1 by the following example.

**Example 3.** In the network model in Fig. 3(a), the problem is to allocate excitation signals such that  $G_{73}$  becomes generically identifiable.  $\mathcal{X}_7^0 = \{e\}$  is the only external signal that is initially present. Firstly, a disconnecting set from  $\mathcal{X}_7^0 \cup \{w_3\} = \{w_3, e\}$  to the other in-neighbors of  $w_7$ , i.e.  $\mathcal{W}_7 \setminus \{w_3\} = \{w_4, w_5, w_6, w_8\}$ , is constructed as  $\mathcal{D} = \{w_4, w_7\}$ , indicated in Fig. 3(b). Based on Theorem 5, generic identifiability of  $G_{73}$  requires three vertex disjoint paths from external signals to  $\mathcal{D} \cup \{w_3\} = \{w_3, w_4, w_7\}$ , while  $\mathcal{D}$  remains a disconnecting set from the external signals to  $\mathcal{W}_7 \setminus \{w_3\}$ . Following step 2 in Algorithm 1, we find a path  $e \rightarrow w_4$  from  $\mathcal{X}_7^0$  to  $\mathcal{D} \cup \{w_3\}$  (colored blue in Fig. 3(b)). Thus we only need to allocate extra

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#### Algorithm 1 Signal allocation for a single module

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**INPUT:** A network model set  $\mathcal{M}$  with graph  $\mathcal{G}$ , a target module  $G_{ji}$ , and a set of initial external signals  $\mathcal{X}_j^0$  for  $G_{ji}$ ;

**OUTPUT:** A new model set  $\mathcal{M}_{out}$  with its graph  $\mathcal{G}_{out}$

- 1: Compute a minimum disconnecting set  $\mathcal{D}$  from  $\mathcal{N}_{w_i}^+ \cup \mathcal{X}_j^0$  to  $\mathcal{W}_j \setminus \{w_i\}$ ;
  - 2: Based on Lemma 4, compute a set  $\mathcal{P}$  containing the maximum number of vertex disjoint paths from  $\mathcal{X}_j^0$  to  $\mathcal{D} \cup \{w_i\}$  while the paths are internally vertex disjoint with  $\mathcal{D} \cup \{w_i\}$ ;
  - 3: Let  $\bar{\mathcal{D}} \subseteq \mathcal{D} \cup \{w_i\}$  denote all the ending vertices of the paths in  $\mathcal{P}$ ;
  - 4: **if**  $|\mathcal{P}| < |\mathcal{D}| + 1$  **then**
  - 5: Find the set  $\bar{\mathcal{W}} \subseteq \mathcal{W}$  such that  $\bar{\mathcal{D}}$  is a disconnecting set from  $\bar{\mathcal{W}}$  to  $\mathcal{W}_j \setminus \{w_i\}$ ;
  - 6: Build a subgraph  $\bar{\mathcal{G}} \subseteq \mathcal{G}$  by removing all vertices in  $\bar{\mathcal{P}}$  and their corresponding edges;
  - 7: Find the set  $\mathcal{W}_{exp} \subseteq \bar{\mathcal{W}}$  such that in  $\bar{\mathcal{G}}$ , there are  $|\mathcal{D}| + 1 - |\mathcal{P}|$  vertex disjoint paths from  $\mathcal{W}_{exp}$  to  $(\mathcal{D} \cup \{w_i\}) \setminus \bar{\mathcal{D}}$ ;
  - 8: In  $\bar{\mathcal{G}}$ , assign distinct excitation signals to every vertex in  $\mathcal{W}_{exp}$ , which leads to a new model set  $\mathcal{M}_{out}$  with a new graph  $\mathcal{G}_{out}$ ;
  - 9: Return  $\mathcal{M}_{out}$  with the graph  $\mathcal{G}_{out}$ ;
  - 10: **else**
  - 11:  $\mathcal{M}_{out} \leftarrow \mathcal{M}$  and  $\mathcal{G}_{out} \leftarrow \mathcal{G}$ ;
  - 12: Return  $\mathcal{M}_{out}$  with the graph  $\mathcal{G}_{out}$ ;
  - 13: **end if**
- 

excitation signals to create two vertex disjoint paths to  $\{w_3, w_7\}$ , which should be vertex disjoint with  $e \rightarrow w_4$ . As in step 5, the potential locations to allocate excitation signals is  $\bar{\mathcal{W}} = \{w_1, w_2, w_3, w_4, w_7, w_8\}$ , which satisfies that  $\bar{\mathcal{D}}$  remains a disconnecting set from  $\bar{\mathcal{W}}$  to  $\mathcal{W}_7 \setminus \{w_3\}$ . After removing  $e \rightarrow w_4$ , we choose  $\mathcal{W}_{exp} = \{w_1, w_8\} \subseteq \bar{\mathcal{W}}$  to be excited. Now there are two vertex disjoint paths from  $\mathcal{W}_{exp}$  to  $\{w_3, w_7\}$ , and the paths are also vertex disjoint with  $e \rightarrow w_4$ , as indicated by the blue paths in Fig. 3(c). Then  $G_{73}$  is generically identifiable.  $\triangle$

Even if Algorithm 1 requires fewer signals than the approach in Lemma 8, there is no guarantee for the minimal number of additional signals when  $\mathcal{X}_j^0 \neq \emptyset$ . However, since

the number of allocated signals in Algorithm 1 is  $|\mathcal{D}| + 1 - b_{\mathcal{X}_j^0 \rightarrow \mathcal{D} \cup \{w_i\}}$ , the following bound on the number of additional signals required can be derived.

*Corollary 10.* Given a network model set  $\mathcal{M}$  with its graph  $\mathcal{G}$ . Consider a set of initial external signals  $\mathcal{X}_j^0$  that have no unknown directed edge to  $w_j$ . Let  $\mathcal{D}$  be a minimum disconnecting set from  $\mathcal{N}_{w_i}^+ \cup \mathcal{X}_j^0$  to  $\mathcal{W}_j \setminus \{w_i\}$ , and  $c$  denote the number of excitation signals available for allocation. The number of additional excitation signals  $c$  is sufficient to make  $G_{ji}$  generically identifiable if

$$c \geq |\mathcal{D}| + 1 - b_{\mathcal{X}_j^0 \rightarrow \mathcal{D} \cup \{w_i\}}.$$

The presented methods can also handle the presence of known modules, which affects the construction of  $\mathcal{W}_j$ . The consequence of known modules is shown in this example.

*Example 4.* Consider a network model set with the same topology as the one in Fig. 1,  $G_{43}$  and  $G_{42}$  are known in the model set, as shown in Fig. 4. Based on the formulation of  $\mathcal{W}_4$  in Theorem 2,  $w_3$  and  $w_2$  do not belong to set  $\mathcal{W}_4$  now, as we have  $\mathcal{W}_4 = \{w_1\}$  in Fig. 4. Thus, in this example, the minimum disconnecting set is an empty set, and generic identifiability of  $G_{41}$  can be achieved by a single excitation on  $w_1$ , compared to two excitation signals in Fig. 1(b). ■

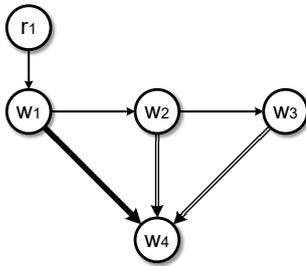


Fig. 4. Generic identifiability of  $G_{41}$  with known modules (double-line arrows) can be achieved by only  $r_1$ .

## 5. CONCLUSION

The synthesis problem to achieve generic identifiability of one module in a dynamic network by allocating excitation signals is addressed in this work. A new condition is derived based on the concept of disconnecting set, and then synthesis approaches are developed to allocated excitation signals at the vertices in the disconnecting set. It has been shown that generic identifiability of a single module can be guaranteed by the synthesis approaches.

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