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Brief Paper

Delay Structure Conditions for Identifiability of Closed Loop Systems*

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Key Words—System identification; closed loop system; identifiability; linear systems.

Abstract—When identifying a linear system operating under a linear feedback law, specific restrictions have to be imposed on the presence of delays in the plant and/or the controller in order to obtain identifiability. In this paper the delay structure conditions are investigated under which a least squares direct type prediction error identification method is able to consistently estimate the open loop plant. It is shown that the commonly required condition of strictly properness of the loop transfer function can be relaxed to a condition that relates to the absence of algebraic loops in the closed loop plant models.

1. *Introduction*

WHEN IDENTIFYING a dynamical system, operating under a linear feedback law, on the basis of recorded input and output data, specific conditions have to be satisfied in order to be able to arrive at a consistent estimate of the open loop system. Dependent on the specific system configuration that is analysed and the specific identification method applied, conditions are formulated that reflect a sufficient complexity of the controller, a sufficient excitation of the closed loop system, a sufficiently "large" model set, and conditions on the presence of delays in the plant and its model $G(q)$ and/or in the controller $F(q)$, see e.g. Ljung *et al.* (1974), Söderström *et al.* (1975, 1976), Ng *et al.* (1977), Gustavsson *et al.* (1977) and Anderson and Gevers (1982).

In the current paper specific attention will be given to the aforementioned delay structure conditions. In the literature restrictions have been formulated concerning strictly properness of the plants $G(q)$, see e.g. Ljung *et al.* (1974), Söderström *et al.* (1975), or to strictly proper loop transfers $G(q)F(q)$, Söderström *et al.* (1976), Ng *et al.* (1977), Gustavsson *et al.* (1977) and Anderson and Gevers (1982). However, in modelling closed loop systems, it may be very suitable to be able to use plant models and controller representations that allow direct feedtrough terms, e.g. to lump very fast dynamic behaviour of the corresponding transfers. In this paper it will be shown that in this respect, the current conditions are too restrictive and that they can be further relaxed. As an identification method we will restrict attention to so-called direct identification, applying a weighted least squares prediction error method to the plant input and output data.

First the basic closed loop configuration to be analysed will be considered, together with the basic identification concepts involved. Next, in Section 3, the main results are presented, followed in Section 4 by a discussion and illustration of the formal results in relation to the existence of algebraic loops in the closed loop models. Some notational conventions: $\mathbb{R}(z)$ is the field of rational functions in z, and $\text{rank}_{\mathbb{R}(z)}$ the rank of a matrix over this field. With q we denote the (forward) shift operator, $qw(t) = w(t+1)$.

2. *Basic concepts*

We will consider a linear time-invariant dynamical system, written in a discrete-time representation:

$$S: y(t) = G_0(q)u(t) + H_0(q)e(t), \tag{1}$$

with $y(t) \in \mathbb{R}^p$ a p-dimensional output signal, $u(t) \in \mathbb{R}^m$ a m-dimensional input and $e(t) \in \mathbb{R}^p$, with $\{e(t)\}$ a sequence of independent random variables with zero mean and covariance matrix $\Lambda > 0$. $G_0(z) \in \mathbb{R}^{p\times m}(z)$ and $H_0(z) \in \mathbb{R}^{p\times p}(z)$ are proper rational transfer function matrices with $H_0(z)$ stable and stably invertible, and additionally $H(\infty) := \lim_{z\to\infty} H(z) = I_p$.

The system (1) is assumed to be operating under a linear feedback law:

$$\chi: u(t) = F(q)y(t) + L(q)r(t), \tag{2}$$

with proper transfer functions $F(z) \in \mathbb{R}^{m\times p}(z)$, $L(z) \in \mathbb{R}^{m\times q}(z)$; $r(t)$ is a q-dimensional external input signal which can either be considered as an additional measurable signal or as a noise disturbance on the regulator output (e.g. setpoint variations). A block diagram of the considered closed loop configuration is sketched in Fig. 1.

In order to identify the open loop system S, a model set is considered denoted by:

$$\mathcal{M}: y(t) = G_\theta(q)u(t) + H_\theta(q)\varepsilon(t), \quad \theta \in \Theta \subset \mathbb{R}^d, \tag{3}$$

with $G_\theta(z)$, $H_\theta(z)$ appropriate rational transfer function matrices as discussed before, depending on a real-valued parameter vector θ that lies within a set Θ of admissible values, and $\varepsilon(t)$ the one step ahead prediction error. The notation $S \in \mathcal{M}$ will be used to indicate that there exists a $\theta_0 \in \Theta$ such that $G_{\theta_0}(z) = G_0(z)$ and $H_{\theta_0}(z) = H_0(z)$ for almost all $z \in C$.

With respect to the identification criterion we will focus on a prediction error based method applied in a so-called direct identification scheme:

$$\hat{\theta}_N = \arg\min_{\theta\in\Theta} V_N(\theta), \tag{4}$$

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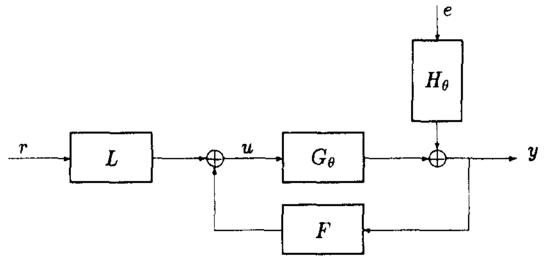

FIG. 1. The closed loop system configuration.

with

$$V_N(\theta) = \frac{1}{N}\sum_{t=1}^{N} \varepsilon^T(t,\theta)Q\varepsilon(t,\theta), \qquad (5)$$

with $\varepsilon(t,\theta)$ the one step ahead prediction error of a model represented by parameter θ, and Q a weighting matrix satisfying $Q = Q^T > 0$. The mapping from measured data to $\hat{\theta}_N$ will be denoted by the identification criterion J_N. Note that in this direct identification approach any information regarding the experimental conditions under which the data has been obtained is neglected.

The closed loop identifiability problem is formulated as a consistency problem. The question is whether the system S can be estimated consistently when the data are obtained under specific closed loop operating conditions. The formulation of this problem inherently shows that this identifiability property of a system S will be dependent on (a) the model set $\mathcal{M}$ chosen, (b) the identification criterion J involved, and (c) the experimental conditions χ. In correspondence with the definitions used in literature we will state the following formal definition of the problem (Ljung *et al.*, 1974).

Definition 2.1. A system S is said to be system identifiable under $\mathcal{M}, J_N$ and χ if $\hat{\theta}_N \to D_T(S,\mathcal{M})$ with probability 1 as $N\to\infty$ with $D_T(S,\mathcal{M}) := \{\theta\in\Theta \mid G_\theta(z) = G_0(z)$ and $H_\theta(z) = H_0(z)$ for almost all $z\in\mathbb{C}\}$. □

When S is system identifiable under $\mathcal{M}, J_N$ and χ, then the fact that it operates under feedback does not cause any additional problems for identifying the system consistently. Note that in comparison with Ljung *et al.* (1974) not only the identification strategy—direct identification—is specified in the definition of system identifiability, but also the specific identification criterion (4), (5). This is in correspondence with considering the concept of identifiability to be specifically related to the identification criterion applied (Van den Hof, 1989).

3. *Delay structure conditions*

In all results on identifiability conditions for systems operating in closed loop, it is assumed that delays are present in the transfer function $G_0(z)$ and/or in the controller $F(z)$ formalized by the requirement that $G_0(\infty)F(\infty) = 0$, and that accordingly one has to restrict the model set to

$$G_\theta(\infty)F(\infty) = 0, \quad \text{for all } \theta\in\Theta, \qquad (6)$$

see e.g. Ljung *et al.* (1974), Söderström *et al.* (1976) and Gustavsson *et al.* (1977) and Anderson and Gevers (1982). Apart from the fact that these conditions can be fruitfully utilized in the derivations of the identifiability results, they also appear to be quite appealing. Take for instance the SISO situation ($m = p = 1$): if there is no delay in the input output model ($G_\theta(\infty)\neq 0$) nor in the controller ($F(\infty)\neq 0$) then there exist algebraic relations between $u(t)$ and $y(t)$ caused by both the i–o model and the controller. A direct identification method based on $\{u(t), y(t)\}$ now will never be able to distinguish between these two unknown relations, leading to a non-identifiable situation. Condition (6) generalizes this situation to the multivariable case; it deals with the exclusion of algebraic loops in the closed loop configuration. In this paper it will be shown that in a number of situations condition (6) is too strong and that it can be relaxed while—more or less—keeping up with the intuitive appeal of absence of algebraic loops. For purpose of illustrating (6) the following example is added.

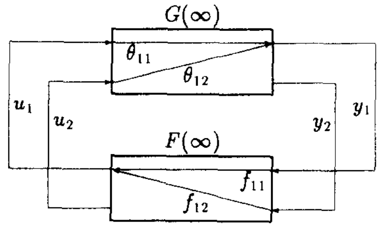

FIG. 2. Diagram of algebraic relations between inputs and outputs in Example 3.1.

Example 3.1. Consider a two input–two output situation with $G_\theta(q)$ and $F(q)$ satisfying

$$G_\theta(\infty) = \begin{bmatrix}\theta_{11} & \theta_{12}\\ 0 & 0\end{bmatrix}, \quad F(\infty) = \begin{bmatrix} f_{11} & f_{12}\\ 0 & 0\end{bmatrix}, \qquad (7)$$

with given constants $f_{11}, f_{12}\in\mathbb{R}$. The algebraic links between inputs and outputs are depicted in Fig. 2.

The product $G_\theta(\infty)F(\infty) = \begin{bmatrix}\theta_{11}f_{11} & \theta_{11}f_{12}\\ 0 & 0\end{bmatrix}$ which shows that condition (6) generally will not be satisfied. Figure 2 shows that there exists an algebraic loop between u_1 and y_1 which causes lack of identifiability. However note that when $f_{11} = 0$ the algebraic loop is cancelled, while the product $G_\theta(\infty)F(\infty)$ still has a nonzero element $\theta_{11}f_{12}$. Consequently, according to the existing results one can not prove system identifiability in this situation for $f_{11} = 0$. □

The intuitive idea that the presence of algebraic loops in the closed loop system is responsible for lack of identifiability will be exploited in the next sections in order to relax condition (6).

4. *Main result*

The conditions that have to be imposed in order to arrive at identifiability are twofold: one condition on the delay structure of the models in the set, and two conditions on the complexity of the experimental situation and the complexity of the controller.

Theorem 4.1. Let S be a dynamical system according to (1), operating under a linear feedback law χ (2), such that $\{r(t)\}$ and $\{e(t)\}$ are uncorrelated and the closed loop system is asymptotically stable. Let a weighted least squares direct identification method J_N be applied (4), (5) to a model set $\mathcal{M}$ (3) such that $S\in\mathcal{M}$. Assume that

1. (a) $I - G_\theta(\infty)F(\infty)$ is nonsingular for all $\theta\in\Theta$; and
   (b) tr $Q[T_\theta(\infty) - I]\Lambda \geq 0$ for all $\theta\in\Theta$; with

$$T_\theta(z) := [I - G_\theta(z)F(z)][I - G_0(z)F(z)]^{-1}$$

2. $\{r(t)\}$ is persistently exciting of sufficient order;
3. $\operatorname{rank}_{\mathbb{R}(z)}\begin{bmatrix} I_p & 0\\ F(z) & L(z)\end{bmatrix} = m + p,$

then S is system identifiable under $\mathcal{M}, J_N$ and χ. □

Proof. The proof is added in the Appendix. □

The theorem very much resembles the original closed loop identifiability result as presented in Söderström *et al.* (1976). However condition 1 in Theorem 4.1 appears to be a relaxed version of the classical one: $G_\theta(\infty)F(\infty) = 0$ for all $\theta\in\Theta$. In the sequel of this paper we will analyse the consequence of this relaxed condition 1, for the situation that $Q = \Lambda = I$. A sufficient—and simply verifiable—condition can now be formulated.

Proposition 4.2. Consider the situation of Theorem 4.1. If there exists a $p\times p$ non-singular matrix U such that for all $\theta\in\Theta$, $UG_\theta(\infty)F(\infty)U^{-1}$ is an upper triangular matrix with diagonal entries zero, then condition 1 in Theorem 4.1 is satisfied. □

Proof. The proof is added in the Appendix. □

Taking a look at this proposition, it shows that the delay structure condition clearly is less strict than the classical condition. Note also that for all θ, all the eigenvalues of the matrix product $G_\theta(\infty)F(\infty)$ have to be equal to zero.

When we consider again the situation shown in Example 3.1, it can simply be verified that the condition formulated in Proposition 4.2 is satisfied only if $\theta_{11}f_{11} \equiv 0$; this means that either the (1,1) element of $G_\theta(\infty)$ should not be parametrized ($\theta_{11} \equiv 0$), or that $f_{11} = 0$. Both alternatives result in an elimination of the algebraic loop in Fig. 2. The intuitive idea that the presence of algebraic loops in the set of closed loop models is responsible for lack of identifiability, is exploited further in the following section.

5. *Interpretation in terms of algebraic loops*

First we have to formally define how to characterize the absence of algebraic loops. For this purpose we employ results from the theory of graphs. Through the notation $F(*)$ we denote a matrix having elements either $*$ or 0; i.e. the matrix contains only "structural" information and no numeric values. An element $*$ refers to a connection, whereas an element 0 refers to a non-connection between column nodes and row nodes of the matrix F. The "structural" matrix—or adjacency matrix (Gondran and Minoux, 1986)—of $F(\infty)$ is denoted by $F(*)$, and that of $G(\infty)$ by $G(*)$.

Definition 5.1. A closed loop system consisting of an open loop plant $G(q)$ and a controller $F(q)$ is said to contain no algebraic loop if there exists a $p \times p$ permutation matrix P such that $PG(*)F(*)P^T$ is an upper triangular matrix with diagonal entries zero. □

For evaluation of the presence of algebraic loops in a closed loop system we have to evaluate the product $G(*)F(*)$. This matrix product is a mapping: $\mathbb{R}^p \to \mathbb{R}^p$ and indicates whether there exist algebraic relations between output signals of the process. The definition states that there is an absence of algebraic loops if the output signals can be permuted in such a way that only algebraic relations are allowed between output signals that are of the form: $y_k \to y_{k+i}$ for $k = 1, \ldots, p-1$ and $i > 0$. For absence of algebraic loops it should not be allowed to have algebraic relations of the form: $y_k \to y_{k+i} \to \cdots \to y_k$. The definition above is equivalent with the graph-theoretical property that the directed graph with nodes $y_1, \ldots, y_p$ defined by the matrix $G(*)F(*)$ is acyclic, see e.g. Lawler (1976). This means that in this graph there does not exist a connection $y_i \to y_i$ for any $i = 1, \ldots, p$.

Now we can formulate the following corollary.

Corollary 5.2. Consider the situation of Theorem 4.1. If the closed loop system consisting of parametrized model $G_\theta(q)$ and controller $F(q)$ contains no algebraic loops, then condition 1 of Theorem 4.1 is satisfied. □

Proof. The corollary follows directly from Proposition 4.2 and Definition 5.1. □

Note that in the evaluation of algebraic loops, the parametrized matrix $G_\theta(\infty)$ is used in terms of its "structure-matrix" $G(*)$. This implies that all elements in $G_\theta(\infty)$ that are dependent on a parameter, have a "$*$" in the corresponding matrix $G(*)$.

The corollary formulates the condition that there should not arise any algebraic loops if the parametrized model set is controlled by the controller F. This condition is easily verifiable by checking the permutation requirement on $G(*)F(*)$ as formulated in Definition 5.1, or by visual inspection of an arrow diagram, as will be illustrated in Example 5.3.

It has to be stressed that the absence of algebraic loops, as formulated in the corollary, is a sufficient condition, and not a necessary one. Theorem 4.1 may also hold even when algebraic loops are present. In order to illustrate this remark, we consider the following example.

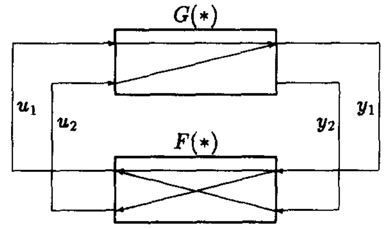

FIG. 3. Diagram of algebraic relations between inputs and outputs in Example 5.3.

Example 5.3. Consider a two-input–two-output situation with

$$G_\theta(\infty) = \begin{bmatrix} \theta_1 & \theta_2 \\ 0 & 0 \end{bmatrix}, \qquad F(\infty) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}. \tag{8}$$

The algebraic relations beteen inputs and outputs are represented in the diagram sketched in Fig. 3. Note that

$$G(*)F(*) = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \begin{pmatrix} * & * \\ * & 0 \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}.$$

This matrix can definitely not be brought to an upper diagonal form through the permutation operation as meant in Definition 5.1. Consequently—as confirmed by the diagram in Fig. 3—there do exist algebraic loops, e.g. $y_1 \to u_1 \to y_1$. Since $G_\theta(\infty)F(\infty) = \begin{pmatrix} \theta_1 - \theta_2 & \theta_1 \\ 0 & 0 \end{pmatrix}$, the condition of Proposition 4.2 can only be satisfied if $\theta_1 \equiv \theta_2$, i.e. $G_\theta(\infty)$ is parametrized as $\begin{pmatrix} \theta_1 & \theta_1 \\ 0 & 0 \end{pmatrix}$. In the latter situation there is no limitation put forward for obtaining identifiability, and apparently a cancellation of effects of different loops at the output takes place. □

Checking on absence of algebraic loops, is of course the simplest test in a given situation. If there are no algebraic loops one can be sure that identifiability can be obtained (if the other conditions are satisfied). If there do exist algebraic loops, one has to evaluate the condition of the nonsingular transformation in Proposition 4.2, or even the more general condition 1 in Theorem 4.1. It has to be stressed that delay-structure conditions are necessary in the present context. This is caused by the fact that in the direct prediction error identification method, no use is made of the knowledge that the data is obtained under closed loop conditions with a possibly known controller. Incorporating this knowledge into the identification scheme creates the possibility of identifying even nonproper plants under closed loop observations, see Schrama (1991).

6. *Conclusions*

In this paper the question has been discussed under which conditions a prediction error direct identification method is able to consistently estimate a linear system on the basis of closed loop observations. The classical delay-structure condition for identifiability of closed loop systems, reflected by $G_\theta(\infty)F(\infty) = 0$, has been relaxed and has been shown to be satisfied if there is absence of algebraic loops in the set of closed loop models.

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Appendix

Proof of Theorem 4.1.

Combining (1), (2), (3), the prediction can be written as:

$$\varepsilon(t,\theta) = R_{\theta,r}(q)r(t) + R_{\theta,e}(q)e(t), \tag{9}$$

with

$$R_{\theta,r}(q) = H_\theta(q)^{-1}[T_\theta(q)G_0(q) - G_\theta(q)]L(q), \tag{10}$$

and

$$R_{\theta,e}(q) = H_\theta(q)^{-1}T_\theta(q)H_0(q). \tag{11}$$

Since $\{r(t)\}$ and $\{e(t)\}$ are uncorrelated it follows that under weak conditions (see Ljung, 1987) $\arg\min_{\theta\in\Theta} V_N(\theta)\to \arg\min_{\theta\in\Theta} \bar V(\theta)$ with probability 1 as $N\to\infty$ with

$$\bar V(\theta) = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N} E\varepsilon^T(t,\theta)Q\varepsilon(t,\theta).$$

Using the notation $\varepsilon_1(t,\theta) = R_{\theta,r}(q)r(t)$ and $\varepsilon_2(t,\theta) = R_{\theta,e}(q)e(t)$, we can write $\bar V(\theta) = \bar V_1(\theta) + \bar V_2(\theta)$ with

$$\bar V_1(\theta) = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N} E\varepsilon_1^T(t,\theta)Q\varepsilon_1(t,\theta),$$

and

$$\bar V_2(\theta) = \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N} E\varepsilon_2^T(t,\theta)Q\varepsilon_2(t,\theta).$$

First we will characterize the solution of minimizing $\bar V_2(\theta)$ with respect to θ. Secondly we will show that there always exists an argument $\theta^*\in\arg\min_{\theta\in\Theta}\bar V_2(\theta)$ that satisfies $\bar V_1(\theta^*) = 0$. Since $\min_{\theta\in\Theta}(\bar V_1(\theta) = 0$ it follows that $\arg\min_{\theta\in\Theta}\bar V(\theta)$ is completely characterized by the set of parameters θ^*.

1. *Characterization of* $\arg\min \bar V_2(\theta)$. With Assumption 1a it follows that $T_\theta(z)$ will be a proper transfer function, and combination with (11) shows that also $R_{\theta,e}(z)$ will be proper, with $R_{\theta,e}(\infty) = T_\theta(\infty)$. Using a Laurent expansion of $R_{\theta,e}(z)$ around $z=\infty$ it follows that $\varepsilon_2(t,\theta)$ can be written as

$$\varepsilon_2(t,\theta) = T_\theta(\infty)e(t) + \sum_{j=1}^{\infty} D_j e(t-j), \tag{12}$$

or

$$\varepsilon_2(t,\theta) = e(t) + \sum_{j=0}^{\infty} D_j e(t-j),$$

with

$$D_0 := T_\theta(\infty) - I. \tag{13}$$

Using the specific properties of e, it follows that

$$\bar V_2(\theta) = \operatorname{tr} Q\Lambda + \lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N}\sum_{j=0}^{\infty} Ee^T(t-j) \times D_j^T Q D_j e(t-j) + 2\operatorname{tr} QD_0\Lambda. \tag{14}$$

It follows directly that the second term on the right hand side of (14) ≥ 0 with equality if and only if $D_j = 0$ for $j = 0,\ldots,\infty$. Since by Assumption 1b the third term satisfies $2\operatorname{tr} QD_0\Lambda \geq 0$, it follows that $\bar V_2(\theta)\geq \operatorname{tr} Q\Lambda$, for all $\theta\in\Theta$. Since $\bar V_2(\theta_0) = \operatorname{tr} Q\Lambda$, and $\theta_0\in\Theta$, it follows that $\min_{\theta\in\Theta}\bar V_2(\theta) = \operatorname{tr} Q\Lambda$, and consequently $\arg\min_{\theta\in\Theta}\bar V_2(\theta) = \{\theta\in\Theta \mid \bar V_2(\theta) = \operatorname{tr} Q\Lambda\}$.

Now consider again (14). If the minimum of $\bar V_2(\theta)$ is obtained, it follows that the second term on the right hand side of the equation has to be equal to 0, leading to $\{D_j\}_{j=0,\ldots,\infty} = 0$, or equivalently $R_{\theta,e}(q) = T_\theta(\infty) = I$. In that situation the minimum is indeed obtained and consequently

$$\arg\min_{\theta\in\Theta}\bar V_2(\theta) = \{\theta\in\Theta \mid R_{\theta,e}(q) = I\}. \tag{15}$$

2. *Existence of* $\theta^*\in\arg\min\bar V_2(\theta)$ *such that* $\bar V_1(\theta^*) = 0$. If r is persistently exciting of sufficient order, it follows that $\bar V_1(\theta^*) = 0$ if and only if $R_{\theta,r}(q)\equiv 0$. Within the solution set (15) there exists at least one element θ^* that satisfies $R_{\theta^*,r}(q)\equiv 0$, e.g. $\theta^* = \theta_0$.

Combining the results under 1 and 2 shows that $\arg\min_{\theta\in\Theta}\bar V(\theta)$ is determined by the parameters satisfying the equations $R_{\theta,e}(q)\equiv I$, and $R_{\theta,r}(q)\equiv 0$. This is exactly the same starting point which is taken in Söderström *et al.* (1976) for proving the classical result on system identifiability, showing that under condition 3 of Theorem 4.1 the solution set is given by $\theta = \theta_0$, which proves the result. □

Proof of Proposition 4.2.

Assume that there exists a nonsingular matrix U such that $UG_\theta(\infty)F(\infty)U^{-1}$ is strictly upper triangular. Applying the same operation of U to $[I - G_\theta(\infty)F(\infty)]$ shows that $U[I - G_\theta(\infty)F(\infty)]U^{-1} = W_\theta$, with W_θ being an upper triangular matrix with ones on the diagonal. The equation above represents an eigenvalue decomposition showing that $I - G_\theta(\infty)F(\infty)$ has all eigenvalues in 1. Consequently this matrix is nonsingular for all $\theta\in\Theta$ (condition 1a of Theorem 4.1). Applying this same operation to $T_\theta(\infty)$ shows that $UT_\theta(\infty)U^{-1} = W_\theta W_{\theta_0}^{-1}$, which again is an upper triangular matrix with ones on the diagonal, showing that $\operatorname{tr}[T_\theta(\infty)] = p$. This verifies condition 1b of the theorem. □