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Graphical conditions for network identifiability of a single module with unmeasured input or output

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***Abstract*— Conditions for network identifiability of a single module in a dynamic network typically concern the presence and locations of external signals, the correlation structure of noises and the selection of network signals that can be measured. In a generic setting, graphical conditions can be formulated on the graph of the considered model set. The existing results in the literature require the input and the output of the target module to be measured. In this paper, the conditions for generic identifiability are generalized to the case where the input or the output of the target module cannot be measured. The analysis is based on the concepts of disconnecting sets and vertex disjoint paths.**

I. INTRODUCTION

Due to the increasing complexity of current technological systems, the study of large-scale interconnected dynamic systems receives considerable attention recently. As a modeling framework for dynamic networks, we consider the network of transfer functions introduced in [1], [2], where vertices represent the internal signals, and directed edges denote transfer functions which are called modules.

Besides the literature considering identification and identifiability of a full network [3]–[6], identifiability and identification of a single module in a dynamic network has also been the subject of much recent attention. To identify a module, typically given a network model set with a certain network topology, one of the main steps is to choose an appropriate predictor model and thus a set of measured signals and excitation signals, such that the target module can be consistently estimated. The difference between identifiability and identification analysis, as noted by [7], is that the latter typically requires informative signals, which can only be achieved if structural conditions in identifiability analysis are guaranteed.

Identification of a single module has been addressed in [2] and [7], where a multiple-input-single-output (MISO) approach is taken. In the MISO model which contains the target module, besides the measured output, all its in-neighbors are also measured. An important step is made in [8], which shows that instead of using all the inputs of the MISO model, different subsets of measured signals can be sufficient for consistent identification of the target module. Several extensions of [8] are made in [9]–[11] to further increase the freedom of selecting the signals to be measured. In [9], [10] this includes even an extension to allowing non-proper modules. The work in [12], [13] extends the d-separation criterion from the field of probabilistic graphical models to select signals for identification, where non-proper modules are also allowed. However, the connection between d-separation and the result in [8] has not yet been fully formalized in the literature. For the so-called direct method of identification, using node signals as predictor inputs, an extension to deal with correlated disturbances in the network is provided in [14]. For indirect methods, using external excitation signals as predictor inputs, analysis of the problem is provided in [5], [7], [15], [16], where, due to the nature of the identification method, a close resemblance occurs with the property of identifiability.

Algebraic conditions for identifiability of dynamic networks have been developed in [4], [17], where all internal signals are assumed to be measured but only a subset of them is excited. In [5], [15] the step was made to consider identifiability in a generic sense, leading to path-based conditions based on the network structure, and covering single module identifiability as a special case. The problem of single module generic identifiability has been further addressed in [18] and [16], while in [19] attention has been paid to the synthesis problem, i.e. the allocation of external excitation signals for generic identifiability of a full network. To allocate excitation signals for single module identifiability, the path-based conditions in [18] have later been extended to vertex-based graphical conditions in [20].

While in the current literature different measurement and excitation schemes have been analysed, almost all contributions require measuring both the input and the output of the target module. The only work going beyond this setting is [16], where generic identifiability of a single module is considered, by providing sufficient conditions in the form of a combination of path-based and algebraic conditions. The latter conditions can be tested explicitly on the network structure only in particularly formulated special cases. In the current paper we follow a related setting, and consider the situation of an unmeasured input or output, while a subset of node signals is measured and a subset of signals is excited. Our target is to formulate explicitly verifiable graph-based conditions for generic identifiability of a single module.

We extend the result in [20] where single module identifiability has been analyzed in a more restrictive setting requiring all signals to be measured. The situation with unmeasured input or unmeasured output is analyzed subsequently in Sections III and IV. Sufficient graphical conditions are derived based on the concepts of vertex disjoint paths and

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disconnecting sets, and thus they can be tested by inspecting the structure of the network model set only. The derived conditions also show potential for solving the synthesis problem, i.e. for deciding which signals to measure and to excite such that a module is generically identifiable.

II. Problem formulation

The dynamic network model describes the relationship among *internal signals* $\mathcal{W} \triangleq \{w_1(t), \cdots, w_L(t)\}$, a vector of measured excitation signals $r(t)$, noise signals $\{v_1(t), \cdots, v_L(t)\}$, a vector of measured output $y(t)$:

$$\begin{aligned} w(t) &= G(q)w(t) + Rr(t) + v(t), \\ y(t) &= Cw(t), \end{aligned} \tag{1}$$

where R and C are selection matrices, i.e. they have exactly one entry of 1 in each column and each row, and have zeros elsewhere. Note that they can be non-square depending on the dimension of $r(t)$ and $y(t)$. Thus $y(t)$ contains the measured internal signals, and a distinct excitation signal corresponds to a distinct internal signal. $G(q)$ is a matrix of transfer operators with delay operator q^{-1}, i.e. $q^{-1}w_i(t) = w_i(t-1)$. $v(t)$ is a vector of umeansured stationary stochastic processes with power spectrum $\Phi_v(w)$, which is modeled as filtered unmeasured white noise $e(t)$:

$$v(t) = H(q)e(t), \tag{2}$$

where $e(t)$ has a positive definite covariance matrix Λ. Depending on if $\Phi_v(w)$ is full rank, $H(q)$ can be either square or has more rows than columns [4].

Combining (1) and (2) leads to the complete dynamic network model, and the matrices $G(q)$ and $H(q)$ satisfy the following assumptions:

Assumptions:

- $G(q)$ has zero diagonal elements, and $G(q)$ is proper and stable;
- The network is well-posed in the sense that all principal minors of $\lim_{z\to\infty}(I - G(z))$ are non-zero;
- $(I - G(q)^{-1}$ is stable;
- $H(q)$ is minimum phase and monic if square, and the assumptions for the non-square case, i.e. $\Phi_v(w)$ is singular, can be found in [4].

Both excitation signals $r(t)$ and white noise signals $e(t)$ are called *external signals*, and they are collected into a set $\mathcal{X}$. *Modules* are used to refer to the transfer functions in $G(q)$. The dynamic network model also leads to mappings from external signals to internal signals:

$$X(q) \triangleq \begin{bmatrix} R & H(q) \end{bmatrix}, T_{\mathcal{W}\mathcal{X}}(q) \triangleq (I - G(q))^{-1}X(q).$$

Note that the assumption on stability of $G(q)$ is to ensure an invertably stable noise model $(I - G(q))^{-1}H(q)$ of the external-to-internal model.

Structural information of the network model is used in this work to analyze identifiability. The structure of a network model can be represented by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} \triangleq \mathcal{W} \cup \mathcal{X}$ is a set of vertices representing the internal signals and the external signals, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes a set of directed edges representing the non-zero entries in $(G(q), R, H(q))$, e.g. an directed edge $w_i \to w_j$ exists if and only if entry $G_{ji}(q)$ is not zero. The external signals are also represented by vertices as $e_l(t)$ may influence multiple internal signals. $y(t)$ is not considered in the graph because it contains a subset of internal signals, and thus different color will be used to indicate which internal signals are measured, e.g. in Fig. 1.

The network model (1) and (2) is completely specified by a quadruple $(G(q), R, H(q), C, \Lambda)$. Identifiability concerns the uniqueness of $(G(q), R, H(q), C, \Lambda)$ given $CT_{\mathcal{W}\mathcal{X}}(q)$ and $\Phi_v(w)$ [4], where $CT_{\mathcal{W}\mathcal{X}}(q)$ and $\Phi_v(w)$ can typically be estimated from the data of $w(t)$, $y(t)$ and $r(t)$ under certain conditions using an open-loop identification [21]. The uniqueness of the covariance matrix Λ can be guaranteed by restrictions on the feedthrough terms of the model [4]. It is assumed that the model in this work satisfies these assumptions, and thus only the uniqueness related to $(G(q), R, H(q), C)$ given $CT_{\mathcal{W}\mathcal{X}}(q)$ is considered here.

Then the definition of identifiability can be formalized as follows. As identifiability is a property of a model set and assuming no prior knowledge on the non-zero entries of $G(q)$ and $H(q)$, define a model set $\mathcal{M}$ by a rational parameterization of all non-zero entries in $G(q)$ and $H(q)$:

$$\mathcal{M} \triangleq \{(G(q,\theta), R, H(q,\theta), C) | \theta \in \Theta \subseteq \mathbb{R}^n\},$$

where R and C are known as they only contain ones and zeros, and their structure is given. Now, the identifiability concept from [4] is combined with the generic notion from [15] [5] to define single module generic identifiability as follows:

Definition 1 *$G_{ji}(q,\theta)$ is generically identifiable in $\mathcal{M}$ if it holds that*

$$CT_{\mathcal{W}\mathcal{X}}(q,\theta_1) = CT_{\mathcal{W}\mathcal{X}}(q,\theta_2) \implies G_{ji}(q,\theta_1) = G_{ji}(q,\theta_2),$$

for almost all θ_1 and θ_2 in Θ.

The notation *almost all* means excluding a set of Lebesgue measure zero from Θ. Note that the effect of C matrix is to extract a subset of rows of $T_{\mathcal{W}\mathcal{X}}(q,\theta)$, depending on which internal signals are measured, and the R matrix, which is incorporated into $CT_{\mathcal{W}\mathcal{X}}(q,\theta)$, shows which signals are excited.

This paper develops graphical conditions on $\mathcal{G}$ of $\mathcal{M}$ such that module $G_{ji}(q,\theta)$ is generically identifiable when its input or output is not measured. The problems with unmeasured input or unmeasured output are addressed respectively. Note that all proofs of the results are collected in the Appendix.

A. Notations and definitions

The following notations are used throughout the paper. $w_i(t)$ denotes both a signal and a vertex in $\mathcal{G}$, and its dependency on t is often omitted. Similarly dependency of transfer matrices, e.g. $T_{\mathcal{W}\mathcal{X}}(q,\theta)$, on θ and q is also omitted and is written as $T_{\mathcal{W}\mathcal{X}}$. Subset $\mathcal{X}_j \subseteq \mathcal{X}$ is the set of all external signals excluding the white noises having directed edges to w_j. Subset $\mathcal{W}_j \subseteq \mathcal{W}$ denotes all internal signals having directed edges to $w_j(t)$, and $\bar{\mathcal{W}}_j \subseteq \mathcal{W}_j$ denotes the elements in $\mathcal{W}_j$ which have unknown edges (modules) to $w_j(t)$. Under the assumption of this work, $\bar{\mathcal{W}}_j = \mathcal{W}_j$ holds. Given two subsets $\bar{\mathcal{W}} \subseteq \mathcal{W}$ and $\bar{\mathcal{X}} \subseteq \mathcal{X}$, notation $T_{\bar{\mathcal{W}}\bar{\mathcal{X}}}$ denotes a submatrix of $T_{\mathcal{W}\mathcal{X}}$ with the rows and columns corresponding to signals in $\bar{\mathcal{W}}$ and $\bar{\mathcal{X}}$. If $\bar{\mathcal{W}}$ contains only one signal w_k, $T_{\bar{\mathcal{W}}\bar{\mathcal{X}}}$ is simply wrriten as $T_{k\bar{\mathcal{X}}}$. The above notation applies similarly to submatrices of other matrices and vectors.

In graph $\mathcal{G}$, for any directed edge $w_i \to w_j$, w_i is called an *in-neighbor* of w_j, and w_j is an *out-neighbor* of w_i. A (directed) *path* from w_i to w_j is a sequence of vertices and out-going edges starting from w_i to w_j without repeating any vertex. A single vertex is also regarded as a directed path to itself. *Internal vertices* are the vertices in a path excluding the starting and the ending vertices.

Two directed paths are called *vertex disjoint* if they do not share any vertex, including the starting and ending vertices, otherwise they *intersect*. Given two subsets of vertices $\mathcal{V}_1$ and $\mathcal{V}_2$, $b_{\mathcal{V}_1 \to \mathcal{V}_2}$ denotes the maximum number of vertex disjoint paths from $\mathcal{V}_1$ to $\mathcal{V}_2$. $\mathcal{V}_1$ is said *linked to* $\mathcal{V}_2$ if $|\mathcal{V}_1| = |\mathcal{V}_2|$ and $b_{\mathcal{V}_1 \to \mathcal{V}_2} = |\mathcal{V}_1|$. A vertex set $\mathcal{D}$ is a *disconnecting set* from $\mathcal{V}_1$ to $\mathcal{V}_2$ if it intersects with all paths from $\mathcal{V}_1$ to $\mathcal{V}_2$ [22], also written as a $\mathcal{V}_1 - \mathcal{V}_2$ disconnecting set. We also say that $\mathcal{D}$ disconnects from $\mathcal{V}_1$ to $\mathcal{V}_2$. Note that $\mathcal{D}$ may also include vertices in $\mathcal{V}_1 \cup \mathcal{V}_2$.

III. Identifiability with unmeasured input

Generic identifiability of G_{ji} with unmeasured input but measured output is considered in this section. The identifiability result is motivated by the following example.

Example 1 Consider the network in Fig. 1a where identifiability of G_{21} is of interest while w_1 is unmeasured. The mapping from r_1 to w_2 can be computed and equals G_{21}, and thus G_{21} can be uniquely recovered from the available external-to-internal mapping. In Fig. 1b there is a

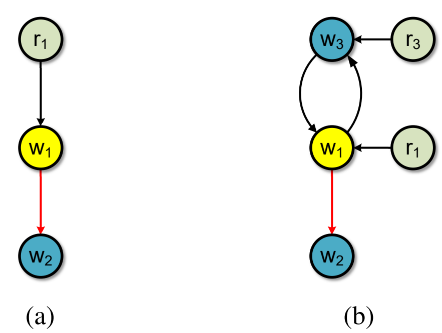


Fig. 1. Two networks with G_{21} (red) as the target module and w_1 (yellow) unmeasured. G_{21} is generically identifiable in both cases under measured internal signals (blue) and excitation signals (green).

loop around w_1, the mapping from r_1 to w_2 is

$$T_{w_2 r_1} = \frac{G_{21}}{1 - G_{13}G_{31}},$$

and thus G_{21} cannot be recovered by $T_{w_2 r_1}$ alone as in Fig. 1(a). However, the loop transfer can be found as

$$T_{w_3 r_3} = \frac{1}{1 - G_{13}G_{31}},$$

and thus G_{21} can be recovered uniquely from

$$T_{w_2 r_1}(T_{w_3 r_2})^{-1},$$

where both mappings are available because w_3 and w_2 are measured. ■

The above example leads to the following intuition: if w_i is the only in-coming internal signal of w_j and is unmeasured, w_j is measured and there is no loop around w_i, G_{ji} can be found by directly exciting w_i. When loops around w_i exist, signals on the loops should be excited and measured such that the loop transfers can be found, and then G_{ji} can be recovered uniquely. When w_j has multiple in-coming internal signals, based on [20], signals in a disconnecting set from w_i to $\mathcal{W}_j \setminus \{w_i\}$ should additionally be measured and excited. The above observation can be formalized as follows.

Theorem 1 *In a network model set $\mathcal{M}$ with its graph, consider G_{ji} and suppose that w_j is measured but w_i cannot be measured. Module G_{ji} is generically identifiable in $\mathcal{M}$ if w_i is directly excited by a excitation signal r_i, and there exists a subset $\bar{\mathcal{X}}_j \subseteq \mathcal{X}_j \setminus \{r_i\}$, and a vertex set $\mathcal{D}$ such that*

1) *all signals in $\mathcal{D}$ are measured;*
2) *$\mathcal{D}$ is a disconnecting set from $\bar{\mathcal{X}}_j \cup \{w_i\}$ to $\mathcal{W}_i \cup (\mathcal{W}_j \setminus \{w_i\})$, and $r_i \notin \mathcal{D}$;*
3) *$\bar{\mathcal{X}}_j \cup \{r_i\}$ is linked to $\mathcal{D} \cup \{w_i\}$.*

In condition (2) of the above theorem, a disconnecting set $\mathcal{D}$ from $\{w_i\}$ to $\mathcal{W}_i$ intersects all loops around w_i, and thus measuring and exciting signals in $\mathcal{D}$ as in condition (1) and (3) respectively matches the intuition from Example 1, where vertex w_3 on the loop around w_1 is excited. In addition, the requirement that $\mathcal{D}$ also disconnects from w_i to the other inputs $\mathcal{W}_j \setminus \{w_i\}$ of w_j is generalized from [20], which separates the target module and the other modules in the MISO model containing the target module. This guarantees the uniqueness of G_{ji} when w_j has multiple in-coming modules. Also note that $\bar{\mathcal{X}}_j$ in condition (3) includes noise signals which thus play the same role as excitation signals.

The fact that an unmeasured input requires excitation for identifiability is in agreement with earlier results [16]. It is shown in [20] how the path-based conditions that result from the work of [5], are related to the disconnecting set conditions in Theorem 1.

The existence conditions in Theorem 1 can be difficult to apply for an analysis problem, while the result is more suitable for a synthesis problem to allocate excitation signals and sensors such that a given setup becomes generically identifiable. The following synthesis approach is straightforward based on Theorem 1.

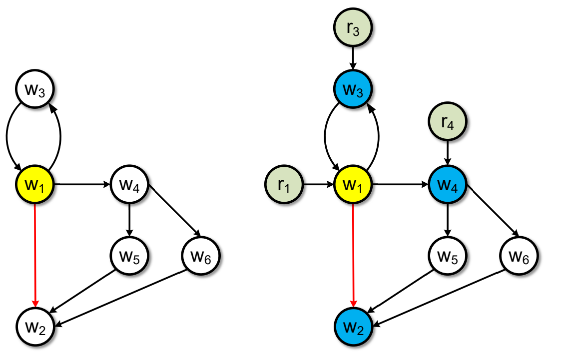
Corollary 2 *Consider module G_{ji} and assume that w_i cannot be measured. Let $\mathcal{D}$ be any disconnecting set from $\{w_i\}$ to*

$\mathcal{W}_i \cup (\mathcal{W}_j \setminus \{w_i\})$ *with* $w_i \notin \mathcal{D}$*.* G_{ji} *is generically identifiable in* $\mathcal{M}$ *if*

1) *a distinct excitation signal is directly assigned to each distinct element in* $\mathcal{D} \cup \{w_i\}$*;*
2) w_j *and signals in* $\mathcal{D}$ *are measured.*

An example of applying the above synthesis approach is shown next.

Example 2 Consider the network in Fig. 2a, where w_1 cannot be measured and generic identifiability of module G_{21} is of interest. To allocate sensors and excitation signals for



(a) (b)

Fig. 2. Sensors and excitation signals need to be allocated to achieve generic identifiability of G_{21} in (a), while its input w_1 cannot be measured. An allocation setup is achieved in (b) using Corollary 2 with the selected measured internal signals (blue).

the network in Fig. 2a, the set $\mathcal{W}_i \cup (\mathcal{W}_j \setminus \{w_i\})$ is now $\{w_3, w_5, w_6\}$, including all other in-neighbors of output w_2 and the in-neighbor of input w_1. Then a disconnecting set $\mathcal{D}$ in Corollary 2 from w_1 to $\{w_3, w_5, w_6\}$ can be chosen as $\{w_3, w_4\}$. Thus if the signals in $\mathcal{D} \cup \{w_2\}$ are measured and the ones in $\mathcal{D} \cup \{w_1\}$ are directly excited, G_{21} is generically identifiable. The resulting setup is shown in Fig. 2b. ■

IV. Identifiability with unmeasured output

This section considers generic identifiability of G_{ji} when w_j cannot be measured, while the input can be measured. Firstly, we consider the following example.

Example 3 In Fig. 3 there are two networks with target module G_{21} and its unmeasured output w_2. In Fig. 3a, G_{21} can be recovered by the available submatrices of $T_{\mathcal{W}\mathcal{X}}$ as follows:

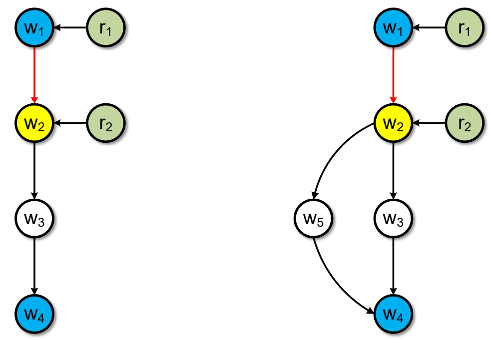
$$G_{21} = T_{4r_1} T_{4r_2}^{-1},$$

where $T_{4r_1} = G_{21}G_{32}G_{43}$, $T_{4r_2} = G_{32}G_{43}$ and they are both available because w_1 and w_4 are measured.

Even if the network in Fig. 3b has one extra path from w_1 to w_4 than the network in Fig. 3a, G_{21} can still be uniquely recovered using the same equation

$$G_{21} = T_{4r_1} T_{4r_2}^{-1},$$

where $T_{4r_1} = G_{21}(G_{32}G_{43} + G_{52}G_{45})$ and $T_{4r_2} = G_{32}G_{43} + G_{52}G_{45}$.



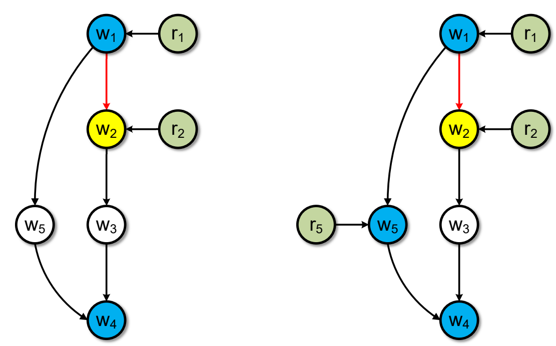
(a) (b)

Fig. 3. Networks with the target module G_{21} and output w_2 (yellow) cannot be unmeasured. G_{21} is generically identifiable in both (a) and (b).

The network in Fig. 4a also has an extra path from w_1 to w_4 compared to Fig. 3a, and the path does not intersect with the output w_2 of the target module. In this netowrk,

$$T_{4r_1} = G_{51}G_{54} + G_{21}G_{32}G_{43}, \quad T_{4r_2} = G_{32}G_{43},$$

where G_{21} cannot be uniquely found from T_{4r_1} and T_{4r_2} due to the extra unknown term $G_{51}G_{54}$. However, if w_5 is also measured and excited as in Fig. 4b, G_{21} becomes generically identifiable, as the term $G_{51}G_{54}$ in T_{4r_1} can be found from $T_{5r_1}T_{4r_5}$.



(a) (b)

Fig. 4. G_{21} is not generically identifiable in (a) but becomes generically identifiable if w_5 is excited and measured as in (b).

■

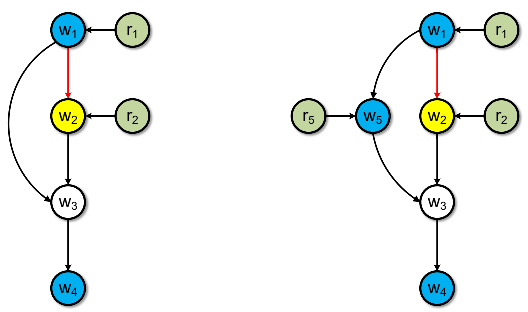
Example 3 shows that when the output is unmeasured but directly excited, G_{ji} may be identified using an additional measured signal w_k such that there exist paths from w_i to w_k via the target module. However, if there exist other extra paths from w_i to w_k which do not contain the target module as in Fig. 3b and Fig. 3c, internal vertices of these extra paths, e.g. vertex w_5 in Fig. 3c, need to be measured and excited, such that G_{ji} can be uniquely found.

Additionally, we consider a different example.

Example 4 In Fig. 5a, there exist two paths from w_1 to a measured vertex w_4, and thus the term T_{4r_1} is an addition of two terms:

$$T_{4r_1} = G_{31}G_{43} + G_{21}G_{32}G_{43}.$$

Based on the intuition from Example 3, a vertex on the extra path $w_1 \to w_3 \to w_4$ should be measured and excited such that the term $G_{31}G_{43}$ can be found, then G_{21} can be found as $(T_{4r_1} - G_{31}G_{43})T_{w_4r_2}^{-1}$. However, this is not possible in this example, as even if w_3 is measured and excited, the paths from r_1 to w_3 are not unique and thus the term $G_{31}G_{43}$ cannot be found.



(a) (b)

Fig. 5. Networks with G_{21} under interest and output w_2 (yellow) cannot be unmeasured. G_{21} cannot be generically identifiable in (a), while with one extra vertex w_5 in (b) G_{21} can be generically identifiable.

However, if there is one additional internal vertex w_5 which is measured and excited as in Fig. 5b, $G_{31}G_{43}$ can be computed as $T_{5r_1}T_{4r_5}$. As a result, G_{21} can be uniquely found given the mappings from the available external signals to the measured internal signals. ∎

The above example leads to some extra information. The vertices of the extra paths from w_i to w_k, which need to be excited and measured, are the internal vertices of their certain subpaths, such that these vertices do not intersect with the paths containing the target module. For example, in Fig. 5b, vertex w_5 does not intersect with path $w_1 \to w_2 \to w_3 \to w_4$, while it intersects with the extra path from w_1 to w_4. By contrast, no such vertex exists in Fig. 5a. Also note that as shown in Fig. 5a, if the output cannot be measured, it can be unfeasible to find an experiment setup where G_{21} is generically identifiable.

The situation from the previous examples can be formalized as follows.

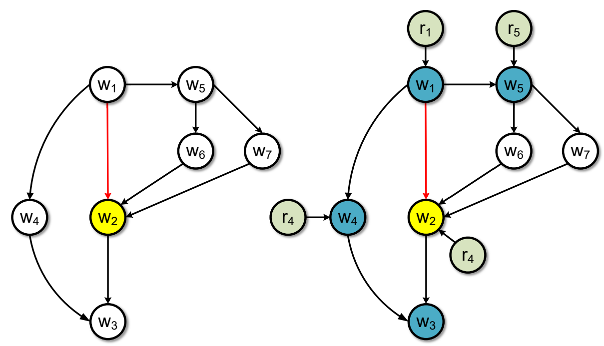
Theorem 3 *In a network model set $\mathcal{M}$ with its graph, suppose that output w_j cannot be measured while input w_i is measured. Module G_{ji} is generically identifiable in $\mathcal{M}$ if w_j is directly excited by a excitation signal r_j, and there exists a vertex w_k, a vertex set $\mathcal{D}$, and a subset $\bar{\mathcal{X}}_k \subseteq (\mathcal{X}_k \cap \mathcal{X}_j) \setminus \{r_j\}$ such that*

1) *w_j has a path to w_k;*
2) *$\mathcal{D} \cup \{w_j\}$ is a disconnecting set from $\{w_i\} \cup \bar{\mathcal{X}}_k$ to $\mathcal{W}_k \cup (\mathcal{W}_j \setminus \{w_i\})$, and $w_i \notin \mathcal{D}$;*
3) *$\mathcal{D}$ does not disconnect from $\{w_j\}$ to $\mathcal{W}_k$;*
4) *w_k and signals in $\mathcal{D}$ are measured;*
5) *$\bar{\mathcal{X}}_k \cup \{r_j\}$ is linked to $\mathcal{D} \cup \{w_i, w_j\}$.*

In the above theorem, condition (1) and the requirement that $\mathcal{D} \cup \{w_j\}$ is a disconnecting set from $\{w_i\}$ to $\mathcal{W}_k$ are reflected in Example 3, where an extra measured vertex is required to form paths from w_i via G_{ji} to w_k, and the vertices in $\mathcal{D}$ intersect with the other extra paths from w_i to w_k. Condition (3) matches Example 4, and roughly speaking, it ensures that $\mathcal{D}$ does not intersect with all the paths from w_i via G_{ji} to w_k. Similar to Theorem 1, the condition that $\mathcal{D} \cup \{w_j\}$ disconnects from $\{w_i\}$ to $\mathcal{W}_j \setminus \{w_i\}$ is generalized from [20].

Theorem 3 may also be interpreted and used from a synthesis point of view. However, the design of the synthesis procedure is more complex than the derivation of Corollary 2 due to the condition (3), and thus it is left for future work. An example is provided to show the potential of using Theorem 3 for the synthesis problem.

Example 5 Consider Fig. 6a, to achieve generic identifiability of G_{21} using Theorem 3, w_k can only be w_3. In this case, $\mathcal{W}_2 \setminus \{w_1\} = \{w_6, w_7\}$ and $\mathcal{W}_3 = \{w_2, w_4\}$. Thus a set $\mathcal{D} = \{w_4, w_5\}$ can be found such that $\mathcal{D} \cup \{w_2\}$ is a disconnecting set from input w_1 to $(\mathcal{W}_2 \setminus \{w_1\}) \cup \mathcal{W}_3$. Then besides directly exciting output w_2 and measuring w_3, measuring and exciting the signals in $\mathcal{D} \cup \{w_1\}$ leads to generic identifiability of G_{21}, as in Fig. 6b. ∎



(a) (b)

Fig. 6. A sensor and measurement scheme in (b) for network in (a) such that G_{21} is generically identifiable.

V. Conclusion

The problems of generic identifiability of a single module with the unmeasured input or unmeasured output are addressed respectively in this work. Graphical conditions are developed for analysis, which also shows potential for synthesis problems.

When the input of the target module cannot be measured, it is found that the vertices on the loops around the input should be measured and excited, and these vertices can be found as a disconnecting set from the input to its in-neighbors. When the output cannot be measured, the input needs to have a path via the target module to an additional measured signal. If there are extra paths from the input to the measured signal which do not pass through the target module, vertices on the these paths should also be measured and excited, which can be characterized by disconnecting sets.

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APPENDIX

A. Algebraic result of disconnecting sets

The analysis in this work relies on the algebraic result of a disconnecting set in a network model. Consider two sets $\bar{\mathcal{X}} \subseteq \mathcal{X}$ and $\bar{\mathcal{W}} \subseteq \mathcal{W}$. Given a disconnecting set $\mathcal{D} \subseteq \mathcal{V}$ from $\bar{\mathcal{X}}$ to $\bar{\mathcal{W}}$, the vertices of $\mathcal{G}$ can be divided into three disjoint sets $\mathcal{D}$, $\mathcal{P}$ and $\mathcal{S}$ [5]:

- $\mathcal{S}$ contains all vertices reachable by $\bar{\mathcal{X}}$ through the paths which do not intersect with $\mathcal{D}$. Since $\bar{\mathcal{X}} \subseteq \mathcal{S}$ and the other external signals can not be reached by $\bar{\mathcal{X}}$, we divide $\mathcal{S}$ into two disjoint subsets as $\mathcal{S} = \bar{\mathcal{X}} \cup \mathcal{S}_w$ where $\mathcal{S}_w \subseteq \mathcal{W}$;
- $\mathcal{P} \triangleq \mathcal{V} \setminus (\mathcal{S} \cup \mathcal{D})$. Additionally $\mathcal{P}$ is divided into two disjoint subsets as $\mathcal{P} = \mathcal{P}_x \cup \mathcal{P}_w$ where $\mathcal{P}_x \subseteq \mathcal{X}$ and $\mathcal{P}_w \subseteq \mathcal{W}$.

Let $\bar{\mathcal{G}}$ denote a subgraph of $\mathcal{G}$, and it is formulated from $\mathcal{G}$ by removing all vertices in $\mathcal{P}_u \cup \mathcal{S}$ and their corresponding edges, and by additionally removing all in-coming edges of $\mathcal{D}$. Thus $\bar{\mathcal{G}}$ represents a *dynamic subnetwork* of $\mathcal{G}$ by regarding the signals in $\mathcal{D}$ as the external signals and the ones in $\mathcal{P}_w$ as the internal signals. Then the following result can be obtained.

Lemma 4 *In a network model set $\mathcal{M}$ with its graph, given a disconnecting set $\mathcal{D}$ from $\bar{\mathcal{X}} \subseteq \mathcal{X}$ to $\bar{\mathcal{W}} \subseteq \mathcal{W}$, there exists a proper transfer matrix $K \in \mathbb{R}(q)^{|\bar{\mathcal{X}}|\times|\mathcal{D}|}$ such that*

$$T_{\bar{\mathcal{W}}\bar{\mathcal{X}}} = K T_{\mathcal{D}\bar{\mathcal{X}}}, \tag{3}$$

where K is the mapping from $\mathcal{D}$ to $\bar{\mathcal{W}}$ in the dynamic subnetwork $\bar{\mathcal{G}}$.

Proof: Based on the formulation of the three disjoint sets $\mathcal{D}$, $\mathcal{P}$ and $\mathcal{S}$, we have divided the vertices of $\mathcal{G}$ into the following sets: $\mathcal{W} = \mathcal{P}_w \cup \mathcal{S}_w \cup \mathcal{D}$ and $\mathcal{X} = \bar{\mathcal{X}} \cup \mathcal{P}_x$. It also holds that $\bar{\mathcal{W}} \subseteq \mathcal{D} \cup \mathcal{P}_w$, and there is no directed edge from any vertex in $\mathcal{S}$ to any vertex in $\mathcal{P}$. Algebraically, this means that there exists permuted G, T and X (we do not change the notation for the matrices after permutation for simplicity) such that

$$G = \begin{bmatrix} G_{\mathcal{P}_w\mathcal{P}_w} & G_{\mathcal{P}_w\mathcal{D}} & 0 \\ G_{\mathcal{D}\mathcal{P}_w} & G_{\mathcal{D}\mathcal{D}} & G_{\mathcal{D}\mathcal{S}_w} \\ G_{\mathcal{S}_w\mathcal{P}_w} & G_{\mathcal{S}_w\mathcal{D}} & G_{\mathcal{S}_w\mathcal{S}_w} \end{bmatrix},$$

$$X = \begin{bmatrix} 0 & X_{\mathcal{P}_x\mathcal{P}_u} \\ X_{\mathcal{D}\bar{\mathcal{X}}} & X_{\mathcal{D}\mathcal{P}_u} \\ X_{\mathcal{S}_w\bar{\mathcal{X}}} & X_{\mathcal{S}_w\mathcal{P}_u} \end{bmatrix}, \quad T_{\mathcal{W}\mathcal{X}} = \begin{bmatrix} T_{\mathcal{P}_w\bar{\mathcal{X}}} & T_{\mathcal{P}_w\mathcal{P}_u} \\ T_{\mathcal{D}\bar{\mathcal{X}}} & T_{\mathcal{D}\mathcal{P}_u} \\ T_{\mathcal{S}_w\bar{\mathcal{X}}} & T_{\mathcal{S}_w\mathcal{P}_u} \end{bmatrix},$$

where matrix $G_{\mathcal{P}_w\mathcal{D}}$, and other matrices similarly, denotes the submatrix of G with the columns and the rows corresponding to the signals in $\mathcal{D}$ and $\mathcal{P}_w$ respectively. Submatrices $G_{\mathcal{P}_w\mathcal{S}_w}$ and $X_{\mathcal{P}_w\bar{\mathcal{X}}}$ are zero because there is no directed edge from $\mathcal{S}$ to $\mathcal{P}$.

Having the equation

$$(I - G)T_{\mathcal{W}\mathcal{X}} = X, \tag{4}$$

and for the block $X_{\mathcal{P}_w\bar{\mathcal{X}}}$ of X, the following relationship can be obtained from the above equation:

$$\begin{bmatrix} I-G_{\mathcal{P}_w\mathcal{P}_w} & -G_{\mathcal{P}_w\mathcal{D}} & 0\end{bmatrix}\begin{bmatrix} T_{\mathcal{P}_w\bar{\mathcal{X}}}\\ T_{\mathcal{D}\bar{\mathcal{X}}}\\ T_{\mathcal{S}_w\bar{\mathcal{X}}}\end{bmatrix}=0$$
$$\Longrightarrow (I-G_{\mathcal{P}_w\mathcal{P}_w})T_{\mathcal{P}_w\bar{\mathcal{X}}}=G_{\mathcal{P}_w\mathcal{D}}T_{\mathcal{D}\bar{\mathcal{X}}}. \tag{5}$$

Because $I-G_{\mathcal{P}_w\mathcal{P}_w}(z)$ is proper and the network is well-posed, i.e. $\lim_{z\to\infty}\det(I-G_{\mathcal{P}_w\mathcal{P}_w}(z))\neq 0$, it holds that $I-G_{\mathcal{P}_w\mathcal{P}_w}(z)$ is invertible and proper. Thus, equation (5) leads to

$$T_{\mathcal{P}_w\bar{\mathcal{X}}}=(I-G_{\mathcal{P}_w\mathcal{P}_w})^{-1}G_{\mathcal{P}_w\mathcal{D}}T_{\mathcal{D}\bar{\mathcal{X}}}. \tag{6}$$

Recall that $\bar{\mathcal{W}}\subseteq\mathcal{P}_w\cup\mathcal{D}$. We discuss the two possible cases:

(i) For any element $w_k\in\bar{\mathcal{W}}$ and $w_k\in\mathcal{P}_w$, the row in equation (6) corresponding to w_k leads to

$$T_{w_k\bar{\mathcal{X}}}=[(I-G_{\mathcal{P}_w\mathcal{P}_w})^{-1}G_{\mathcal{P}_w\mathcal{D}}]_{w_k\star}T_{\mathcal{D}\bar{\mathcal{X}}}, \tag{7}$$

where $[(I-G_{\mathcal{P}_w\mathcal{P}_w})^{-1}G_{\mathcal{P}_w\mathcal{D}}]_{w_k\star}$ is the mapping from $\mathcal{D}$ to w_k in the dynamic subnetwork $\bar{\mathcal{G}}$;

(ii) For any element $w_p\in\bar{\mathcal{W}}$ and $w_p\in\mathcal{D}$, we have

$$T_{w_p\bar{\mathcal{X}}}=\begin{bmatrix}0 & \cdots & 0 & 1 & 0\cdots & 0\end{bmatrix}T_{\mathcal{D}\bar{\mathcal{X}}}, \tag{8}$$

where the row of $T_{\mathcal{D}\bar{\mathcal{X}}}$ corresponding to w_p is extracted. The unit vector in the above equation is the mapping from $\mathcal{D}$ to w_p in the dynamic subnetwork $\bar{\mathcal{G}}$.

Combing (i) and (ii) proves the lemma. ■

The definition of subgraph $\bar{\mathcal{G}}$ and Lemma 4 will be used extensively to derive the other results in this work.

B. Proof of Theorem 1

Proof: The theorem is proven by showing the uniqueness of G_{ji} given $T_{j\bar{\mathcal{X}}}$ and $T_{\mathcal{D}\bar{\mathcal{X}}}$, where $\bar{\mathcal{X}}\triangleq\bar{\mathcal{X}}_j\cup\{r_i\}$.

Consider jth row of $(I-G)T=X$ and its columns corresponding to signals in $\bar{\mathcal{X}}$. After permutation, it leads to

$$\begin{bmatrix}-G_{ji} & -G_{j\mathcal{W}_j\setminus\{w_i\}} & 1 & 0\end{bmatrix}\begin{bmatrix}T_{i\bar{\mathcal{X}}}\\ T_{\mathcal{W}_j\setminus\{w_i\}\bar{\mathcal{X}}}\\ T_{j\bar{\mathcal{X}}}\\ \star\end{bmatrix}=\bar{X},$$

where $G_{j\mathcal{W}_j\setminus\{w_i\}}$ is the vector of the uninteresting in-coming modules of w_j and $\bar{X}$ is a known vector. The above equation leads to

$$G_{ji}T_{i\bar{\mathcal{X}}}+G_{j\mathcal{W}_j\setminus\{w_i\}}T_{\mathcal{W}_j\setminus\{w_i\}\bar{\mathcal{X}}}=T_{j\bar{\mathcal{X}}}-\bar{X}, \tag{9}$$

where $T_{i\bar{\mathcal{X}}}$ and $T_{\mathcal{W}_j\setminus\{w_i\}\bar{\mathcal{X}}}$ may not be available as the signals in $\mathcal{W}_j$ may not be measured. However, we show that the two terms can be represented by a combination of available terms as follows.

Due to condition (3) and the fact that r_i only has a directed edge to w_i, $\bar{\mathcal{X}}_j$ have vertex disjoint paths to $\mathcal{D}$ without intersecting with w_i, and thus all paths from $\bar{\mathcal{X}}_j$ to w_i intersect with $\mathcal{W}_i$ and consequently $\mathcal{D}$ due to condition (2). Thus $\mathcal{D}\cup\{r_i\}$ is a $\bar{\mathcal{X}}-\{w_i\}$ disconnecting set. Then based on $r_i\notin\mathcal{D}$ and Lemma 4, it holds that

$$T_{i\bar{\mathcal{X}}_j}=[K]_{w_i\star}\begin{bmatrix}\mathbf{e}_{r_i}\\ T_{\mathcal{D}\bar{\mathcal{X}}}\end{bmatrix}, \tag{10}$$

where $\mathbf{e}_{r_i}$ is a row vector contains one entry of 1 and zeros elsewhere, and it denotes the mapping from $\bar{\mathcal{X}}$ to r_i as $r_i\in\bar{\mathcal{X}}$. Consider the subgraph $\bar{\mathcal{G}}$ defined in Section A. Since $\mathcal{D}$ is a disconnecting set from $\{w_i\}$ to $\mathcal{W}_i$, $\mathcal{D}$ intersects all the loops around w_i. Then $\bar{\mathcal{G}}$ does not contain any loop around w_i as $\mathcal{D}$ do not have in-coming edges in $\bar{\mathcal{G}}$. Thus it holds that

$$[K]_{w_i\star}=\begin{bmatrix}1 & K_{w_i\mathcal{D}}\end{bmatrix}, \tag{11}$$

where 1 is the mapping from r_i to w_i in $\bar{\mathcal{G}}$ as there is no loop around w_i and r_i directly excites w_i. $K_{w_i\mathcal{D}}$ is the mapping from $\mathcal{D}$ to w_i in $\bar{\mathcal{G}}$. Thus it holds that

$$T_{i\bar{\mathcal{X}}_j}=\begin{bmatrix}1 & K_{w_i\mathcal{D}}\end{bmatrix}\begin{bmatrix}\mathbf{e}_{r_i}\\ T_{\mathcal{D}\bar{\mathcal{X}}}\end{bmatrix}. \tag{12}$$

Furthermore, due to condition (2) and Lemma 4, it holds that

$$T_{\mathcal{W}_j\setminus\{w_i\}\bar{\mathcal{X}}}=\bar{K}T_{\mathcal{D}\bar{\mathcal{X}}},$$

for some proper transfer matrix $\bar{K}$. The above equation, (9) and (12) leads to

$$\begin{bmatrix}G_{ji} & (G_{ji}K_{w_i\mathcal{D}}+G_{j\mathcal{W}_j\setminus\{w_i\}}\bar{K})\end{bmatrix}\begin{bmatrix}\mathbf{e}_{r_i}\\ T_{\mathcal{D}\bar{\mathcal{X}}}\end{bmatrix}=T_{j\bar{\mathcal{X}}}-\bar{X}_j,$$

where matrix $\begin{bmatrix}\mathbf{e}_{r_i}\\ T_{\mathcal{D}\bar{\mathcal{X}}}\end{bmatrix}$ is generically full row rank due to condition (3) [5] [18], and the terms $T_{\mathcal{D}\bar{\mathcal{X}}}$ and $T_{j\bar{\mathcal{X}}}$ are given due to condition (1). Consequently a unique G_{ji} can be found from the above equation generically, which concludes generic identifiability of G_{ji}. ■

C. Proof of Theorem 3

Proof: Given the conditions, the uniqueness of G_{ji} is shown from the available submatrices of $T_{\mathcal{W}\mathcal{X}}$ which can be obtained from an appropriate disconnecting set and its corresponding factorization in Lemma 4.

Define $\bar{\mathcal{X}}\triangleq\bar{\mathcal{X}}_k\cup\{r_j\}$. Based on condition (2), it holds that $\mathcal{D}\cup\{w_i,w_j,r_j\}$ is a $\bar{\mathcal{X}}-\mathcal{W}_k$ disconnecting set. Condition (5) and $\bar{\mathcal{X}}_k\subseteq\mathcal{X}_j$ imply that vertices in $\bar{\mathcal{X}}_k$ have no directed edge to w_j. Then all paths from $\bar{\mathcal{X}}_k$ to w_j must intersect with w_i or $\mathcal{W}_j\setminus\{w_i\}$, and the ones intersecting with $\mathcal{W}_j\setminus\{w_i\}$ must also intersect with $\mathcal{D}$ based on condition (2). This concludes that $\{w_i\}\cup\mathcal{D}$ is a $\bar{\mathcal{X}}_k-\{w_j\}$ disconnecting set, and thus $\mathcal{D}\cup\{w_i,r_j\}$ remains a $\bar{\mathcal{X}}-\mathcal{W}_k$ disconnecting set.

Consider the kth row of $(I-G)T=X$ and its subset of columns corresponding to $\bar{\mathcal{X}}$, which leads to the following equation after permutation:

$$\begin{bmatrix}-G_{k\mathcal{W}_k} & 1 & 0\end{bmatrix}\begin{bmatrix}T_{\mathcal{W}_k\bar{\mathcal{X}}}\\ T_{k\bar{\mathcal{X}}}\\ \star\end{bmatrix}=\bar{X},$$

where $G_{k\mathcal{W}_k}$ is a row vector of all in-coming modules of w_k, and $\bar{X}$ is a known row vector as $\bar{\mathcal{X}}$ is a subset of $\mathcal{X}_k$. The above equation leads to

$$G_{k\mathcal{W}_k}T_{\mathcal{W}_k\bar{\mathcal{X}}}=T_{k\bar{\mathcal{X}}}-\bar{X}, \tag{13}$$

where $T_{\mathcal{W}_k\bar{\mathcal{X}}}$ is possibly unavailable as the signals in $\mathcal{W}_k$ may not be measured. However, $T_{\mathcal{W}_k\bar{\mathcal{X}}}$ can be represented by available terms as follows. Since $\mathcal{D} \cup \{w_i, r_j\}$ is a disconnecting set from $\bar{\mathcal{X}}$ to $\mathcal{W}_k$ and $\mathcal{D} \cap \{w_i, r_j\} = \emptyset$ ($r_j \notin \mathcal{D}$ is implied by condition (5)), based on Lemma 4, there exist proper transfer matrices K_i, K_d and K_j such that

$$T_{\mathcal{W}_k\bar{\mathcal{X}}} = \begin{bmatrix} K_i & K_d & K_j \end{bmatrix} \bar{T}, \ \bar{T} \triangleq \begin{bmatrix} T_{i\bar{\mathcal{X}}} \\ T_{\mathcal{D}\bar{\mathcal{X}}} \\ \mathbf{e}_{r_j} \end{bmatrix}, \tag{14}$$

where $\bar{T}$ is available as w_i, $\mathcal{D}$ and r_j are measured. Note that $\mathbf{e}_{r_j}$ is a row vector with zeros and only one entry of 1, and it denotes the mapping from $\bar{\mathcal{X}}$ to r_j as $r_j \in \bar{\mathcal{X}}$.

The K terms in (14) also have a specific structure. Consider the subgraph $\bar{\mathcal{G}}$ defined in Section A by taking $\mathcal{D} \cup \{w_i, r_j\}$ as the $\bar{\mathcal{X}} - \mathcal{W}_k$ disconnecting set. Based on condition (2), all paths from w_i to $\mathcal{W}_k$ intersect with $\mathcal{D} \cup \{w_j\}$ in $\mathcal{G}$. Since in $\bar{\mathcal{G}}$, vertices in $\{w_i\} \cup \mathcal{D}$ have no in-neighbor, no path exists from w_i to $\mathcal{D}$ in $\bar{\mathcal{G}}$. Thus all paths from w_i to $\mathcal{W}_k$ must pass w_j through the directed edge $w_i \to w_j$ or the other in-neighbors $\mathcal{W}_j \setminus \{w_i\}$ of w_j. The paths via $\mathcal{W}_j \setminus \{w_i\}$ must intersect with $\mathcal{D}$ based on condition (2). Since in $\bar{\mathcal{G}}$, vertices in $\mathcal{D}$ have no in-neighbor, all paths from w_i to $\mathcal{W}_k$ in $\bar{\mathcal{G}}$ must contain the directed edge $w_i \to w_j$. As r_j also has a directed edge to w_j, K_i and K_j have the following relationship:

$$K_i = K_j G_{ji}. \tag{15}$$

Furthermore, it can be shown that K_j is not zero as follows. Consider the disjoint sets $\mathcal{S}$, $\mathcal{P}$ formulated in Section A, it holds that $\mathcal{W}_k \subseteq \mathcal{P}_w$, and $w_j \in \mathcal{P}_w$ as all paths from $\bar{X}_k \cup \{r_j\}$ to w_j intersect with $\mathcal{D} \cup \{w_i, r_j\}$. Due to condition (1) and condition (3), a path from w_j to $\mathcal{W}_k$ exists in $\mathcal{G}$ and it does not intersect with $\mathcal{D}$. This implies that the path also does not intersect with w_i, otherwise a path from w_j via w_i to $\mathcal{W}_k$ exists without passing $\mathcal{D}$, which contradicts condition (2). The path also does not intersect with set $\mathcal{S}$, otherwise a path from $\bar{\mathcal{X}}_k \subseteq \mathcal{S}$ to $\mathcal{W}_k$ exists without intersecting with $\mathcal{D} \cup \{w_j\}$, which contradicts condition (2). The above facts imply that a path from w_j to $\mathcal{W}_k$ exists in $\mathcal{G}$ without intersecting $\mathcal{S} \cup \mathcal{D} \cup \{w_i, r_j\}$. Based on the formulation of $\bar{\mathcal{G}}$, the path also exists in $\bar{\mathcal{G}}$, which implies that the mapping K_j in $\bar{\mathcal{G}}$ from r_j via w_j to $\mathcal{W}_k$ is not zero.

Combining (13), (14) and (15) leads to

$$\begin{bmatrix} G_{k\mathcal{W}_k} K_j G_{ji} & G_{k\mathcal{W}_k} K_d & G_{k\mathcal{W}_k} K_j \end{bmatrix} \bar{T} = T_{k\bar{\mathcal{X}}} - \bar{X}_k, \tag{16}$$

where the term on the LHS and $\bar{T}$ are available due to condition (1) and (3). In addition, condition (4) implies that matrix $\bar{T}$ is generically full row rank [5] [18], and consequently (16) leads to unique solutions of $G_{k\mathcal{W}_k} K_j G_{ji}$ and $G_{k\mathcal{W}_k} K_j$. As K_j is non-zero and thus $G_{k\mathcal{W}_k} K_j$ is non-zero, a unique G_{ji} can be found, which proves generic identifiability of G_{ji}. ■