An Outlook on Robust Model Predictive Control Algorithms: Reflections on Performance and Computational Aspects

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Abstract

In this paper, we discuss the model predictive control algorithms that are tailored for uncertain systems. Robustness notions with respect to both deterministic (or set based) and stochastic uncertainties are discussed and contributions are reviewed from the model predictive control literature. We present, classify and compare different notions of the robustness properties of state of the art algorithms, while a substantial emphasis is given to the closed-loop performance and computational complexity properties. Furthermore, connections between (i) the theory of risk and (ii) robust optimization research areas and robust model predictive control are discussed. Lastly, we provide a comparison of current robust model predictive control algorithms via simulation examples illustrating closed loop performance and computational complexity features.

I. Introduction

A. General Outline

Model predictive control (MPC) technology is a mature research field developed over four decades both in industry and academia addressing the question of (practical) optimal control of dynamical systems under process constraints and economic incentives. Its popularity is mainly attributed to two significant properties of MPC algorithms; first one is the (explicit) constraint handling capabilities while providing (sub-)optimal operation, see, e.g., [1], [2], [3]; and the second superiority is the ease of extending the algorithms to multi-input multi-output (MIMO) systems. Many different approaches were developed, such as; Model Algorithmic Control in 1978 ([4]), with finite impulse response models, Dynamic Matrix

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Control in 1980 ([5]), with step response models, Generalized Predictive Control in 1987 ([6]), with transfer function models. Lately, MPC methods developed by considering the state-space models have become the standard way of formulating predictive control problems. Throughout the different algorithms, however, the essence of predictive control is the same and can be stated as, [7], optimizing over manipulated inputs to control the forecasts of future process behaviour. Stated rigorously, [8], [9], MPC is a form of control in which the current control action is obtained by solving, at each sampling instant, a finite or infinite horizon open-loop optimal control problem. In this technique an optimal control sequence is obtained by using the current state of the plant as the initial state of the plant and the first control in this sequence is applied to the plant, while at the next sampling (or decision) instant the whole procedure is repeated.

The process of selecting an optimal control action can be summarized in two distinct steps ([10], [11]),

i) shaping the beliefs of future output performances (forecasts);
ii) the choice of to-be-applied control action as a function of these forecasts.

A general approach to obtain output forecasts is through dynamic models describing the process behaviour. During the initial development of MPC, empirical linear input-output models were utilized. If the operating window is relatively small, such models are proved to be sufficient. However, if the operating conditions vary drastically, e.g., batch processes, then nonlinear models should be used, which effects the complexity of the MPC problem[1]. In either case the developed models will be far from perfect; leading to mismatch between the forecasts and the true behaviour. As a result, the commissioned MPC controllers are kept non-operational frequently due to the model deterioration or lack of maintenance of the model, ([14]). It is both natural and logical to include the effect of (modeled) uncertainty into the prediction model, hence into the optimal control action. In different words, selecting a control action on the basis of the nominal forecasts leads to undesired operation due to definite dispersion from the expectations in the controlled variables. However, uncertainty also radically effects the optimal control actions in closed-loop predictions, casting them to become pessimistic (or aggressive), hence the resulting performance levels are also effected ([15]).

A well established way to overcome or reduce the effects of uncertainty is by applying feedback techniques. In many instances, robust control theory ([16]) provides *sufficient* tools for achieving robust operation. However, this design choice often leads to over-utilization of the available resources as it might not be necessary to execute a pessimistic control law at each time instant. For industrial applications,

[1] Here we do not consider the difficult questions of how and at which complexity level the process model should be constructed. We refer the interested reader to [12], [13] as introductory discussion on modeling uncertain behaviour.

especially in process control industry where economic concerns are directly effecting the operation decisions, the pessimistic control methods are in general rejected and robustness is achieved in an ad-hoc manner ([17]). In recent years, a huge effort has been put in developing computationally efficient (or tractable) and less pessimistic (or adjustable) robust optimization tools that have parameter ambiguity and stochastic uncertainties within the formulation of the optimization (equivalently MPC) problems ([18]).

It is important to distinguish three different robustness aspects of MPC algorithms in the way of treating uncertain effects,

1) robust feasibility,
2) robust stability,
3) robust (closed-loop) performance.

The robust feasibility is about the constraint satisfaction in the face of uncertainty, while the robust stability is tracked via the cost function through Lyapunov based stability arguments. We have a considerable understanding on robust constraint satisfaction or robust stability while the interplay between the uncertainty and the closed-loop performance is yet to be rigorously analyzed. Although there exist some methods to synthesize predictive controllers that operate in a computationally acceptable way ([19]), many of the current robust MPC methods lead to computationally challenging optimization problems, while causing unacceptable levels of performance deterioration. The performance deterioration, or even the total absence of performance, due to overly conservative methods is causing a gap between academic works and industrial implementations. High performance is achieved if the uncertain effects are compensated when it is required, while robustness requirements demand to act in a pre-emptive manner. Hence incorporating *only the necessary* uncertain process predictions into the control action by incorporating risk management techniques is of great interest for predictive control applications.

Combining robust control and predictive control regarding the robust constraint satisfaction, stability and performance aspects with quantitative guarantees is still an open problem. There are a multitude of techniques, detailed in the next sections, to reshape robust MPC (RMPC) (or similarly stochastic MPC (SMPC)) problems. The main dilemma is due to the open-loop nature of predictions, leading to loss of incorporation of *future* uncertainty into the control actions. Dynamic programming (DP) techniques provide a way out of this problem, however the *curse of dimensionality*, specifically for moderate or large scale systems or uncertainty spaces, drastically effects the computational aspects, see [20].

Another important point regarding the industrial acceptance of RMPC algorithms is the computational aspect. It is a requirement that RMPC problems should be consisting of relatively simple and reliable algorithms resulting in desired performance levels of industrial needs ([17]). In the case of industrial implementation, the algorithms should be ([21]);

- easy to interpret and interact by the operator with sufficiently large amount of information on controlled variables and/or constraint violation risks, ([22]);
- relatively simple to solve computationally in repeated fashion, since the industrial platforms are generally not tailored for highly complex and efficient numerical algorithms.

The RMPC techniques developed within the academic practice are severely lacking both of these properties, ([23]). In addition, the resulting closed-loop performances are conservative. Hence, it is no surprising that RMPC has not found applicability in process industry.

The goal of this work is to provide;

- a comprehensive review, discussion and classification of the current RMPC formulations in literature and signify the connections with risk theory and robust optimization research areas;
- a comparison of these formulations regarding their closed loop performance and computational load.

The remainder of the paper is organized as follows. After stating the notations used in this paper at the end of this section, we start discussion with presenting a generic MPC problem, introducing the common ways of incorporating and quantifying the uncertainty in MPC problems and stating the different methods to find the robust counterpart problem in Section II. Section III discusses the RMPC methods and contributions that are introduced with deterministic treatment of uncertainty, either worst-case or uncertainty budget approaches. The SMPC approaches towards uncertain dynamics are discussed in Section IV. We present the moment, probabilistic and randomized MPC contributions and show the possibility of incorporating the theory of risk into the MPC setting. The effectiveness (closed-loop performance) of the methods from literature are demonstrated by means of simulation examples in Section V. Finally, Section VI presents the conclusions.

B. Notation

The field of reals and sets of nonnegative reals, integers and nonnegative integers are denoted by $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, respectively. For a vector $x \in \mathbb{R}^{n\times 1}$ (or in short $x \in \mathbb{R}^n$), $x^\top$ denotes the transpose of that vector. A sequence of vectors $x_i \in \mathbb{R}^n$, until (time) index k, is defined as $x_{[0,k]}^\top := \begin{bmatrix} x_0^\top & x_1^\top & x_2^\top & \dots & x_k^\top \end{bmatrix} \in \mathbb{R}^{nk}$. For a (time) sequence of vectors $x_{[0,t]}$, $x_{i|k}$ denotes the (predicted) vector x_{k+i} for $i,k,t \in \mathbb{Z}_{\geq 0}$ and $k+i \leq t$. Furthermore, $x_{[a,b]|k}$ denotes the sequence of vectors $x_{[k+a,k+b]}$.

For normed vector spaces, $||\cdot||_p$ denotes the standard $p-$norm in $\mathbb{R}^n$. The unit ball in $\mathbb{R}^n$ corresponding to the $p-$norm is denoted with $\mathscr{B}_p^n := \{x \in \mathbb{R}^n |\ ||x||_p \leq 1\}$. If p is not explicitly specified then $||\cdot||$ is taken as the Euclidean norm $||\cdot||_2$, i.e., $\|\cdot\| = \|\cdot\|_2$.

The spectral radius of a matrix A, i.e., $\rho(A)$, is defined as $\rho(A) := \max_i |\lambda_i(A)|$, where λ_i is the i^{th} eigenvalue of matrix A. The identity matrix, with dimension $n \times n$, is denoted by I_n.

Given two vector spaces $\mathscr{X}$, $\mathscr{Y} \subseteq \mathbb{R}^n$, the Minkowski sum of these sets is defined by

$$\mathscr{X} \oplus \mathscr{Y} = \{x + y \in \mathbb{R}^n |\ x \in \mathscr{X},\ y \in \mathscr{Y}\},$$

while the Pontryagin difference of these two sets, assuming $\mathscr{X} \subset \mathscr{Y}$,

$$\mathscr{Y} \ominus \mathscr{X} = \{y \in \mathscr{Y} |\ y + x \in \mathscr{Y},\ x \in \mathscr{X}\}.$$

For stochastic variables, we assume that there is an underlying probability space $(\mathbb{R}^{n_\zeta}, \mathscr{F}, \mathbb{P})$ equipped with the event space $\mathbb{R}^{n_\zeta}$, the $\sigma-$algebra $\mathscr{F}$ defined over the Borel sets of $\mathbb{R}^{n_\zeta}$ and well defined probability measure $\mathbb{P} : \mathfrak{P}^{\mathbb{R}^{n_\zeta}} \to [0,1]$, where $\mathfrak{P}^{\mathbb{R}^{n_\zeta}}$ is the power set of $\mathbb{R}^{n_\zeta}$, see [24], [25] for rigorous treatment of probability spaces. Here, for brevity of discussion, we assume that there is no measurability issues with the stochastic variables evolving over dynamics and used operators. We denote the probability of a random variable $\tilde{w}$, to take values between $\underline{w}$ and $\bar{w}$ as $\mathbb{P}\{\underline{w} \leq \tilde{w} \leq \bar{w}\}$, which is equal to the integral of the probability density function (pdf), $f_{\tilde{w}}(w)$, over the interval $[\underline{w}, \bar{w}]$, i.e.,

$$\mathscr{P}\{\underline{w} \leq \tilde{w} \leq \bar{w}\} = \int_{\underline{w}}^{\bar{w}} f_{\tilde{w}}(w) dw,$$

while for the multi-dimensional case, $\tilde{w} \in \mathbb{R}^{n_\zeta}$, $n_\zeta \geq 1$, the probability is defined over multiple integrals over the considered region $\bar{\mathscr{W}}$. Lastly the mean, the variance, the moment of order n and the covariance matrix of random variables $\tilde{x}$ and $\tilde{y}$ are denoted with $\mu_{\tilde{x}}$, $\sigma^2_{\tilde{x}}$, $\mathbb{E}\{\tilde{x}^n\}$, $\Sigma_{\tilde{x},\tilde{y}}$, respectively.

II. Robustness and MPC Problems

A. A Common RMPC Problem

In this work, we detail our discussion on robustness and MPC for discrete-time linear uncertain systems, denoted with Σ, together with its nominal counterpart Σ^{nom}.

$$\Sigma : \quad \begin{cases} x_{k+1} = & A(\delta_k)x_k + B(\delta_k)u_k + Fw_k, \\ y_k = & C(\delta_k)x_k + D(\delta_k)u_k + v_k, \end{cases} \qquad \Sigma^{nom} \quad : \begin{cases} \bar{x}_{k+1} = & A_0\bar{x}_k + B_0u_k, \\ \bar{y}_k = & C_0\bar{x}_k + D_0u_k, \end{cases} \tag{1}$$

where x_k (or $\bar{x}_k$) $\in \mathbb{R}^n$, and $u_k \in \mathbb{R}^{n_u}$, and y_k (or $\bar{y}_k$) $\in \mathbb{R}^{n_y}$ are the uncertain (or nominal) state, the control input and the uncertain (or nominal) output at discrete time instant $k \in \mathbb{Z}_{\geq 0}$, respectively. The plant Σ is subject to three types of uncertainties denoted by $\delta_k \in \Delta \subseteq \mathbb{R}^{n_\delta}$, $w_k \in \mathscr{W} \subseteq \mathbb{R}^{n_w}$ and $v_k \in \mathscr{V} \subseteq \mathbb{R}^{n_v}$ for $k \in \mathbb{Z}_{\geq 0}$. These uncertainties are either the model uncertainties or the disturbances effecting the state and output equation. These uncertainties might independently take values from bounded sets, i.e., $w_k \in \mathscr{W}$ for $k \in \mathbb{Z}_{\geq 0}$, or can be stochastic vector sequences with known pdfs, e.g., $f_{\tilde{w}_k}(w_k) : \mathfrak{P}^{\mathbb{R}^{n_w}} \to [0,1]$. They may also depend on the instantaneous values states x_k or inputs u_k, e.g., $\delta_k(x_k, u_k)$, implicitly, as discussed thoroughly in [26]. The matrices $A(\delta_k)$, $B(\delta_k)$, $C(\delta_k)$, for all δ_k realizations, are real matrices with dimensions $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times n_u}$, $\mathbb{R}^{n_y\times n}$, respectively. We assume that the uncertain and the nominal systems Σ and Σ^{nom} are stabilizable and observable, see [27] for the definitions of stabilizability and observability. To generalize, we use ζ_k for all uncertain variables, i.e., $\zeta_k^\top = \begin{bmatrix}\delta_k^\top & w_k^\top & v_k^\top\end{bmatrix}$. The i^{th}-step prediction of the state at the k^{th} time instant is denoted with $x_{i|k}$ which, by Eq. (1), depends on the initial point x_k, the system dynamics Σ, the exogenous inputs $u_{[0,i-1]|k}, w_{[0,i-1]|k}$ and the internal uncertainties $\delta_{[0,i-1]|k}$.

In this paper, we consider that the system is subject to hard or soft state (or output) and hard input constraints. The reason for allowing state (or output) constraints to be soft is that these constraints are in general performance requirements while the input constraints are induced from actuator limitations. The constraints are represented here as inequalities

$$c_{ij}(x_{j|k},u_{j|k},\zeta_{j|k},y_{j|k}) \leq 0, \quad \text{for } j\in\mathbb{Z}_{[0,N_p-1]}, \ \text{for } i\in\mathbb{Z}_{[1,N_c^j]}, \tag{2}$$

where N_p denotes the prediction horizon of the MPC controller and N_c^j is the number of constraints for the time step j. In general these constraints are much more explicit, such as set (or zone) membership constraints, i.e., $x_{j|k}\in\mathbb{X}_{j|k}\subseteq\mathbb{R}^n$, $u_{j|k}\in\mathbb{U}_k\subseteq\mathbb{R}^{n_u}$. The computational complexity of the MPC problem is highly dependent on the prediction horizon (N_p), the total number of constraints $N_c := \sum_j N_c^j$ and the convexity properties of the constraints $c_{ij}(\cdot)$, hence each of them effect the resulting closed-loop performance and computational properties of the MPC algorithm.

We cast two distinct MPC problems for systems Σ and Σ^{nom}, denoted with $\mathscr{P}(k)$ and $\bar{\mathscr{P}}(k)$, respectively, at time $k\in\mathbb{Z}_{\geq 0}$. The mismatch between the solutions (or the resulting trajectories) of $\mathscr{P}(k)$ and $\bar{\mathscr{P}}(k)$ is the price paid for robustness:

$$\mathscr{P}(k):\begin{cases}\min\limits_{u_{[0,N_p-1]|k}} & \mathscr{R}^{cost}(J(x_k,u_{[0,N_p-1]|k},\zeta_{[0,N_p-1]|k}))\\ \quad\text{s.t.} & x_{j+1|k}=A(\delta_{j|k})x_{j|k}+B(\delta_{j|k})u_{j|k}+Fw_{j|k},\\ & y_{j|k}=C(\delta_{j|k})x_{j|k}+D(\delta_{j|k})u_{j|k}+v_{j|k},\\ & \mathscr{R}^{const}(c_{ij}(x_{j|k},u_{j|k},\zeta_{j|k},y_{j|k}))\leq 0,\\ & j\in\mathbb{Z}_{[0,N_p-1]},\quad i\in\mathbb{Z}_{[1,N_c^j]},\quad x_{0|k}=x_k,\end{cases} \tag{3a}$$

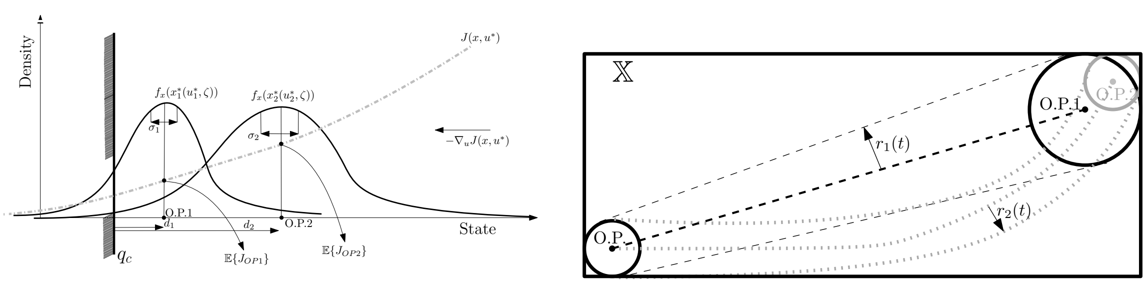
$$\bar{\mathscr{P}}(k):\begin{cases}\min\limits_{u_{[0,N_p-1]|k}} & J(\bar{x}_k,u_{[0,N_p-1]|k},0)\\ \quad\text{s.t.} & \bar{x}_{j+1|k}=A_0\bar{x}_{j|k}+B_0u_{j|k},\\ & \bar{y}_{j|k}=C_0\bar{x}_{j|k}+D_0u_{j|k},\\ & c_{ij}(\bar{x}_{j|k},u_{j|k},\bar{y}_{j|k})\leq 0,\\ & j\in\mathbb{Z}_{[0,N_p-1]},\quad i\in\mathbb{Z}_{[1,N_c^j]},\quad \bar{x}_{0|k}=x_k,\end{cases} \tag{3b}$$

where $J(\cdot)$ is assumed to be the standard quadratic cost function for convenience,

$$\begin{aligned} J(x_k, u_{[0,N_p-1]|k}, \zeta_{[0,N_{p-1}]|k}) := & \sum_{j=0}^{N_p-1} J_r(x_{j|k}, u_{j|k}) + J_f(x_{k+N_p}), \\ = & \sum_{j=0}^{N_p-1} x_{j|k}^\top Q x_{j|k} + u_{j|k}^\top R u_{j|k} + x_{N_p|k}^\top Q_f x_{N_p|k}, \end{aligned} \tag{3c}$$

In Eq. (3c), J_r is named as the *running cost* or the *cost to go* and J_f is the terminal cost, i.e., [28], [3]. In $\mathscr{P}(k)$ we introduce a (risk) functional $\mathscr{R}^{cost} : \mathscr{J}(x,u,\zeta) \to \bar{\mathscr{J}}(x,u)$ (or similarly $\mathscr{R}^{const}(\cdot) : \mathscr{C}(x,u,\zeta,y) \to \bar{\mathscr{C}}(x,u,y)$), which is mapping an uncertain function $\mathscr{J}$ to a deterministic function $\bar{\mathscr{J}}$. The deterministic functions $\bar{\mathscr{J}}$ and $\bar{\mathscr{C}}$ are called as the robust counterparts of the cost ($\mathscr{J}$) or the constraints ($\mathscr{C}$). For linear systems, due to high parametrization, there are easy ways for expressing the robust counterparts of cost and constraint functions, i.e., $\mathscr{R}^{cost}(J)$ or $\mathscr{R}^{const}(c_{ij})$, which is not true for nonlinear system dynamics.

The RMPC problems are dealing with robust stabilization of the system Σ while satisfying the constraints $c_{ij}(\cdot)$ in a risk-aware manner, by solving $\mathscr{P}(k)$. It is expected from a robust predictive controller



(a) Selection of steady state operating point.

(b) Steering in between the operating points.

Fig. 1: The steady state and transition operations with uncertainty.

to adjust the average performance levels versus the constraint violation possibility. Consider the operating a process close to a constraint as depicted in 1a. The distance from the constraint boundary to operating point 1 (O.P.1) or O.P.2, i.e., d_1 and d_2 is a nominal performance metric. Furthermore, the inherent standard deviation in operating conditions σ_{OP1} or σ_{OP2} are relevant for the robustness of the operation. As the controller suppresses the dispersion from O.P.2, the nominal operation can be pushed towards the constraint, such as operating point O.P.1 on average. *Lastly, as also mentioned in [29], nonlinear predictive control techniques can be preferred to control the process, since some uncertainty realizations are not necessarily adversary acts, instead many realizations are pushing the operation to the desired direction in input-state-output space. Is this relavant here?*. For the case of transitions between two operating points, as depicted in Figure 1b 1b, the time span that the servo action to achieve its goals versus the robust constraint satisfaction is another robustness metric. As the controller becomes more aggressive, which is expected to decrease the transition time, the spread of possible trajectories in general grows considerably[2], i.e., for $r_1(t), t \in \mathbb{R}_{[0,t_f^1]}$ and $r_2(t), t \in \mathbb{R}_{[0,t_f^2]}$, in general one satisfies $t_f^1 \leq t_f^2$ only if $r_1(t') \geq r_2(t')$, for $t' \in \bar{t}, \bar{t} = \min(t_f^1, t_f^2)$.

Regarding the robust stability property, three meta-approaches are used in the literature, [8], to achieve guaranteed stability. These approaches are briefly discussed below:
(i) The first approach consists of designing an MPC controller for the nominal system by neglecting uncertainty and achieving robustness in an ad-hoc manner. This approach has close connections with the reasoning where the uncertainty realizations are assumed to be equal to the mean value of the distribution over the prediction time steps, see also Certainty Equivalence principle;
(ii) In the second approach we consider some (or all) of the possible realizations in the uncertainty set to generate the possible trajectories and cast the system to be robust with respect to all of these realizations;
(iii) The last approach consists of solving the closed-loop MPC problem through using a dynamic feedback over the predictions. This case eventually leads to a dynamic programming problem ([30]) which is the most promising case from the perspective of robustness however the computational demand to solve the associated problem increases exponentially, so much that, this approach is not valid for many of the real world examples ([31]).

In the case of first approach, the uncertainty is replaced with an instance thus we handle a nominal MPC problem, however, the effect of other possible realizations are not taken into account, hence no provisions of risk are taken. This leads to possibly frequent and highly effective constraint violations or instability. In practice these drawbacks are suppressed by improving the prediction model, which shrinks the uncertainty set. Furthermore one can make use of the larger MPC problems (with more constraints or scenarios) since nominal MPC problems are computationally much more simpler than the cases where uncertainties are present. To compensate for the effects of the uncertainty, one can incorporate some realizations of uncertainty into the predictions, the case (ii). However, an undesired effect of this choice is that it leads to poor control actions, hence poor responses, since the optimal input depends heavily on the selected (or robustified-against) uncertainty realizations, i.e., $u_k^*(\bar{\zeta}^1_{[0,N_p-1]|k}) \neq u_k^*(\bar{\zeta}^2_{[0,N_p-1]|k})$, where $u_k^*(\bar{\zeta}^i_{(\cdot)})$ is the optimal solution of MPC problem $\mathscr{P}(k)$ with respect to the uncertain variables $\bar{\zeta}^i_{[0,N_p-1]|k}$. In the next section we introduce the possible uncertainty descriptions.

[2]In Section V we implement various robust MPC techniques for a batch reactor and a CSTR system in simulation environment to compare the closed-loop performance metrics such as the ones mentioned here

B. Uncertainty Descriptions & Uncertainty Quantification

The internal model principle states that the uncertainty model should resemble the true uncertainty effecting the system to counteract adverse effects of it. However, the complexity to describe the uncertainty is a crucial factor on the feasibility and the practical applicability of the RMPC algorithms.

Here we group the uncertainties as model based (internal) uncertainties and environment based (exogeneous) uncertainties. The mismatch between the process and the mathematical model (internal uncertainty) can be induced from, [16], unmodeled dynamics, time varying effect in the process or changing loads, while the external uncertainties, the disturbance signals and the output noise, are effecting the control input, the state evolution or the measurement signals.

1) Modeling the Internal Uncertainties: We classify the model mismatch possibilities as;

- **Multi-model Uncertainty:** Equally acceptable class of models, a countable set of models likely to represent the true system, see [32] as an example,

$$\left[\begin{array}{c|c} A(\delta) & B(\delta) \\ \hline C(\delta) & D(\delta) \end{array}\right] \in \left\{ \left[\begin{array}{c|c} A(\delta_i) & B(\delta_i) \\ \hline C(\delta_i) & D(\delta_i) \end{array}\right] | \delta_i \in \Delta, i = 1,2,\ldots,N_{MM} \right\}. \tag{4}$$

- **Unknown-but-bounded uncertainty description:** By assigning a nominal model with describing the uncertainty set Δ in terms of; (i) affine relations on the system matrices, such as, for the polytopic affine uncertainties case, which is quite common in control relevant literature

$$\left[\begin{array}{c|c} A(\delta) & B(\delta) \\ \hline C(\delta) & D(\delta) \end{array}\right] \in \Delta := \left\{ \left[\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array}\right] + \sum_{i=1}^{N_v} \delta_i \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], \sum_{i=1}^{N_v} \delta_i = 1, \delta_i \geq 0 \right\},$$

or for the ellipsoidal affine uncertainty case, which is not commonly preferred in MPC literature,

$$\left[\begin{array}{c|c} A(\delta) & B(\delta) \\ \hline C(\delta) & D(\delta) \end{array}\right] \in \Delta := \left\{ \left[\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array}\right] + \sum_{i=1}^{N_v} \delta_i \left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], (\delta - \delta_c)^\top Q_\delta (\delta - \delta_c) \leq 1, \right\},$$

(ii) parameteric (structured or unstructured, dynamic or static) uncertainty description, ([33]), such as

$$\Sigma : \left\{ \begin{array}{rl} x_{k+1} = & A_0 x_k + B_0 u_k + B^\delta \delta_k, \\ y_k = & C_0 x_k, \\ q_k = & C^\delta x_k + D^\delta u_k, \\ \delta_k = & \Delta q_k, \end{array} \right.$$

where Δ is a(n) (un-)structured, dynamic (or static) operator with norm bounds on the (possibly time-varying) elements; (iii) uncertainty in impulse response coefficients ([34]) such as $\Delta^h = \{h_k | h_k^l \leq h_k \leq h_k^u\}$, where $y_k = \sum_{\tau=0}^{k} h_{k-\tau} u_\tau$, or uncertainty in frequency response values $\Delta(j\omega) = \{\delta(j\omega) \in \mathbb{C}(j\omega), \omega \in [0,\infty)\}$, where $Y(j\omega) = G(j\omega)(I + \Delta(j\omega))U(j\omega)$ and $G(j\omega)$ is the nominal transfer function. In some cases uncertain systems modeled via the linear fractional representation (LFR) are preferred in comparison to other uncertainty models, due to the ease of incorporation of a larger set of possible dynamics, see the discussion in [35], [36] or [16].

- **probability distribution functions for uncertainties:** by assigning probabilistic information to the parameters of uncertain models to describe the internal uncertainties. One possible benefit of utilizing stochastic descriptions is that the models/parameters identified from data yield statistical information which is difficult to express in deterministic uncertainty models such as the previous cases.

2) Modeling the External Uncertainties: External uncertainties are the unknown signals affecting the system in an exogenous way, w_k and v_k vectors in Equation (1). In this paper we note three distinct approaches for modeling the external uncertainty while considering only the effect of w_k, since a similar reasoning is valid also for noise vector v_k.

- **The uncertainties with discrete set of realizations:** The uncertainties with an event space of finite number of realizations can effect the plant exogenously, such as different load configurations,

$$\Sigma : \begin{cases} x_{k+1} = & Ax_k + Bu_k + Fw_k, \\ y_k = & Cx_k + Du_k, \end{cases} \tag{5}$$

  where $w_k \in \mathscr{W} := \{w^1, w^2, \dots, w^{N_s}\}$. Since the number of realizations are finite, one can come up with all of the possible instances, called also as the scenario tree.

- **Unknown-but-bounded disturbances:** Uncertainties with bounded support can also be affecting the processes, the case for Σ in (5) with $w_k \in \mathscr{W} \subset \mathbb{R}^{n_w}$. In literature, these uncertainty sets are mainly considered to be either polytopic or ellipsoidal sets, due to the numerical properties of these set classes, see [37] for an extensive discussion on the set-theoretic methods.
- **Statistical descriptions:** The distributional uncertainties can also be used instead of bounded disturbances. In this case the uncertain state evolution can be modeled as driven by an exogenous signal that is assuming realizations from its pdf. such as the estimated (initial) state that is deviating from the true state with a finite covariance value.

Remark 1. *The uncertainty models are in general not equivalent to each other, that is the uncertain dynamics describe different behaviours (or different processes) for different uncertainty descriptions. This results in MPC algorithms belonging to different complexity classes. The MPC literature is very scarce with regard to rigorous uncertainty modeling, in general the uncertainties are not modeled at all, instead uncertainty descriptions are taken as a given.*

C. Robust Counterparts of MPC Problems

One can distinguish three different sources of uncertain predictions in MPC problems; i) wrong initial condition estimation due to measurement noise or lack of sensors measuring all of the states; ii) perturbations to the dynamics; iii) plant-prediction model mismatch. In any type of uncertainty, the state predictions result in a collection of trajectories (with or without stochastic properties). To incorporate possible prediction errors to the MPC formulation, one needs to apply a risk-mapping ($\mathscr{R}^{cost}$ or $\mathscr{R}^{const}$ in Equation (3a)) to the functions of uncertain state, i.e., the cost and constraint functions. There are five different common approaches to map an uncertain optimization problem, which are the certainty equivalence, the scenario based, the worst case based, the moment based, or the quantile (probabilistic) based mappings, see Figure 2. The resulting MPC problem, and hence the control law, differs according

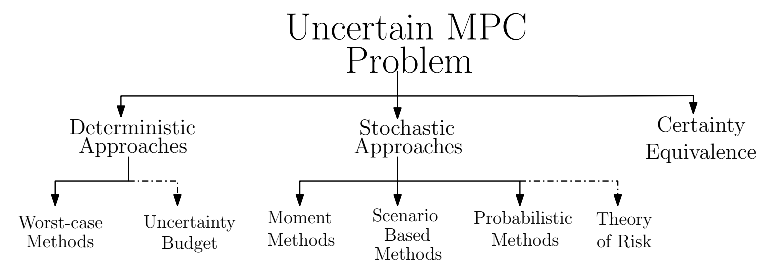


Fig. 2: Commonly used projection techniques for uncertain optimization problems, in particular RMPC.

to the used risk mapping. These risk mappings re-shape the feasible area of the optimization problem, in general drastically tightening it, see Figure 3 for the worst case, the probabilistic and the moment based cases, but also the selected method effects the computational complexity properties of the robust counterpart problem. We summarize the effects of these risk mapping methods in Table I, where the computational complexity of the nominal MPC problem is denoted with $\mathscr{O}(\bar{\mathscr{P}})$, the uncertain constraint (w.l.o.g. the cost) function with $c(x_{j|k}, u_{j|k}, y_{j|k}, \zeta_{j|k}) \leq 0$ or simply $c(\zeta) \leq 0$, the functions mapping the uncertain constraints to robust counterparts with $\mathscr{R}_i$, where i is the indicator of the respective mapping, the probability of satisfying the constraint with $\mathbb{P}\{c(\zeta) \leq 0\}$; 'coherent' corresponds to the risk function $\mathscr{R}_i$ being a coherent risk metric, implying a convex, monotonic and closed mapping, see [38] for the rigorous definition of coherence. All of the techniques result in a trade-off between various aspects, the constraint satisfaction guarantees, the resulting risk aversity, the computational complexity and the effort to model the uncertainty set. In the robust MPC literature, a rigorous discussion encompassing all aspects of the risk-mappings in Table I is missing, even for the linear systems case.

Some of these techniques are already well studied, such as the worst-case or the scenario based approach, leading to many survey articles and books available in literature, see Table II. Furthermore we

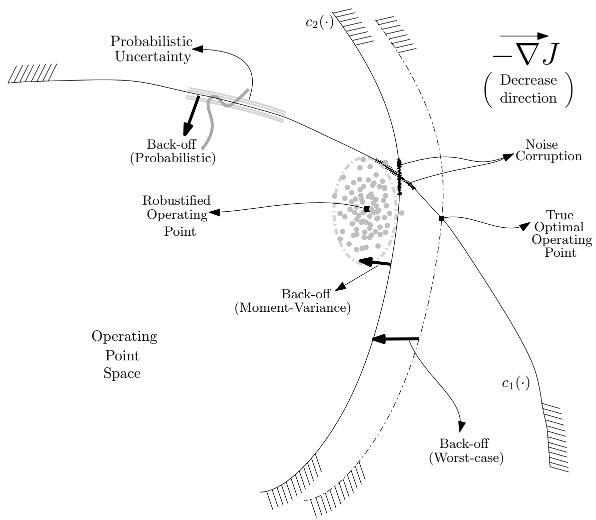


Fig. 3: The operating point space and various forms of mapping the uncertainty to a deterministic counterpart.

mention two topics that are closely related to the RMPC research topic, the contributions from the robust optimization and the (asymmetric) risk metrics research areas.

Until recently many of the RMPC problem constructions followed solely the robustness with respect to worst case approach. We also start our discussion with this approach.

III. Deterministic Approach to Robustness in MPC

In this section we first introduce the worst case (also called as the min-max) MPC approach, then we discuss the uncertainty budgets and their relation with MPC. Lastly, we present and classify contributions from MPC with deterministic robustness properties.

A. Worst Case MPC Problems

The worst-case (WC) optimization approach can be summarized with the following three distinct statements;

- The control action is calculated either in a "here-and-now" fashion, the uncertainty reveals itself after the control decisions are made, or "wait-and-see" fashion, the control action is decided after some of the uncertain variables reveal themselves to the decision maker.
- The decisions are made for, and only for, a known/decided subset of the uncertainty. If the whole (true) uncertainty set, say Δ, is taken, then we call it the worst case MPC (WC-MPC), else, if a strict subset is taken into account $\bar{\Delta} \subset \Delta$, then we name it the budgeted worst case MPC (BWC-MPC).

Approach		Uncertainty and Risk Function	Constraint Guarantees	Risk Aversity	Computational Complexity	Modelling Uncertainty
Scenario Based Approach	Nominal $\mathscr{P}^{CE}$	$\mathscr{R}_{CE}: c(\zeta) \to c(\bar{\zeta})$, $\bar{\zeta} = \mathbb{E}\{\zeta\}$ Coherent	No guarantees	Optimistic Risk-loving	$\mathscr{O}(\mathscr{P}^{CE}) = \mathscr{O}(\bar{\mathscr{P}})$	*Easy* Calculate $\mathbb{E}\{\zeta\}$
	Randomized $\mathscr{P}^{rand}$	$\mathscr{R}_{rand}: c(\zeta) \to c(\bar{\zeta}^s)$, $s = 1, \ldots, N_s$ Coherent	Probabilistic guarantees $\mathbb{P}\{c(\zeta) \leq 0\} \geq \beta(N_s)$ with prob. $1-\varepsilon(N_s)$	Risk Decision Select N_s As $N_s \gg 1$ Pessimistic	$\mathscr{O}(\mathscr{P}^{rand}) \approx N_i \mathscr{O}(\bar{\mathscr{P}})$ Parallelizable	*Easy* Generate scenario tree
Worst Case Based Approach	Full Set $\mathscr{P}^{WC}$	$\mathscr{R}_{WC}: c(\zeta) \to \max_{\zeta \in \Delta} c(\zeta)$, Coherent	$\mathbb{P}\{c(\zeta) \leq 0\} = 1$	Highly pessimistic	$\mathscr{O}(\mathscr{P}^{WC}) \gg \mathscr{O}(\bar{\mathscr{P}})$ Reformulations exist	*Difficult* All possible realizations
	Uncertainty Budget Set $\mathscr{P}^{BWC}$	$\mathscr{R}_{BWC}: c(\zeta) \to \max_{\zeta \in \bar{\Delta}} c(\zeta)$, Coherent	$\mathbb{P}\{c(\zeta) \leq 0\} = \alpha_{BWC}(\bar{\Delta})$ $\lim_{\bar{\Delta} \to \Delta} \alpha_{BWC} \to 1$	Risk Decision Select $\bar{\Delta}$ As $\bar{\Delta} \to \Delta$ Pessimistic	$\mathscr{O}(\mathscr{P}^{BWC}) \gg \mathscr{O}(\bar{\mathscr{P}})$ Reformulations exist	*Easy* Select realizations
Moment Based Approach	Mean $\mathscr{P}^{M}$	$\mathscr{R}_{M}: c(\zeta) \to \mathbb{E}\{c(\zeta)\}$, Coherent	$\mathbb{P}\{c(\zeta) \leq 0\} = \alpha_M$ α_M depends on $c(\cdot)$	Optimistic Risk-loving	(Online) $\mathscr{O}(\mathscr{P}^{M}) = \mathscr{O}(\bar{\mathscr{P}})$ (Offline) $\mathscr{O}(\mathbb{E}\{c(\cdot)\})$	*Easy* Convergent with samples
	Mean Variance $\mathscr{P}^{MV}$	$\mathscr{R}_{MV}: c(\zeta) \to f_{MV}(c(\cdot), \zeta)$ $f_{MV}(\cdot) = \mathbb{E}\{c(\zeta)\} + \lambda_v \mathbb{D}\{c(\zeta)\}$, Not coherent Not monotonic	$\mathbb{P}\{c(\zeta) \leq 0\} = \alpha_{MV}$ $\lim_{\lambda_v \to \infty} \alpha_{MV} \to 1$	Risk Decision Select λ_v As $\lambda_v \gg 1$ Pessimistic	(Online) $\mathscr{O}(\mathscr{P}^{MV}) = \mathscr{O}(\bar{\mathscr{P}})$ (Offline) $\mathscr{O}(\mathbb{E}\{c(\cdot)\}, \mathbb{D}\{c(\cdot)\})$	*Easy* Convergent with samples
Probabilistic Approach	$\mathscr{P}^{CC}$	$\mathscr{R}_{CC}: c(\zeta) \to q_{\alpha_{CC}}(c(\zeta))$, Not coherent Not convex	$\mathbb{P}\{c(\zeta) \leq 0\} = \alpha_{CC}$	Risk Decision Select α_{CC} As $\alpha_{CC} \to 1$ Pessimistic	(Online) $\mathscr{O}(\mathscr{P}^{CC}) \gg \mathscr{O}(\bar{\mathscr{P}})$ (Offline) Calculate $\mathbb{P}\{c(\zeta) \leq 0\}$	*Difficult* Calculate true pdf of ζ, and propogate ζ

TABLE I: Common approaches and properties of projecting an uncertain MPC problem to a robust counterpart MPC problem.

	Nominal MPC	Robust MPC	Stochastic MPC	Practical Aspects Industrial Applications	Robust Optimization	Theory of Risk
Surveys	[7], [8], [9], [21], [23] [34], [39], [40]	[7], [8], [9], [17], [20] [21], [23], [33], [34], [39] [40]	[17], [23], [31]	[41], [42], [43], [44]	[18], [45]	[18], [45], [46], [47], [48]
Books	[29], [49], [50], [51], [52], [53], [54], [55], [56], [57]	[29], [49], [50], [51], [52], [53], [54], [56]	[49], [52], [56]	[29], [49], [50], [54], [57]	[15]	[15]

TABLE II: Surveys and books on MPC with robustness properties.

- Any realization from the uncertainty set can not violate the constraints or destabilize the closed-loop system.

All of these statements are highly effective on the resulting control action and introduces the high pessimism in closed-loop response. This is due to the fact that the true trajectories of the system and predicted, but highly unlikely, set of trajectories deviate from each other, see Figure 4. Allowing the

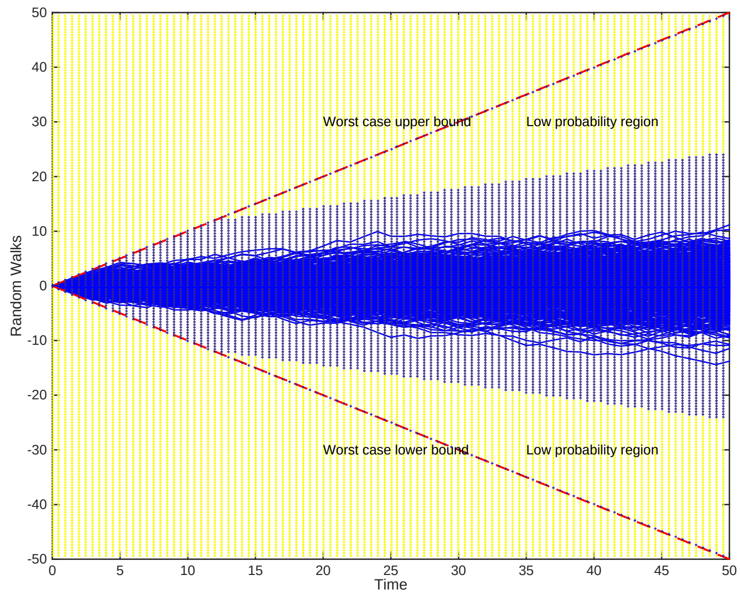


Fig. 4: Fan of trajectories and the conservative worst-case bounds on the trajectories, for the discrete time integrator system effected by additive uncertainty with bounds $[-1, 1]$, inspired from [58].

designer to decide on the set of uncertainty realizations, i.e., $\bar{\Delta}$, might eventually improve the closed-loop performance. By actively selecting the set of uncertainties, one can quantitatively discuss the trade-off of including a larger set of uncertainty, or equivalently adjusting the operating point.

In many cases WC-MPC problems are generally presented with here-and-now strategy and full uncertainty set case ([59]). The inherent pessimism within the construction is difficult to avoid, which leads to a tremendous effort in incorporating closed-loop predictions into the RMPC algorithms and computational problems. To reduce the pessimism, one can allow the control actions to be parameterized as function of future uncertainties or cast them as control policies, the wait-and-see formulation. Once the uncertainty set is decided and the control law structure (sequence or policies) is selected, then the distinction between different min-max MPC techniques depends on how the stability and constraint satisfaction are guaranteed. Various methods are proposed in the literature which we group in Figure 5.

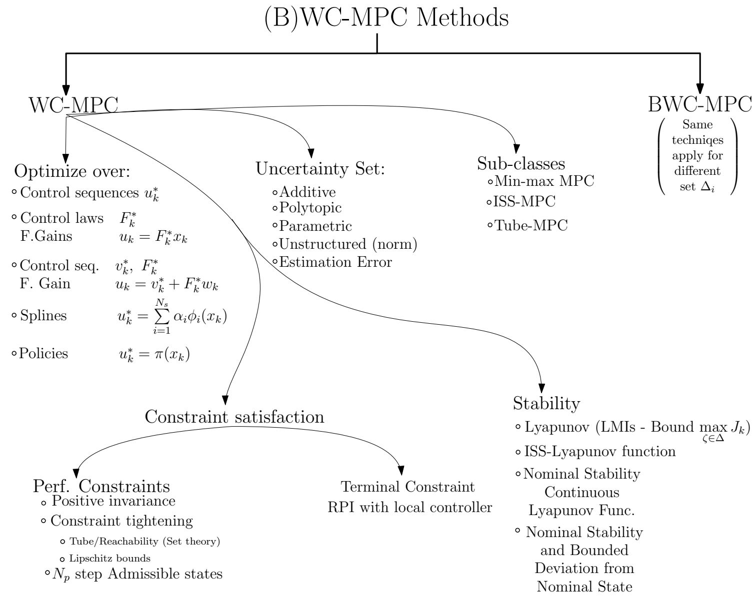


Fig. 5: Diagram visualizing the commonly used techniques for WC-MPC methods, with the approaches utilized for achieving robust constraint satisfaction or stability.

One example of WC-MPC problems is given as in $\mathscr{P}^{WC}$,

$$\mathscr{P}^{WC}: \begin{cases} \min\limits_{u_{[0,N_p-1]|k}} \max\limits_{\zeta_{[0,N_p-1]|k}} & \sum\limits_{j=0}^{N_p-1} x_{j|k}^\top Q x_{j|k} + u_{j|k}^\top R u_{j|k} + x_{N_p|k}^\top Q x_{N_p|k}, \\ \text{s.t.} & x_{j+1|k} = A(\delta_{j|k}) x_{j|k} + B(\delta_{j|k}) u_{j|k} + F w_{j|k}, \\ & y_{j|k} = C(\delta_{j|k}) x_{j|k} + D(\delta_{j|k}) u_{j|k} + v_{j|k}, \\ & c_{ij}(x_{j|k}, u_{j|k}, \zeta_{j|k}, y_{j|k}) \leq 0, \ \forall \zeta_{[0,j]|k} \in \Delta_{[0,j]|k}, \\ & \zeta_{j|k}^\top = \begin{bmatrix} \delta_{j|k}^\top & w_{j|k}^\top & v_{j|k}^\top \end{bmatrix} \in \Delta_{j|k}, \\ & i = 1, \ldots, N_c^j, \quad j = 0, \ldots, N_p - 1, \quad x_{0|k} = x_k, \end{cases} \tag{6}$$

hence the risk mappings $\mathscr{R}^{cost}$ and $\mathscr{R}^{const}$ are selected as

$$\mathscr{R}_{WC}^{cost}(J) = \max_{\zeta \in \Delta} J(\zeta), \quad \mathscr{R}_{WC}^{const}(c_{ij}) = \max_{\zeta \in \Delta} c_{ij}(\zeta).$$

Here two important aspects of risk mappings $\mathscr{R}^{(\cdot)}$ are striking, (i) one minimizes the worst case cost and (ii) the constraints $c_{ij}(\cdot)$ should be satisfied for all realizations of $\zeta_{[0,j]|k}$. From these points and regarding the closed-loop performance, the min-max-MPC formulation is;

- leading to excessive backing-off from the nominally optimal operating point, since the constraints are forced to be satisfied for all uncertainty realizations within the prediction horizon. This results in a shrinking feasible set of control actions, see Table I. Furthermore the min-max MPC algorithms are highly fragile to uncertainties with outlier realizations or unstable dynamics;

- considering the worst case cost hence further deteriorating the average control performance. The worst-case cost is not a good representative of the true performance measure, in general, since

$$u_k^*(\arg\min_u \max_{\zeta\in\Delta} J(\zeta)) \neq u_k^*(\arg\min_u \mathscr{R}(J(\zeta))),$$

where $\mathscr{R}_{(\cdot)}$ is a risk-mapping other than worst case method with full set of uncertainty, see Table I.

From the computational point of view, the main difficulty is in the inner maximization step (for the cost function) or the satisfaction of constraints for all realizations, leading to semi-infinite constraints for uncertainties with continuous domains. For convex problems, the constraint satisfaction is relatively well discussed in literature ([60]). The tube-based MPC is introduced into the RMPC (and later to SMPC) area by incorporating reachability analysis. For guaranteeing robust stability the stability conditions should be assured while guaranteeing the constraint satisfaction after the prediction horizon and bounded gains from the uncertain effects. A common technique to establish robust stability is to; 1) use a robust positively invariant (RPI) set, i.e., a set $\mathscr{X}_{RPI}$ satisfying the invariance property over time iterations;

$$x \in \mathscr{X}_{RPI} \to f(x,u,\zeta) \in \mathscr{X}_{RPI}, x \in \mathscr{X}_{RPI}, \exists u \in \mathscr{U}, \forall \zeta \in \Delta,$$

where $f(x,u,\zeta)$ is the dynamics of the considered system. The set $\mathscr{X}_{RPI}$ is computationally difficult to find for large scale nonlinear processes; 2) construct a (locally continuous) Lyapunov function (inside the RPI set) that is sufficiently decreasing after each step. This approach is then further investigated within the input-to-state-stability (ISS) reasoning.

Remark 2. *The invariant sets play a crucial role in MPC problems and here we mention some important contributions on the area of invariant set calculations. In [61], authors present the construction of minimal invariant sets for linear systems with additive external uncertainties. A comprehensive treatment on the invariant sets of dynamical systems is also conducted in [62]. The paper [63] investigates the polytopic invariant set approximations from the ellipsoidal invariant sets for nonlinear systems effected by parametric uncertainties and exogenous inputs*[3]*. The wrapping effect, the phenomena observed when sets are over-approximated with simpler (to parametrize) sets, severely affects the invariant sets' size over the approximation iterations. One effective treatment of this issue is discussed in [66], through the use of zonotopes.*

[3]We skip detailed discussion on calculation of invariant sets for dynamical systems affected by additive or multiplicative uncertainties, however interested reader is referred to [64], [65] and also [37] for further information on the set-theoretic methods in control theory applications.

Min-max MPC methods are well-established by now, many different approaches and numerical methods exists in the literature to overcome known drawbacks of the this type of robust MPC. Here, in the next section, we discuss briefly two approaches that stem from original min-max MPC approach, the input-to-state MPC and tube based MPC techniques. The ISS-MPC approaches are tuned towards establishing stable operation via MPC controllers for nonlinear systems, though ISS-MPC results for linear systems case also exists. Tube based MPC is essentially taking advantage of highly parameterized state prediction trajectories, which are expressed as sets that construct the tube, which is then used to reformulate the min-max problem as a nominal MPC problem with tighter requirements for stability, in the sense of tighter constraint or the Lyapunov function decrease.

*1) **Input-to-state stability and RMPC:*** The notion of input-to-state stability ([67]) is frequently used to establish robust stability properties of uncertain (generally nonlinear) systems. One can make use of the ISS-Lyapunov functions to guarantee robust stability under continuity conditions on dynamics and/or the cost function. For the case of constrained problems, the constraints should also be tightened to account for the uncertain effects. Here we present the case by using explicit (but generally theoretical) Lipschitz bounds on the uncertain trajectories, summarized from [68]. Consider a nonlinear system with additive uncertainty, expressed with its nominal counterpart as

$$\Sigma : \begin{cases} x_{k+1} = & f(x_k, u_k, \zeta_k), \\ y_k = & h(x_k, u_k, \zeta_k), \end{cases} \quad \Sigma_{nom} : \begin{cases} \bar{x}_{k+1} = & \bar{f}(\bar{x}_k, u_k), \\ \bar{y}_k = & \bar{h}(\bar{x}_k, u_k), \end{cases} \tag{7}$$

with $\zeta_k \in \Delta$. Now assume that there exists a Lipschitz gain L_f for the nominal system, i.e.,

$$||\bar{f}(\bar{x}_1, \kappa(\bar{x}_1)) - \bar{f}(\bar{x}_2, \kappa(\bar{x}_2))||_p \leq L_f ||\bar{x}_1 - \bar{x}_2||_p$$

holds for all $\bar{x}_1, \bar{x}_2$ inside the region of interest $\mathscr{X}$ and $\kappa(\bar{x}) \in \mathbb{U}$ is an admissible control law. If the uncertainties are bounded, i.e., $\zeta_k \in \Delta \subseteq \gamma \mathscr{B}_p(0)$, for all $k \in \mathbb{Z}_{\geq 0}$, then the predictions of the true state $x_{j|k}$ can be encapsulated inside a cone defined through the Lipschitz gain L_f, γ and $\bar{x}_{0|k} = x_k$ which is an overapproximation of the reachable set[4]. We denote the upper bounds as $\gamma^j_{L_f,\gamma} \mathscr{B}_p(0)$, where $\gamma^j_{L_f,\gamma}$ is the calculated upper bound, tighten the constraints with $\gamma^j_{L_f,\gamma}$ as $\bar{\mathscr{X}}_{j|k} := \mathscr{X} \ominus \gamma^j_{L_f,\gamma} \mathscr{B}_p(0)$. Under some technical properties on the dynamics and cost function, the optimal solution u^*_k of $\mathscr{P}^{ISS}$ (Equation (8))

[4]Here we make use of open-loop predictions, which leads to highly conservative results in closed-loop evolution. Parameterizing the control actions as control laws and generating the closed-loop predictions of state inherently reduces the associated pessimism, see [69] for a detailed discussion on the closed-loop ISS-MPC methods. Furthermore, an ISS-MPC technique in which the Lipschitz gain L_f is parameterized in terms of state and input actions is presented in [70], in which the robustness properties, and hence the required constraint tightening and pessimism, can be adjusted in real-time.

steers the state into the $\mathscr{X}_{Ter}$, guarantees recursive feasibility, and robustly stabilizes the closed-loop system,

$$\mathscr{P}^{ISS}: \begin{cases} \min\limits_{u_{[0,N_p-1]|k}} & \sum\limits_{j=0}^{N_p-1} \bar{x}_{j|k}^\top Q \bar{x}_{j|k} + u_{j|k}^\top R u_{j|k} + \bar{x}_{N_p|k}^\top Q \bar{x}_{N_p|k}, \\ \text{s.t.} & \bar{x}_{j+1|k} = f(\bar{x}_{j|k}, u_{j|k}), \\ & \bar{x}_{j|k} \in \bar{\mathscr{X}}_{j|k}, \quad u_{j|k} = \kappa(\bar{x}_{j|k}) \in \mathbb{U}, \quad \bar{x}_{N_p|k} \in \mathscr{X}_{Ter} \\ & j = 0, \ldots, N_p - 1, \quad \bar{x}_{0|k} = x_k, \end{cases} \tag{8}$$

Performance wise, by construction, the Lipschitz constant cased bounds, i.e., $\gamma_{L_f}^j$ for $j \in [1, N_p]$, can be overly conservative, causing over-tightening of the constraints, which potentially leads to infeasibilities or poor average performances. Computationally it is not clear whether the computational load is due to the requirements of ISS-MPC or naturally induced from the nonlinearities of the plant.

*2) **Tube Based RMPC**:* The tube-based MPC is another WC-MPC approach based on the reachability analysis and set-theoretic operations. The interesting observation of tube-based MPC is the separation of the robustness problem from the MPC problem. Consider the linear system Σ and the nominal counterpart as in Equation (1). The idea is to keep the true state x_k close to the nominal state $\bar{x}_k$ over time instants $k \in \mathbb{Z}_{\geq 0}$, while controlling the nominal state $\bar{x}_k$ into a RPI set $\mathscr{X}_{Ter}$ at time $k + N_p$. For this purpose (although not necessary) with a "low-level" controller K, the control signal is constructed as $u_k = \bar{u}_k + K(x_k - \bar{x}_k)$, where $K(x_k - \bar{x}_k)$ term attenuates the effect of the uncertainties affecting the true state x_k but not the nominal (and virtual) state $\bar{x}$. The tube is then all the possible values of the mismatch $x_{j|k} - \bar{x}_{j|k}$ over the prediction horizon $j \in \mathbb{Z}_{[0,N_p]|k}$. Hence by subtracting the tube from the original constraints, the tightened constraints that are used for the are found the MPC problem cast on nominal system Σ^{nom}. Controlling the nominal state, which is defined to be the center of the tube, within the tightened constraint sets guarantees that the true state trajectory is inside the original constraint sets. With several technical but common conditions on the nominal MPC problem, the nominal system is then stabilized, but more importantly due to the fact that the reachable set of true state's deviation from nominal state is stabilized while adhering the constraints, the robust stability is also established. A tube-MPC problem, considering system Σ with additive uncertainty, can be stated as follows;

$$\mathscr{P}^{tube}: \begin{cases} \min\limits_{\bar{u}_{[0,N_p-1]|k}} & \sum\limits_{j=0}^{N_p-1} \bar{x}_{j|k}^\top Q \bar{x}_{j|k} + \bar{u}_{j|k}^\top R \bar{u}_{j|k} + \bar{x}_{N_p|k}^\top Q_f \bar{x}_{N_p|k}, \\ \text{s.t.} & \bar{x}_{j+1|k} = A\bar{x}_{j|k} + B\bar{u}_{j|k}, \\ & \bar{x}_{j|k} \in \bar{\mathbb{X}}_{j|k}, \quad \bar{u}_{j|k} \in \bar{\mathbb{U}}_{j|k}, \quad \bar{x}_{N_p|k} \in \bar{\mathbb{X}}_{Ter|k}, \\ & j = 1, \ldots, N_p - 1, \quad \bar{x}_{0|k} = x_k \end{cases} \tag{9}$$

with the standard stability conditions on the terminal set $\bar{\mathbb{X}}_{Ter|k}$, the terminal cost Q_f and the terminal controller K_f[5]. The tightened constraints $\bar{\mathbb{X}}_{j|k}$, $\bar{\mathbb{X}}_{Ter|k}$, $\bar{\mathbb{U}}_{j|k}$ are calculated from reachable sets of dynamics and the Pontryagin difference operation, following a three-step procedure;

i) Compute the reachable states from the initial state x_k with the plant dynamics Σ and the uncertainty sets, e.g., $w_{j|k} \in \mathscr{W}$, for the prediction horizon as

$$\mathscr{X}_{[1,N_p]|k} := \begin{bmatrix} \mathscr{X}_{1|k} & \dots & \mathscr{X}_{N_p|k} \end{bmatrix}, \quad \mathscr{X}_{i|k} := \{x' | x' = Ax'' + Fw, w \in \mathscr{W}, x'' \in \mathscr{X}_{i-1|k}\}, \tag{10a}$$

with $\mathscr{X}_{0|k} := \{x_k\}$[6];

ii) Calculate the tightened constraint sets by

$$\begin{aligned} \bar{\mathbb{X}}_{[1,N_p-1]|k} &:= \mathbb{X} \ominus \mathscr{X}_{[1,N_p-1]|k}, \quad \mathscr{U}_{[1,N_p-1]|k} := K\mathscr{X}_{[1,N_p-1]|k}, \\ \bar{\mathbb{X}}_{Ter|k} &:= \mathbb{X}_{Ter} \ominus \mathscr{X}_{N_p|k}, \quad \bar{\mathbb{U}}_{[1,N_p-1]|k} := \mathbb{U} \ominus \mathscr{U}_{[1,N_p-1]|k}, \end{aligned} \tag{10b}$$

where $\mathbb{X}$ and $\mathbb{U}$ are the original constraint sets;

iii) Solve the nominal MPC problem $\mathscr{P}^{tube}$ to find $\bar{u}_{[0,N_p-1]|k}$ and apply the first control signal to the nominal system while applying the control signal $u_k = \bar{u}_k + K(x_k - \bar{x}_k)$ to the plant.

Performance-wise, if no precompensator, K, is used, since the reachable state sets are equivalent, the tube-MPC is equivalent with the min-max MPC. However a well-defined feedback structure with K causes the deviations to be suppressed between the nominal and the true state within the predictions and hence results in a tube diameter which is much smaller in comparison to the $K = 0$ case. This is expected to increase the closed-loop performance by allowing the MPC controller act on top of the controller K. On the other hand one important drawback of tube based MPC with controller structure is the shrinking input constraint set due to the compensation action $K(x_k - \bar{x}_k)$. If, as suggested earlier, the compensator is aggressive to suppress the deviation $(x_k - \bar{x}_k)$, achieved by a high gain static feedback matrix K, the input constraints are tightened substantially. This tightening can be so much that the feasible set becomes an empty set for realistic disturbance bounds.

Computationally, if one considers only the MPC problem, then it is simplified since one solves an nominal MPC problem. However tube-MPC approach contains (i) complex set-based operations, Minkowski sums (for reachable set calculations) and Pontryagin differences (for constraint tightening step), and (ii) calculation of a prestabilizing controller K for the process. Two major issues are inherent to set-based operations, (a) the memory requirement for expressing the n-step reachable sets grow quite

[5]The terminal controller K_f is not necessarily equal to the low-level precompensator K.

[6]One can also incorporate nonlinear dynamics, hence input actions, and also compact and bounded sets as the initial condition $\mathscr{X}_{0|k}$ into the reachable set of disturbance effected state formulation.

fast (and approximations induce further conservativeness), (b) the effect of dimensionality, for medium to large sized systems (say more than 100 states), for the mentioned set operations are leading to numerical problems, even in the simplest forms of sets, see [71] for a discussion.

Remark 3. ***Affine State or Disturbance Feedback Structures:*** *An important development in the RMPC area is using low level controllers within the optimization step. The validity of using these controllers can be seen from the tube-MPC construction, the set of reachable states may shrink for a stabilizing controller K. In parallel to this observation, one might seek to optimize the state feedback controller K at each time step, by optimizing over control actions* $\bar{u}_{[0,N_p-1]|k}$ *and gain matrices* $K_{[0,N_p-1]|k}$*, i.e.,*

$$u_{j|k} = \bar{u}_{j|k} + \sum_{i=0}^{j} K_{j,i|k} x_{i|k}.$$

However the resulting MPC problem in $\bar{u}_{j|k}$ *and* $K_{j,i}$ *is shown to be non-convex, hence leads to sub-optimality or computational issues. The remarkable extension comes via adjustable robust optimization (ARO) approach, leading to convex RMPC problems with affine disturbance feedback structures, ([72]),*

$$u_{j|k} = \bar{u}_{j|k} + \sum_{i=0}^{j} K_{j,i|k} w_i.$$

B. Price of Robustness, Uncertainty Budgets and RMPC Problems

In recent years the robust optimization (RO) techniques, e.g., ([15], [18]), highlight one crucial question relevant also in the robust control area, “what is the corresponding performance deterioration with respect to the uncertainty set?”. Here we name this aspect as the price of robustness or uncertainty budgets for robustness. Now consider the approach as in the worst-case MPC reasoning, but replace the uncertainty set Δ with a subset of it, say $\bar{\Delta}$. For an uncertain optimization problem $\mathscr{P}^{WC}(\Delta)$ that is feasible under Δ, then for any subset of the true uncertainty set, i.e., $\bar{\Delta} \subseteq \Delta$, the problem $\mathscr{P}^{BWC}(\bar{\Delta})$ remains feasible and the performance w.r.t. cost function is non-decreasing. Furthermore, since the selected set $\bar{\Delta}$ is directly effecting the resulting control action, i.e.,

$$u^* = \arg\min_u \max_{\zeta \in \bar{\Delta}} (J(x_k, u, \zeta)) = \arg\min_u \mathscr{R}^{cost}_{BWC}(J(x_k, u, \zeta)),$$

one can actively design an uncertainty set to which a RMPC is synthesized with desired performance properties, while providing robustness against $\bar{\Delta}$, see Table I. With this construction one can circumvent the pessimism problems or give quantitative measures on the robustness to the uncertainty with unbounded distributions, [15], hence increasing the degree of freedom of the designer.

A generic RMPC example with uncertainty budgets can be stated as follows,

$$\mathscr{P}^{BWC}:\begin{cases}\min\limits_{\bar{u}_{[0,N_p-1]|k}} \max\limits_{\zeta_{[0,N_p-1]|k}\in\bar{\Delta}_{[0,N_p-1]|k}} & \sum\limits_{j=0}^{N_p-1} x_{j|k}^{\top}Qx_{j|k}+u_{j|k}^{\top}Ru_{j|k}+x_{N_p|k}^{\top}Q_f x_{N_p|k},\\ \text{st.} & x_{j+1|k}=A(\delta_{j|k})x_{j|k}+B(\delta_{j|k})u_{j|k}+Fw_{j|k},\\ & y_{j|k}=C(\delta_{j|k})x_{j|k}+D(\delta_{j|k})u_{j|k}+v_{j|k},\\ & c_{ij}(x_j,u_j,\zeta_j,y_{j|k})\leq 0,\ \forall\zeta_{j|k}\in\bar{\Delta}_{j|k},\\ & i=1,\ldots,N_c^j,\quad j=0,\ldots,N_p-1,\quad x_{0|k}=x_k,\end{cases} \tag{11}$$

where $\bar{\Delta}$ is adjusting the trade-off between guaranteed robustness versus closed-loop performance (risk-aversion), [73]. The associated risk-mappings are defined as

$$\mathscr{R}_{BWC}^{cost}(J)=\max_{\zeta\in\bar{\Delta}}J(\zeta),\quad \mathscr{R}_{BWC}^{const}(c_{ij})=\max_{\zeta\in\bar{\Delta}}c_{ij}(\zeta).$$

The computational complexity of $\mathscr{P}^{BWC}$ problem is still challenging similar to the $\mathscr{P}^{WC}$ problem, the min-max (saddle) nature of the problem is still effective. For generic convex uncertain optimization problems, the robust counterparts are in general intractable, while for some uncertainty classes the robustified problems remain convex, hence tractable ([74], [45]). For examples from literature, we refer to; [75] for the robustified generic convex optimization problems, [76] for the robustified least squares problems, [77] for the robust semidefinite programming problems, [78] for the robust linear programming problems, [79] for the robustified quadratic programming or the robust conic programming problems, and lastly [80] for the approximations of the robust conic programming problems.

C. Contributions from RMPC Literature

In this section we present and classify various contributions in RMPC literature. One group of the earliest works done in close association with WC-MPC problems is called the 'pursuit-evasion problems', i.e., [81], [82], [83], [84], closely related to the developments in the optimal control theory. However, the first works conducted strictly in receding horizon fashion was [85], [59], [86] and [87]; where both [86] and [59] elegantly present the WC-MPC problem for nonlinear and DAE systems, respectively, and [87] focuses on the frequency domain robust performance measures. The article [88] constructs the robust stability proof quite similar to the standard stability arguments for the nominal MPC problems. Another milestone contribution for guaranteeing robust stability property is [89] which uses linear matrix inequalities (LMIs) in RMPC problems for polytopic or parametric internal uncertainties through overbounding the cost function and using positive invariance arguments for the constraints on the future trajectory. Later, specifically for nonlinear systems, input-to-state stability (ISS) arguments are introduced for WC-MPC in [68], which tightens the constraints and forces the nominal system to have an ISS-Lyapunov function within a RPI terminal set inside the tightened constraints. A two-level (sometimes, though incorrectly, called as the closed-loop MPC) controller approach is presented in [90] which consists of a MPC and an $\mathscr{H}_\infty$ controller. Constructing a low level controller (precompansator) on top of a MPC lead to WC-MPC with disturbance feedback control structure [91] and tube based MPC approach, [1]. Lastly we mention the min-max MPC constructions with cost functions cast as linear norms such as [92], [93], [94] and more recently [95] while [96] provides a nice discussion on linear programming in MPC domain.

In Table III we distinguish publications in RMPC area with respect to; underlying system class (linearity), type of uncertainties, the high-level approach (min-max, ISS, tube), control parameterization, how the constraints are robustified and lastly the complexity of simulation examples provided in these papers. To comment on some observations on the Table III, due to the various contributions from output feedback structure and nonlinear systems domains, the current effort is directed towards practical implementations with various complexity reduction formulations. Similarly in many contributions the attention has been directed towards disturbance rejection properties (additive external uncertainties), while the polytopic internal uncertainty case has been treated (mainly) with LMI based techniques. The dynamic uncertainty and the effect of estimator in the loop (the initial condition mismatch case) is still lacking a detailed treatment. The ISS and the tube based RMPC approaches offer various solution strategies by decreasing the complexity back to the nominal MPC problem (with additional computationally complex operations) and the affine disturbance feedback parametrization lead to a huge reduction in the pessimism, while the sub-optimality should be further discussed in comparison to the policy based approaches. Lastly constraint tightening methods operate in coherence with the ISS and tube based approaches, while calculation of these sets (or approximations) is still to be mastered, which necessitates using highly complex (or realistic) examples to observe the drawbacks of the proposed techniques and convince the practitioners.

Although not in the RMPC area, but min-max based methods are also used in MPC design for i) LPV systems, [150], [151], ii) game-theoretic approaches, [152], [153], iii) decentralized or distributed control, [154], iv) (robust) estimation, [155], v) energy-consumption and smart-communication, [156]. Similarly due to the highly structured construction of MPC problems, multi-parametric programming approaches ([157]) were utilized for RMPC problems also. The article [95] constructs the robustly invariant set of an uncertain linear system with polytopic uncertainties for explicit RMPC purposes which is then extended by [125], [126], [158], [141] and [19]. In [118], one of the first practical applications of explicit RMPC for a linearized system, solving the estimation and robust control problem for a batch polymerization process, is presented.

System Dynamics and Control Implementation	Linear: State Feedback	Linear: Output Feedback	Nonlinear: State Feedback	Nonlinear: Output Feedback
	The majority of the papers	[3], [28], [87], [97], [98], [99], [100] [101], [102], [103], [104],	[2], [59], [68], [86], [90], [69], [105], [106], [26], [107] [108], [109], [110], [111], [112], [113], [114], [115], [116], [117]	[115], [118], [119],

Uncertainty types	Additive External	Polytopic or Parametric Internal	Dynamic Internal	State Estimation Measurement Noise	Multi-model and Linearization
	[1], [2], [3], [19], [28], [59], [68], [72], [86], [88], [91], [95], [103], [69], [105], [106], [26], [107], [108], [109], [110], [113], [114], [115], [117], [118], [119], [120], [121], [122], [123], [124], [125], [126], [127], [128], [65], [129], [130], [131], [132], [133], [134],	[59], [86], [89], [95], [100] [101], [102], [69], [105], [26], [107], [108], [109], [110], [114], [115], [118], [121], [135], [136], [137], [138], [139], [140], [141], [142], [143],	[90], [97], [98], [99], [144], [145],	[3], [28], [87], [116], [118],	[32], [104], [112],

High-level Approach	Min-max	Cost overbounds	ISS	Tube	Other
	[19], [88], [86], [59], [95], [100], [26], [107], [108], [111], [121], [122], [125], [135], [140], [142], [143],	[32], [89], [97], [98], [99], [101], [102], [104], [109], [110], [136], [137], [138], [144], [145], [146]	[68], [72], [105], [106], [26], [115], [116], [118], [120], [126], [133], [134], [141],	[1], [2], [3], [28], [103] [112], [117], [127], [128], [65], [129], [130], [131], [132], [147],	[87], [90], [114], [124]

Control Parameterization	Control Sequence	Stabilizing Feedback Gains	Affine Disturbance Feedback Parameterization	Control Policy
	[1], [2], [59], [68], [86], [88], [90], [95], [103], [26], [112], [114], [115], [116], [117], [118], [121], [124], [125], [126], [127], [128], [65], [141], [147],	[32], [89], [97], [98], [99], [100], [101], [102], [104], [136], [137], [138], [144], [148], [146],	[19], [72], [111], [120], [129], [130], [131], [132], [134],	[91], [107], [108], [109], [135], [147]

Robust Constraints	Constraint Tightening via Tube	Constraint Tightening via Lipschitz Bound	Positive Invariance and LMIs	Admissible Sets Explicit Solution
	[1], [2], [3], [28], [103], [112], [117], [127], [128], [65], [129], [130], [131], [132], [147],	[59], [68], [86], [90], [110] [111], [133], [141],	[32], [88], [89], [98], [99], [100], [101], [114], [124], [136], [138], [145], [146],	[19], [91], [95], [69], [105], [106], [107], [108], [109], [115], [116], [118], [120], [121], [125], [135], [142], [143], [126], [134],

Complexity of simulation example	No example	1-2 State	3-4 State	5 or more states
	[59], [91], [69], [105], [107] [109], [111], [114], [120], [134], [138],	[1], [3], [19], [28], [32], [68], [87] [88], [89], [90], [95], [100], [101], [103], [104], [106], [26], [108], [110], [112], [115], [116], [117], [121], [125], [126], [127], [65], [129], [130], [131], [132], [133], [135], [138], [141], [142], [143], [147], [149],	[2], [87], [89], [95], [97], [98], [99], [100], [101], [102], [119], [122], [136], [144], [145], [146],	[104], [112], [118], [119] [124],

TABLE III: An overview of RMPC approaches, via a group of selected papers.

IV. Stochastic Approach to Robustness in MPC

The second direction in establishing robustness properties of the closed-loop systems is established by considering the uncertainties as stochastic variables, see, e.g., [159] or [160] as two main contributions.

In the early days of predictive control, within the GPC framework ([6], [161]) the stochastic optimization problem is transformed into a deterministic optimization problem by the expectation operator on the cost function, $\mathbb{E}\{J(x_k, u, \zeta)\}$. Similarly, we distinguish the approaches to SMPC problems with respect to the treatment of stochastic variables, how the uncertain functions are mapped to deterministic counterparts. There are three inherently different methods to reformulate stochastic cost and constraint functions as deterministic functions;

1) Considering the (centralized) moments of the variables/functions through the expectation operators.
2) Considering probabilistic (chance) constraints on the functions, utilizing known pdfs,
3) Considering finite number of uncertainty scenarios and casting the RMPC problem w.r.t. these scenarios while using the min-max, (ensamble) moments or probabilistic reasoning.

Throughout this section, we assume that each realization of the uncertain variables is independent from another, with known pdf for each variable w_k, v_k, δ_k for $k \in \mathbb{Z}_{\geq 0}$.

A. Moment Based MPC Problems

The moment based optimization is possibly the most frequently used methodology in optimization problems with stochastic elements due to its relatively easy modeling and low complexity implementation, see Table I. The (centralized) moments provide inherent statistical information, such as the mean, variance, skewness or kurtosis which indicate the average magnitude, the spread, the asymmetry or the fat-tailedness of the predicted trajectories, all of which are desired to be controlled.

In the moment based approach, one maps the uncertain optimization problem by considering (linear combinations of) the (centralized) moments of the cost and the constraint functions. Here we only report the cases with first two (centralized) moments.

a) Mean MPC: The expectation of uncertain functions can be easily and analytically expressed for several classes of random variables, which reduces the computational need, while providing a more realistic performance measure than the worst-case cost ([9]).

In its simplest form of mean MPC (M-MPC), one evaluates the mean of the cost and constraints as,

$$\mathscr{P}^M : \begin{cases} \min\limits_{u_{[0,N_p-1]|k}} & \mathbb{E}\left\{ \sum\limits_{j=0}^{N_p-1} x_{j|k}^\top Q x_{j|k} + u_{j|k}^\top R u_{j|k} + x_{N_p|k}^\top Q_f x_{N_{p|k}} \right\}, \\ \text{st.} & x_{j+1|k} = A(\delta_{j|k}) x_{j|k} + B(\delta_{j|k}) u_{j|k} + F w_{j|k}, \\ & y_{j|k} = C(\delta_{j|k}) x_{j|k} + D(\delta_{j|k}) v_{j|k}, \\ & \mathbb{E}\{c_{ij}(x_{j|k}, u_{j|k}, \zeta_{j|k}, y_{j|k})\} \leq 0, \\ & i = 1, \ldots, N_c^j, \quad j = 0, \ldots, N_p, \quad x_{0|k} = x_k, \end{cases} \tag{12}$$

hence

$$\mathscr{R}_M^{cost}(J(\zeta) := \mathbb{E}_\zeta\{J(\zeta)\}, \quad \mathscr{R}_M^{const}(c_{ij}(\zeta)) := \mathbb{E}_\zeta\{c_{ij}(\zeta)\}.$$

Performance wise one needs to distinguish two aspects of M-MPC, (i) the cost reduces to, in general, the nominal cost function under linearity, however (ii) for the robust constraint satisfaction, the effect of uncertainty in constraints of $\mathscr{P}^M$ is disregarded by the expectation operator. This does not imply any quantitative robustness guarantees for some realizations of uncertainty, see Table I. Providing quantitative robustness arguments for the closed-loop performance of the nominal MPC, or the M-MPC, is yet to be discussed further in the MPC literature, see also the discussion in [23]. To improve the robust constraint satisfaction properties, one can introduce variance terms, which effectively backs off the operating point such that the constraints are satisfied with a higher probability.

b) Mean-Variance MPC: Variance operator indicates the spread of the uncertainty realizations, hence if the variance of cost and constraint functions are evaluated, one can quantitatively include the spread (of state predictions) information into the MPC control action. This MPC formulation is called here as the mean-variance MPC (MV-MPC) and an example is given in Equation (13),

$$\mathscr{P}^{MV}: \begin{cases} \min\limits_{u_{[0,N_p-1]|k}} & \mathbb{E}\left\{J(x_{j|k},u_{j|k},\zeta_{j|k})\right\}+\lambda_0\mathbb{D}\left\{J(x_{j|k},u_{j|k},\zeta_{j|k})\right\} \\ \text{st.} & x_{j+1|k}=A(\delta_{j|k})x_{j|k}+B(\delta_{j|k})u_{j|k}+Fw_{j|k}, \\ & y_{j|k}=C(\delta_{j|k})x_{j|k}+D(\delta_{j|k})u_{j|k}+v_{j|k}, \\ & \mathbb{E}\{c_{ij}(x_{j|k},u_{j|k},\zeta_{j|k},y_{j|k})+\lambda_{i,j}\mathbb{D}\{c_{ij}(x_{j|k},u_{j|k},\zeta_{j|k},y_{j|k})\}\leq 0, \\ & i=1,\ldots,N_{c_i^j}, \quad j=0,\ldots,N_p, \quad x_{0|k}=x_k, \end{cases} \tag{13}$$

where $\mathbb{D}\{\cdot\}$ is the variance operator and $\lambda_{i,j}$ (also λ_0) are positive weights, related to the risk aversion of the designer. This yields the risk mappings

$$\mathscr{R}_{MV}^{cost}(J(\zeta):=\mathbb{E}_\zeta\{J(\zeta)\}+\lambda_0\mathbb{D}_\zeta\{J(\zeta)\}, \quad \mathscr{R}_{MV}^{const}(c_{ij}(\zeta)):=\mathbb{E}_\zeta\{c_{ij}(\zeta)\}+\lambda_{i,j}\mathbb{D}_\zeta\{c_{ij}(\zeta)\}.$$

Regarding the performance aspects, ([162]), through including the $\mathbb{D}\{\cdot\}$ term into the cost function, one introduces the effect of uncertainty in the control actions, as λ_0 gets larger, the controller acts to reduce the spread of the cost function. For robust constraint satisfaction, which depends on the weights $\lambda_{i,j}$, the resulting operating point is backed off, as the feasible set is tightened by increasing $\lambda_{i,j}$ or $\mathbb{D}\{\cdot\}$. If $\lambda_{i,j}$ are large then the constraints are tightened substantially, hence allowing many realizations of ζ to occur without violating the constraints. However guaranteeing constraint satisfaction with certainty is not possible upto a large value of $\lambda_{i,j}$ for bounded domain ζ, see Table I.

Yet another aspect is the asymmetry in the distribution functions of the constraints or cost. The deviation-measure term on the cost or constraint functions (here taken as the variance) is not necessarily distinguishing the ‘good’ and ‘bad’ realizations of the uncertainty, see Figure. 1a. Some realizations are steering the operation out of the feasible region, called as the ‘bad’ realizations, while some realizations push the operating point further inside the constrained region, called as the ‘good’ realizations (the good-bad aspect reverses, in general, from the performance point of view). If the risk-neutral approach i.e. the symmetric evaluation of uncertainty realizations, such as the variance, is taken, then the control performance might reduce for a longer (skewness) or fatter (kurtosis) 'good' realization tail, see also Section IV-D or ([38], [163]).

Computationally, moment based MPC problems are divided into two distinct phases. During the offline phase the moments of the cost and constraint functions are calculated (or approximated through histograms) and during the online stage an uncertainty-free (computationally equivalent to the nominal) MPC problem is solved. Expressions of the moments $\mathbb{E}\{J(\cdot)\}$ (or $\mathbb{E}\{c_{ij}(\cdot)\}$) or $\mathbb{D}\{J(\cdot)\}$ (or $\mathbb{D}\{c_{ij}(\cdot)\}$) are easily obtained for linear systems, while for nonlinear systems computationally cheap methods exists, see [164], [165]. This is a huge improvement in comparison to the other RMPC or SMPC methods (WC-MPC or probabilistic MPC), see Table I.

B. Probabilistic Approaches to MPC Problems

The second approach to transform SMPC problems is using the probabilistic constraints, also called as the chance constraints. This type of MPC problems are called as chance-constrained MPC (CC-MPC). In this setting, one finds the control actions to reduce the frequency of constraint violations according to predefined probability levels α_{ij}, see [166] and also Table I. This provide quantitative ways of selecting the reliability of constraint satisfaction, an additional degree of freedom to reduce the conservatism induced by robustness. Indeed in the limiting case satisfying a constraint with probability one is equivalent to min-max formulation. However if one allows a sufficiently small margin of not satisfying the constraints (for the infrequent tail events), this tolerance can improve the closed-loop performance a lot while the system is robustified against frequently observed realizations. One possible CC-MPC problem can be provided as

$$\mathscr{P}^{CC}:\left\{\begin{array}{rl}\min\limits_{u_{[0,N_p-1]|k},\gamma} & \gamma \\ \text{st.} & \mathbb{P}\left\{\sum\limits_{j=0}^{N_p-1} x_{j|k}^\top Q x_{j|k}+u_{j|k}^\top R u_{j|k}+x_{N_p|k}^\top Q_f x_{N_p|k}\leq\gamma\right\}\geq\alpha_0 \\ & x_{j+1|k}=A(\delta_{j|k})x_{j|k}+B(\delta_{j|k})u_{j|k}+Fw_{j|k}, \\ & y_{j|k}=C(\delta_{j|k})x_{j|k}+D(\delta_{j|k})v_{j|k}, \\ & \mathbb{P}\{c_{ij}(x_{j|k},u_{j|k},\zeta_{j|k},y_{j|k})\leq 0\}\geq\alpha_{ij}, \\ & i=1,\ldots,N_{c_i^j},\quad j=0,\ldots,N_p,\quad x_{0|k}=x_k.\end{array}\right. \tag{14}$$

In this case the associated risk mappings $\mathscr{R}_{CC}^{cost}$ and $\mathscr{R}_{CC}^{const}$ are defined as

$$\begin{aligned}\mathscr{R}_{CC}^{cost}(J(\zeta) &:= \mathbb{P}\left\{\sum_{j=0}^{N_p-1} x_{j|k}^\top Q x_{j|k}+u_{j|k}^\top R u_{j|k}+x_{N_p|k}^\top Q_f x_{N_p|k}\leq\gamma\right\}\geq\alpha_0, \\ \mathscr{R}_{CC}^{const}(c_{ij}(\zeta)) &:= \mathbb{P}\{c_{ij}(x_{j|k},u_{j|k},\zeta_{j|k},y_{j|k})\leq 0\}\geq\alpha_{ij}.\end{aligned}$$

The problem $\mathscr{P}^{CC}$ in Equation (14) expresses the constraints $c_{ij}(\cdot) \leq 0$ as chance constraints with probability levels of α_{ij}, the constraint violation margins, which are design parameters selected from the interval $[0,1]$. The cost term in Equation (14) is also chance constrained by a new decision variable γ. In this way, one finds the regions in the uncertainty space for which the probability of obtaining a cost less than or equal to γ is larger than α_0, while satisfying the constraints with probability α_{ij}.

Closed-loop performance aspects of CC-MPC is yet to be established rigorously, though improvements in closed-loop performance compared to WC-MPC are already reported in the literature ([167]). Regarding the constraint satisfaction, the violation margins α_{ij} determine exactly the allowed frequency of violations. The correlations between the constraints, if the constraints are to be satisfied jointly, i.e.,

$$\mathbb{P}\left\{c_{ij}(x_{j|k}, u_{j|k}, \zeta_{j|k}) \leq 0, i = 1, \ldots, N_c^j, j = 0, \ldots, N_p\right\} \geq \alpha,$$

is generally a source of conservatism. For using the individual constraints, the computationally more efficient case, the approximations of the joint constraints are required. These approximation methods are heuristic and generally highly conservative. See [168] or [169] for the exactness of these relaxations.

The robust stability aspect is slightly vague in comparison to the WC-MPC formulations. The main problem is establishing the stochastic stability within the MPC context, while the implications of employing a specific stability approach is still open to discussion[7].

The possible reduction in conservatism comes with a price. In general the chance constraints are non-convex functions, see Table I, hence computational problems should be expected. Furthermore it is an inherently difficult task to optimize w.r.t the joint pdfs of uncertain functions, since the probability levels after propagation of pdf through the dynamics and the gradients of constraints over these pdfs are required to be calculated. This induces severe computational problems for nonlinear dynamical systems. Another point is related to the modeling aspect of the uncertainties. Modeling the exact pdf is an impossible task, while in $\mathscr{P}^{CC}$ one necessarily assumes this information. To overcome this drawback, some methods are introduced to guarantee robustness with respect to a class of pdfs, [171], instead of only one pdf.

C. Randomized or Scenario Based MPC

Sampling the uncertainty space and conducting the optimization problem with respect to the selected realizations is a recently appreciated technique, in MPC domain, for solving uncertain optimization problems ([172]), also see the Monte Carlo sampling methods ([173], [174]). Although providing increasingly good approximating solutions to the WC-MPC problem, due to the requirement of 'excessive' number of scenarios, the scenario based methods were not popular until recently in the MPC community.

[7]For an elegant discussion on various different formulations of stochastic stability, we refer to [170].

Similar to previous methods, the randomized MPC techniques consists of two steps, constructing (or sampling) the uncertain space and establishing robustness properties of MPC towards them. By extracting (or generating) a number of scenarios, one samples the uncertain space. This projects the uncertainty to deterministic values for which the MPC problem can be robustified easily. A great effort has been directed to a-priori guarantees (for convex problems) and a-posterior guarantees on the probability levels of constraint satisfaction for all uncertainties through the finitely many generated scenarios, see Table I.

Here we construct the randomized MPC, for the worst case formulation, as

$$\mathscr{P}^{randWC}: \begin{cases} \min\limits_{u_{[0,N_p-1|k},\gamma} & \gamma, \\ \quad\quad\text{st.} & x^s_{j+1|k} = A(\delta^s_{j|k})x^s_{j|k} + B(\delta^s_{j|k})u_{j|k} + Fw^s_{j|k}, \\ & y^s_{j|k} = C(\delta^s_{j|k})x_{j|k} + D(\delta^s_{j|k})v^s_{j|k}, \\ & \sum\limits_{j=0}^{N_p-1}(x^s_{j|k})^\top Qx^s_{j|k} + u^\top_{j|k}Ru_{j|k} + (x^s_{N_p|k})^\top Qx^s_{N_p|k} \leq \gamma, \\ & c_{ij}(x^s_{j|k}, u_{j|k}, \zeta^s_{j|k}, y^s_{j|k}) \leq 0, \\ & i = 1,\dots,N^j_c,\ j = 0,\dots,N_p,\ s = 1,\dots,N_s,\ x^s_{0|k} = x_k, \end{cases} \tag{15}$$

where $s \in \mathbb{Z}_{[1,N_s]}$ is the index running over the selected samples of uncertainty $\zeta^s_{j|k}$ for $j \in \mathbb{Z}_{[0,N_p]}$ and N_s is the total number of scenarios. Here the risk mappings are taken as

$$\begin{aligned} \mathscr{R}^{cost}_{rand}(J(\zeta)) :=& \quad \sum_{j=0}^{N_p-1}(x^s_{j|k})^\top Qx^s_{j|k} + u^\top_{j|k}Ru_{j|k} + (x^s_{N_p|k})^\top Qx^s_{N_p|k}, \\ \mathscr{R}^{const}_{rand}(c_{ij}(\zeta)) :=& \quad c_i(x^s_{j|k}, u_{j|k}, \zeta^s_{j|k}, y^s_{j|k}) \leq 0, s = 1,\dots,N_s. \end{aligned}$$

This problem both guarantees the robustness w.r.t. N_s selected scenarios and provides a probabilistic bound on the violation of constraints for unconsidered scenarios, i.e.,

$$\mathbf{P}(\mathscr{P}^{randWC}, \varepsilon) := \mathbb{P}\{\mathbb{P}\{c_i(x_{j|k}, u^*_{j|k}, \zeta_{j|k}) \geq 0, \zeta \in \Delta, i = 0,1,\dots,N_{c_i}, j = 0,\dots,N_p\} > \varepsilon\}.$$

The $\varepsilon-$constraint violation probability $\mathbf{P}(\mathscr{P}^{randWC}, \varepsilon)$ is bounded from above and below as, [175],

$$(1-\varepsilon)^{N_s} \leq \mathbf{P}(\mathscr{P}^{randWC}, \varepsilon) \leq \beta(N_s, \varepsilon), \tag{16}$$

where the upper bound $\beta(N_s, \varepsilon)$ is a random variable with the Bernoulli distribution, i.e.;

$$\beta(N_s, \varepsilon) := \sum_{j=0}^{N_s-1} \binom{N_s}{j} \varepsilon^j (1-\varepsilon)^{N_s-j}.$$

In this formulation $\beta(N_s, \varepsilon)$ is the design parameter to set the probability of constraint violation, ε. For small values of $\beta(N_s, \varepsilon)$, N_s has a logarithmic growth and as N_s gets larger $\beta(\varepsilon)$ tends to zero.

Relatively recently the scenario based optimization formulation is also discussed in the context of non-convex problems and also reducing the conservatism in the provided violation probability bounds β in [176], with similar discussion also reported in [177], [178], [179]. One remarkable extension that scenario based approach provides is the relatively easy incorporation of various type of uncertainties effecting the dynamical system. Once the scenarios are constructed, the problem $\mathscr{P}^{randWC}$ in Equation (15) becomes a nominal MPC problem, with a large number of constraints, increasing the applicability with the current solvers, see Table I. Furthermore, similar to the randomized worst case approach ([180]), one can cast the chance constrained MPC problems in randomized fashion ([181]), where the randomized MPC problem has a similar structure.

D. Risk and Deviation Metrics

The so-called coherent risk measures are raising considerable attention in operations research and in economics communities, while incorporation into SMPC is already reported in [163]. Following the previous SMPC constructions, one can observe that; (i) in the moment MPC, the constraints in $\mathscr{P}^{M}$ or $\mathscr{P}^{MV}$ are formulated as risk-natural, not considering asymmetries of the pdfs of constraint functions, hence leading to a possible performance decrease; (ii) the chance constraints are computationally difficult to calculate, due to the non-convexity properties. The use of various other reformulations being developed in the risk theory area, such as semi-deviations or conditional expectations, which can improve both the closed-loop performance and the computational aspects of the resulting SMPC problem.

In [182], the equivalence between chance constraints and a risk indicator, the notion of value at risk (VaR), is established. The $\mathrm{VaR}_\alpha(c_{ij}(\zeta))$ is defined, for a risk level of α as

$$VaR_\alpha(c_{ij}(\zeta)) := \min\{\gamma \in \mathbb{R} | \mathbb{P}\{c_{ij}(\zeta) > \gamma\} \leq 1-\alpha\}.$$

The VaR values consider one side of the (cost or constraint) pdf, so they are not risk-neutral. Same authors have shown that the conditional VaR (CVaR), also called as integrated chance constraints, are behaving far better than the VaR constraints in the optimization problems since; (i) the feasible set is convex, see [183], and (ii) the CVaR values are relatively easy to calculate through scenarios. The α-level CVaR of an uncertain function $c_{ij}(\zeta)$ is defined as,

$$\mathrm{CVaR}_\alpha(c_{ij}(\zeta)) := \frac{1}{1-\alpha} \int_{1-\alpha}^{1} \mathrm{VaR}_\beta(c_{ij}(\zeta)) d\beta.$$

In [182], the authors provide different algorithms for expressing CVaR. One drawback is that integrated chance constraints are more conservative than chance constraint formulations, since $\mathrm{CVaR}_\alpha(c_{ij}(\zeta)) \geq \mathrm{VaR}_\alpha(c_{ij}(\zeta))$ for the positive tail of the pdf, see Figure 6. However this implies that the constraint satisfaction guarantees for $\mathrm{CVaR}_\alpha(c_{ij}(\zeta))$ are also valid for the chance constraints (VaR constraints) with the violation level of α. Similar to replacing the chance constraints (or moment formulations) with

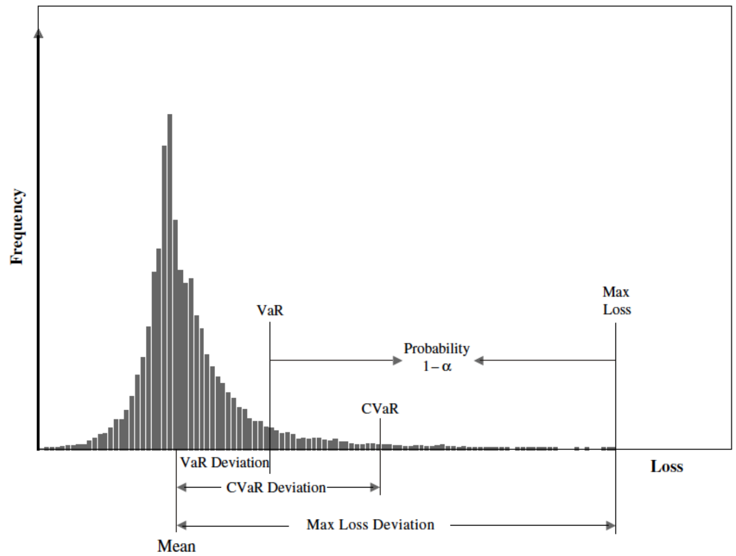


Fig. 6: Various risk and deviation metrics, taken from [48].

integrated chance (CVaR) constrains, one can make use of other risk or deviation measures such as a re-formulation of $\mathscr{P}^{MV}$,

$$\mathscr{P}^{Risk}: \begin{cases} \min\limits_{u_{[0,N_p-1]|k}} & \tilde{\mathbb{E}}\left\{J(x_{j|k}, u_{j|k}, \zeta_{j|k})\right\} + \lambda_0 \tilde{\mathbb{D}}\left\{J(x_{j|k}, u_{j|k}, \zeta_{j|k})\right\} \\ \text{st.} & x_{j+1|k} = A(\delta_{j|k})x_{j|k} + B(\delta_{j|k})u_{j|k} + Fw_{j|k}, \\ & y_{j|k} = C(\delta_{j|k})x_{j|k} + D(\delta_{j|k})v_{j|k}, \\ & \tilde{\mathbb{E}}\{c_i(x_{j|k}, u_{j|k}, \zeta_{j|k}, y_{j|k}) + \lambda_{i,j}\tilde{\mathbb{D}}\{c_i(x_{j|k}, u_{j|k}, \zeta_{j|k}, y_{j|k})\} \leq 0, \\ & i = 1, \ldots, N_{c_i}, \qquad j = 0, \ldots, N_p, \end{cases} \tag{17}$$

where $\tilde{\mathbb{E}}\{\cdot\}$ and $\tilde{\mathbb{D}}\{\cdot\}$ are generalized risk metric and generalized deviation metric, respectively, instead of expectation or variance operators. Rigorous definitions of generalized risk and deviation metrics are provided in [47], [184], while some examples of these metrics from the literature can be given as,

$$\begin{array}{lrl} \text{Entropic VaR:} & EVaR_\alpha := & \min\limits_{\gamma} \left\{ \gamma \in \mathbb{R}_{>0} \middle| \frac{\ln(\frac{M_\zeta(\gamma)}{1-\alpha})}{\gamma} \right\}, \\ \text{Semideviations:} & \sigma_+^2 := & \mathbb{E}\left\{\max\{\zeta - \mathbb{E}\{\zeta\}, 0\}^2\right\}, \\ & \sigma_-^2 := & \mathbb{E}\left\{\max\{\mathbb{E}\{\zeta\} - \zeta, 0\}^2\right\}, \\ \text{Maximum Deviations:} & \rho_+ := & \max\limits_{\zeta} \zeta - \mathbb{E}\{\zeta\}, \\ & \rho_- := & \max\limits_{\zeta} \mathbb{E}\{\zeta\} - \zeta, \\ \text{Mean Absolute Deviation:} & \mathrm{MAD}(\zeta) := & \int\limits_{\Xi} |\zeta - \mathbb{E}\{\zeta\}| d\zeta. \end{array} \tag{18}$$

Since the numerically better behaving risk and deviation indicators will be in use, one can overcome the computational burden while improving the closed-loop performance of robust MPC applications by using the non-standard risk metrics.

E. Contributions From SMPC Literature

In this section we present and classify various relevant entries from the SMPC literature. Similar to the WC-MPC case and following a similar classification with [31], in Table IV we distinguish the publications with respect to the underlying system class (linearity), type of uncertainties effecting the dynamics, how uncertainties are treated or integrated, control parameterization, how the robust counterparts of constraints are expressed and lastly the complexity of simulation examples provided in these papers.

System Dynamics and Control Implementation	Linear		Nonlinear	
	State Feedback	Output Feedback	State Feedback	Output Feedback
	The majority of the papers	[133], [185], [186], [187], [188], [189], [190], [191]	[192], [193], [194], [195], [196], [197], [198], [199], [200], [201]	[202],

Uncertainty types	Additive External	Multiplicative External	Multiplicative Time invariant
	[162], [181], [185], [186], [189], [190], [191], [194], [197], [198], [200], [201], [202], [203], [204], [205], [206], [207], [208], [209], [210], [211], [212], [213], [214], [215], [216], [217], [218], [219], [220], [221], [222], [223], [224], [225], [226]	[163], [192], [194], [195], [196], [199], [200], [201], [213], [221], [227], [228], [229], [230], [231], [232], [233], [234]	[187], [235], [236]

Uncertainty Treatment	Integration/Expectation	Scenario	Probabilistic	Other
	[185], [186], [187], [188], [189], [190], [192], [199], [203], [204], [205], [206], [208], [209], [210], [211], [212], [213], [214], [215], [216], [223], [224], [225], [227], [232], [236], [237],	[162], [163], [176], [178], [179], [181], [190], [193], [194], [195], [196], [200], [201], [202], [217], [218], [219], [221], [228], [229], [230], [233], [234],	[198]	[197], [207], [222], [238],

Control Parameterization	Control Sequence	Affine Disturbance Feedback Parameterization	Control Policy
	[162], [188], [190], [192], [193], [194], [196], [197], [198], [200], [201], [208], [209], [214], [217], [218], [220], [225], [228], [229], [230], [233], [234],	[181], [189], [191], [202], [203], [205], [206], [211], [212], [213], [215], [216], [217], [218], [221], [222], [223], [224], [226], [227], [232], [239], [240], [237], [231],	[185], [186], [190], [191], [195], [199], [207], [219], [238],

Constraints	Chance Constraints via Relaxations	Chance Constraints via Moments	Worst Case Constraints via Samples	Chance Constraints via Samples	Other
	[181], [203], [210], [211], [212], [214], [222], [223], [225], [227], [231], [232], [237],	[187], [188], [192], [193], [202], [209], [210], [223], [226], [240],	[194], [206], [217], [220], [228], [230],	[162], [208], [218], [219], [229] [234],	[163], [185], [186], [189], [190], [191], [195], [196], [197], [198], [199], [204], [205], [207], [216], [221], [233], [236], [238],

Complexity of simulation example	No example	1-2 State	3-4 State	5 or more states
	[189], [193], [199], [225]	[162], [163], [181], [185], [186], [192], [198], [201], [203], [205], [207], [208], [211], [213], [215], [216], [220], [221], [224], [226], [227], [228], [229], [230], [231], [232], [233], [234], [237], [238], [239], [240],	[191], [194], [197], [209], [212], [217], [218], [223], [236],	[187], [188], [200], [202], [206], [210], [214], [219],

TABLE IV: An overview of SMPC approaches, via a group of selected papers.

As mentioned above, the extensions of SMPC methods to nonlinear systems are difficult to achieve, since moments (for $\mathscr{P}^M$ or $\mathscr{P}^{MV}$) and pdf calculations (for $\mathscr{P}^{CC}$) are quite difficult to evaluate. Scenario based methods dominate the cases where it is difficult to obtain the pdfs to treat the uncertainty in the optimization problems. Lastly how constraints should be treated and, similar to the RMPC case, the closed-loop performance aspects to practical examples are still yet to be investigated further.

Next we point out some specific contributions outside of the SMPC research area. The expectation and/or variance of the cost functions are discussed in [241], [242], [243], [46] with or without chance constraints. The chance constrained programming (CCP) approach is primarily discussed in [244] and [245]. The CCP algorithms gather attention for optimal design or operation problems also, see [167], [246], [247] for an elegant discussion and implementation considering nonlinear chemical systems. One of the large scale implementations for SMPC is presented in [214] which, uniquely, considers a linear DAE system with chance constraints.

The randomized algorithms are discussed in depth in [248] or [179], [176]. In order to further improve the bounds or decrease the number N_s of scenarios, one can refine the selection method to sample scenarios, which is not discussed here. One such method is presented in [249], other than the well known techniques such as Latin hypercube or Metropolis-Hastings sampling methods ([167]). Furthermore, scenario based MPC is used for linear parameter varying systems in [219], for a vehicle scheduling study in [195], for finance and econometrics areas in [250], [196], [221], for electricity grids in [251], and lastly for the automotive industry in [220].

V. Simulation Examples on Robust Model Predictive Controllers

In this section, we implement the robust and stochastic MPC algorithms detailed in previous sections to compare their closed-loop and computational performances. The examples are selected mainly from the literature and demonstrate the effect of different risk mappings discussed in Chapter 2. The first example considers a batch process of a reaction, where the moment and worst case based MPC methods show opposite closed-loop behaviour. In the second example we consider an operating point change for a CSTR system and discuss the differences in between transient responses of closed-loop systems controlled with different robust (or stochastic) MPC algorithms. Last two examples, a mass-spring system and a MIMO debutanizer system, demonstrate the effect of tuning variables in moment and worst case risk mappings respectively.

For all of the simulation examples, the optimization routines are implemented via YALMIP ([74]), in Matlab 2015b environment on a computer with 32GB of RAM. The (averaged) computational time, in seconds, are provided in Table V for the simulation examples. The results in Table V are consistent with the theoretical expectations which are summarized Table I. The moment based robustness evaluations ($\mathscr{P}^M$ and $\mathscr{P}^{MV}$) yield much simpler optimization problems to be solved at each iteration in comparison

	Mean MPC	Mean-Variance MPC	Chance Constrained MPC	Worst-case MPC	Budgetted Worst-case MPC
Semi-batch Reaction	9.3904(s)	14.5054(s)	-	4345.1(s)	3682.6(s)
van der Vusse Reaction	10.6251(s)	12.7974(s)	9.6846(s)	2134.2(s)	1482.5(s)
MSD System	7.3677(s)	7.5501(s)	7.9003(s)	20.3868(s)	19.9146(s)
Debutanizer System	12.3937(s)	12.9231(s)	1266.6(s)	3950.8(s)	3973.6(s)

TABLE V: Computational times for different examples

to the chance constrained (due to non-convexity) and set based techniques, i.e., (budgeted) worst case MPC. This is due to computational requirements of the CC-MPC or WC-MPC, caused either by the uncertainty scenarios being in excessive size or explicit maximization of cost or constraint functions with respect to uncertainties.

A. *Predictive Control of a Simple Batch Reaction*

In this example, we consider a simple exothermic chemical reaction, $A \to P$. The reaction is controlled through a cooling action. Cooling the reaction temperature is slowing down the reaction rate (exothermic reaction) hence increasing the final time of the batch. Therefore, extensive use of control is not desired. We assume that the reaction dynamics are not known exactly and the uncertain effects are modeled as additive disturbances, with i.i.d Gaussian characteristics ($\mathscr{N}(0, I_2)$). Since the purpose of this process is to finalize the reaction as fast as possible, a *batch-time minimization* problem, while operating within the allowable temperature levels, the pessimistic nature of deterministic MPC constructions can be easily observed. We implement the mean (M), mean-variance (MV), worst-case (WC), and budgeted-robust (BWC) MPC controllers for this example. The reaction dynamics are taken from [252] as

$$\begin{aligned} \dot{c}_A = -r, \qquad \dot{c}_P &= r, \\ \rho \bar{c} \dot{T} &= -\Delta h_R r + \tfrac{U_W A_W}{V_R}(T_C - T) + f_2 w^2, \\ r &= c_A k_{300} \exp(\tfrac{E}{R}\left(\tfrac{1}{300} - \tfrac{1}{T}\right) + f_1 w^1), \end{aligned} \tag{19}$$

where c_A is the reactant concentration, c_P is the product concentration, T is the reactor temperature and the T_C is the control input, the cooling temperature. The initial values of states are taken as $x_0^\top = \begin{bmatrix} 5000 & 0 & 40 \end{bmatrix}$. The parameter values are given in Table VI. For the set-based methods, we consider bounded support for the uncertainty $w_k \subset \mathscr{W}^{(\cdot)},\ k \in \mathbb{Z}_{\geq 0}$. For WC-MPC case, the disturbances assume

$V_R = 6.28\ [m^3]$	$A_W = 10\ [m^2]$
$U_W = 500\ [W/m^2/K]$	$\rho = 1000\ [kg/m^3]$
$\bar{c} = 2\ [kJ/kgK]$	$\Delta h_R = -10^5\ [kJ/kmol]$
$E = 40000\ [kJ/kmol]$	$R = 8.314\ kJ/kmolK$
$k_{300} = 0.125$ [1/h]	

TABLE VI: Semi-batch Reaction Parameters

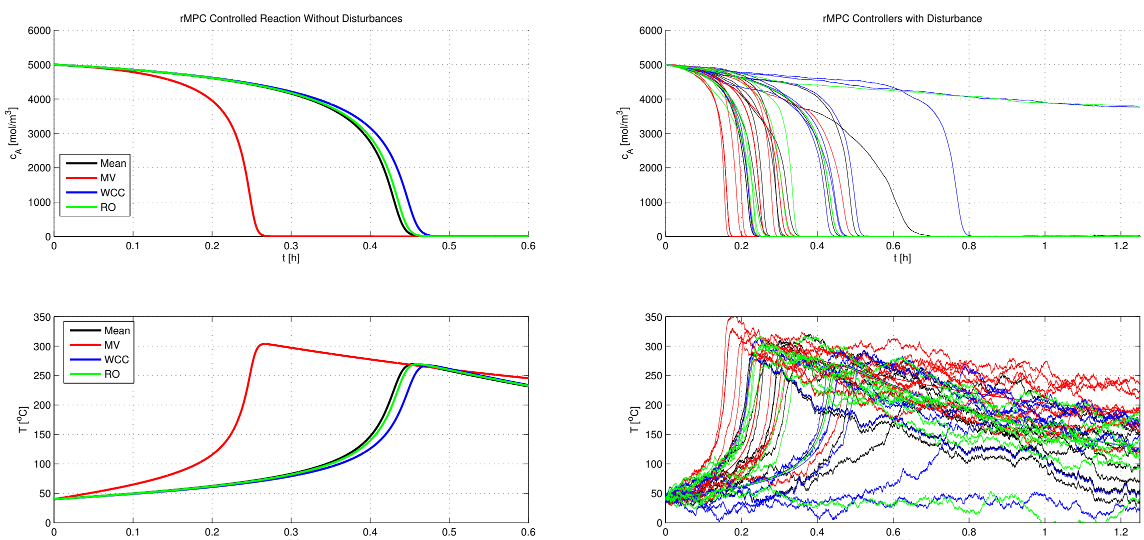
values from the interval $\mathscr{W}^{WC} = [-3,3]\times[-3,3]$, which is equal to 6 standard deviations. For the BWC-MPC case the disturbance set is shrunk to $\mathscr{W}^{BWC} = [-1,1]\times[-1,1]$, since this set incorporates many of the observed realizations already. Furthermore, for the MV-MPC, we set λ_0 to 2.5, which is tuned to improve the distinction from mean case, while $\lambda_{ij},\ i = 1,\ldots,N_c^j,\ j = 1,\ldots N_p$, values are set to zero, hence no constraint tightening is imposed for MV-MPC formulation. The prediction model is set as the linearized and discretized (with sampling time of 5 seconds) dynamics over an operating trajectory, i.e., a high level deterministic dynamic optimization problem with the nonlinear model is solved to provide the operating conditions which minimize the cost function

$$J^1 = \int_0^{0.75} \alpha_a c_a(t) + \alpha_T T_C^2(t)dt,$$

where c_a and T_C are state variables, $\alpha_a = 1$ and $\alpha_T = 0.001$ are the state and input weighting terms in the optimization problem, respectively. The result of the high-level optimization problem is used as the operating points, hence the linearized dynamics are known to the MPC controllers. The prediction horizon for the MPC problem is taken as 5 time steps to discard the possible effects of linearization and mismatch in the operating points. The optimization problem for the MPC algorithm is taken as,

$$\begin{aligned} \min_{u_{[0,N_p-1]|k}} \quad & \mathscr{R}_i\{\textstyle\sum_{j=0}^{N_p-1} x_{j|k}^T Q x_{j|k} + u_{j|k}^T I u_{j|k}\}, \\ \text{s.t.} \quad & \Delta x_{j+1|k} = A_{j|k}\Delta x_{j|k} + B_{j|k}\Delta u_{j|k} + F_{j|k} w_{j|k}, \\ & \begin{bmatrix} x_{k+j} \\ u_{k+j} \end{bmatrix} \in \begin{bmatrix} \mathbb{X} \\ \mathbb{U} \end{bmatrix}, \end{aligned}$$

where the state constraints are taken as $\mathbb{X} := \{(x_1,\ x_2,\ x_3)|\ 0 \le x_1 \le 5000, 0 \le x_2 \le 5000, -10 \le x_3 \le 350\}$, the input constraints are taken as $\mathbb{U} := \{u|\ 0 \le u \le 100\}$,. Lastly the weighting matrix Q is taken as $\mathrm{diag}(1,0,10^4)$. In Figure 7a, we visualize the performance for the RMPC solutions without any disturbance effects, i.e., $w^i(t) = 0$ for $i = 1,2$ and $t \in [0,t_f]$ where t_f is the final time of the reaction. The results indicate the structural differences between the RMPC techniques; the min-max based techniques resulting in longer batch final times, compared to Mean-MPC case, due to the control action cooling down the reaction not to exceed the temperature constraints. Totally opposite behaviour is observed in the MV-MPC case, which greatly speeds up the reaction. This is due to the instability of the system, the variance term dominates the cost function, hence the controller acts aggressively to compensate the actions induced from uncertain perturbations in prediction stages. We also provide the results for the case where disturbances are not set to zero in Figure 7b, which demonstrates that the reactor temperature is kept in allowable bounds, in almost all cases. Similar to the uncertainty-free case, WC-MPC and BWC-MPC react similar to Mean-MPC case at the start of the simulation, but as the state constraints start to be effective, the controllers slow down the reaction, which leads to longer total batch time compared to M-MPC case, while the MV-MPC, again, speeds up the reaction. This behaviour leads to higher temperatures over the reaction trajectory, in one case a violation of temperature constraint is observed. This is an expected result since for the moment MPC formulations, the constraints are allowed to be violated.



(a) The concentration and temperature profiles with RMPC controllers for disturbance free case.

(b) The concentration and temperature profiles with RMPC controllers with the effect of disturbances.

B. Control of CSTR with Changing Operation Point

As a second example, we consider the van der Vusse reaction, taken from [70],

$$\begin{aligned} \dot{C}_a = &\quad -k_1(T)C_a - k_2(T)C_a^2 + (C_{in} - C_a)u_1, \\ \dot{C}_b = &\quad k_1(T)(C_a - C_b) - C_b u_1, \\ \dot{T} = &\quad -f_{react} + \alpha(T_c - T) + (T_{in} - T)u_1, \\ \dot{T}_c = &\quad \beta(T - T_c) + \gamma u_2, \\ f_{react} = &\quad \delta(k_1(T)(C_a H_{ab} + C_b H_{bc}) + k_2(T)C_a^2 H_{ad}), \end{aligned} \tag{20}$$

where the states are the concentration of substance A (C_a), the concentration of substance B (C_b), the temperature of reactor (T) and temperature of cooling jacket (T_c). The inputs are the flow rate to the reactor (u_1) and the cooling power (u_2). The parameter values are given in Table VII. We construct a

$\alpha =$	30.8285 [1/h]	$\beta =$	86.688 [1/h]
$\gamma =$	0.1 [K/kJ]	$\delta =$	0.3556 [Kl/kJ]
$E_1 =$	9758.3 [ ]	$E_2 =$	8560.03[ ]
$k_{10} =$	$1.287 * 10^{12}$ [l/h]	$k_{20} =$	$9.043 * 10^{9}$ l/(mol*h)
$h_{AB} =$	4.2 [kJ/mol]	$h_{BC} =$	-11 [kJ/mol]
$h_{AD} =$	-41.85 [kJ/mol]		

TABLE VII: Van der Vusse Reaction Parameters

scenario where the known disturbances, the inflow rate C_{in} and the inlet temperature T_{in}, are assumed to change over to a different operating point after the first hour of the reaction. Before and after the switching we regulate the state trajectories to the known operating points, ρ_1 and ρ_2; which are assumed as,

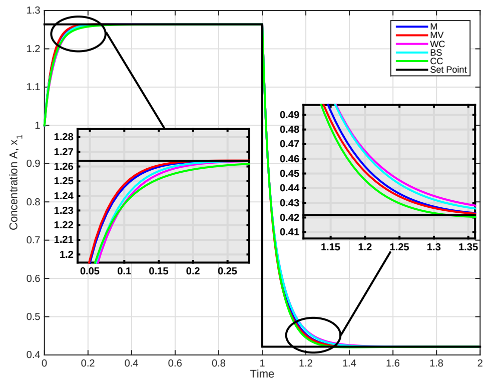
$$\begin{aligned} \rho_1^\top = &\quad \begin{bmatrix} C_{in}(1) & T_{in}(1) & u_1^{eq}(1) & u_2^{eq}(1) & x_1^{eq}(1) & x_2^{eq}(1) & x_3^{eq}(1) & x_4^{eq}(1) \end{bmatrix}, \\ = &\quad \begin{bmatrix} 5.1 & 104.9 & 4.19 & -1113.5 & 1.2639 & 0.9049 & 109.2881 & 108.0037 \end{bmatrix}, \\ \rho_2^\top = &\quad \begin{bmatrix} 1.1 & 109.9 & 4.19 & -1113.5 & 0.4216 & 0.2530 & 101.6720 & 100.3875 \end{bmatrix}, \end{aligned} \tag{21}$$

Furthermore we assume i.i.d. Gaussian additive uncertainties effecting the dynamics with the covariance matrix being equal to $\Sigma := \mathrm{diag}(.1,\ .1,\ 5,\ 5)$, 10% error rate for the concentration and 2.5% for the temperature variables. The cost functional is selected as quadratic cost function with weighting terms

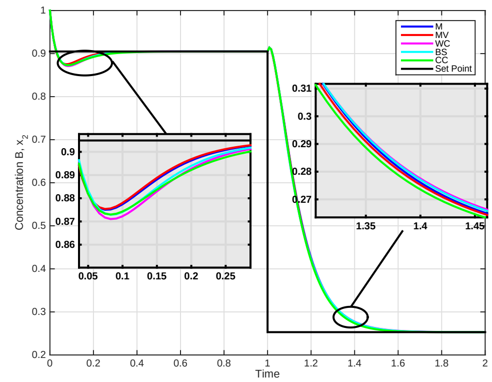
$$Q = 10\ I_4, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 10^{-6} \end{bmatrix}, \quad Q_f = 0.$$

The state and input constraints are selected as polytopes and defined by $0 \le x_1 \le 1.6,\ 0 \le x_2 \le 1.6,\ 80 \le x_3 \le 130,\ 80 \le x_4 \le 130$, and $3 \le u_1 \le 35,\ -9000 \le u_2 \le 0$.

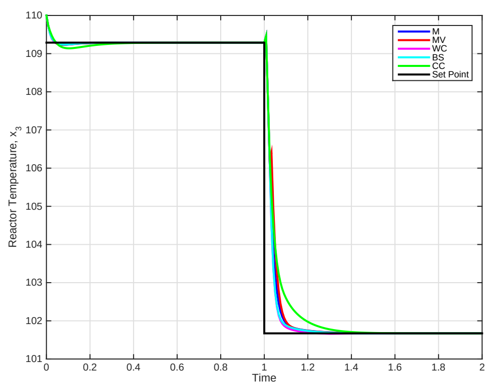
With these information, first we demonstrate the results obtained from the disturbance free case, where we set the uncertainty equal to zero for all time instants, see Figure 8. All robustness methods yield



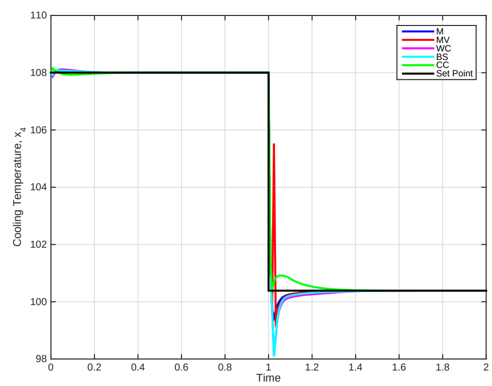
(a) Concentration of component A trajectories.



(b) Concentration of component B trajectories.



(c) Reaction temperature trajectories.

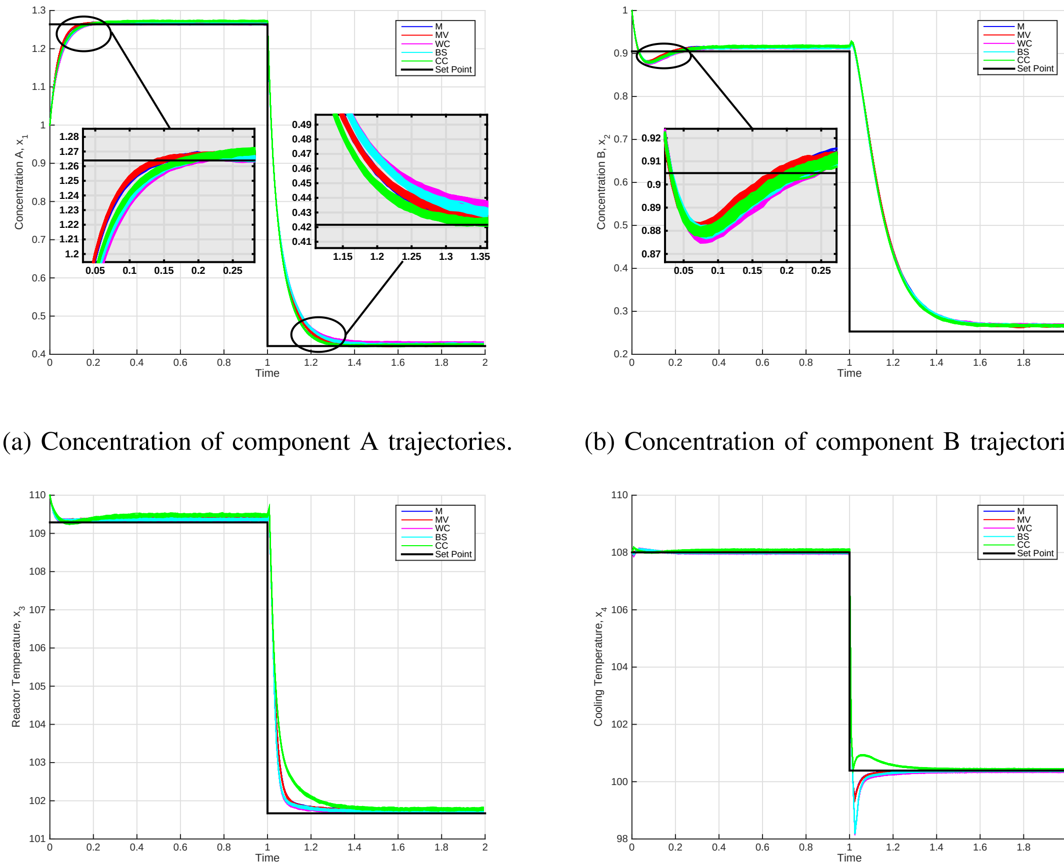


(d) Cooling temperature trajectories.

Fig. 8: The state trajectories for CSTR simulation without disturbances

similar performances, since the constraints allow for a sufficiently large operating window for the MPC controllers. If one shrinks the constraint set, first the WC-MPC, then BWC-MPC and CC-MPC turn infeasible. Also we observe the change of chance constrained MPC behaviour (green trajectories in Figure 8) in the case of unstable equilibrium, the operating point ρ_1, which leads to a slow convergence to the equilibrium and for the case of the stable equilibrium, operating point ρ_2, CC-MPC leads to fast convergence, due to reduced spread of uncertainties. Similar to the previous example, MV-MPC controlled system acts slightly faster than the Mean-MPC case, due to the increased aggressiveness of the controller induced from variance terms; while the budgeting MPC leads to a higher performance in comparison to the WC-MPC, due to smaller set of possible uncertain realizations. Secondly we present

100 different realizations of the perturbed states trajectories with different disturbance sequences for the mentioned RMPC controllers in Figure 9. Lastly we visualize in Figure 10 two sets of state trajectory



(a) Concentration of component A trajectories.

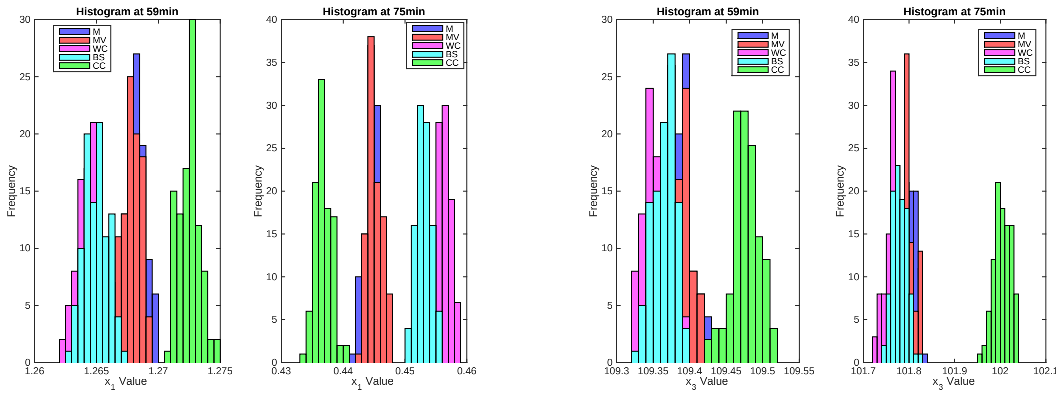
(b) Concentration of component B trajectories.

(c) Reaction temperature trajectories.

(d) Cooling temperature trajectories.

Fig. 9: The state trajectories for CSTR simulation with disturbances

histograms for two time instants. The first time instant is the 59^{th} minute of operation, in which the steady state responses are reported before the switching from one operating point to the second one occurs, and the second one is at the 75^{th} minute of the reaction which is demonstrating the transient behaviour after the switching. From the histograms we observe two distinct characteristics of different robustness techniques. Firstly, both before and after the operating point change from unstable to stable equilibrium, the set based robust MPCs (WC and BWC) cause (unnecessarily) pessimistic closed-loop trajectories, while the chance constrained MPC allows for aggressive control actions, hence drives the state trajectories towards the constraints during and after the operating point transition, similar to the case in Figure 1a and Figure 1b. Secondly, the moment based techniques yield comparable trajectories



(a) Concentration of component A histograms.

(b) Reaction temperature histograms.

Fig. 10: The state (x_1 and x_3) histograms for CSTR simulation at two different time instants.

in both operating points, indicating necessity for further research on the tuning of λ_0 variable. Rigorous analysis for tuning the λ_0 variable for MV-MPC is left as a future study.

C. A Two Mass and Spring System

In this example we make use of a mass-spring system, similar to the case in [89], which consists of two masses connected with a spring. The disturbance corrupted control action is effective on one of the masses and the control goal is to effectively stabilize the system. The dynamics of the process is taken as follows;

$$
\begin{gathered}
x_{k+1} = Ax_k + B(u_k + w_k), \\
A = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -\frac{K}{m_1} & 0.1\frac{K}{m_1} & 1 & 0 \\ \frac{K}{m_2} & -0.1\frac{K}{m_2} & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0.1\frac{1}{m_1} \\ 0 \end{bmatrix},
\end{gathered} \tag{22}
$$

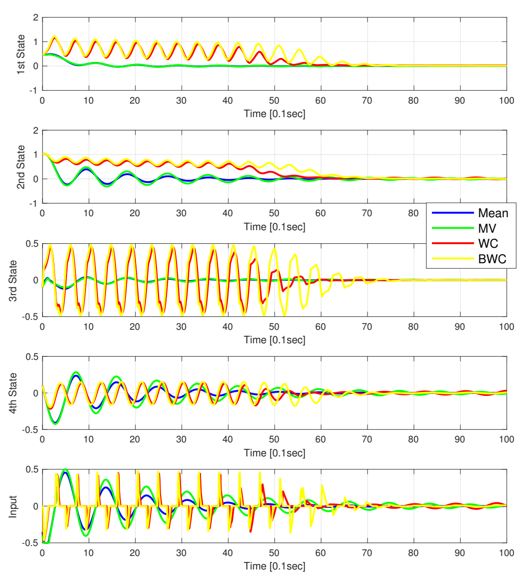
where the parameters, the spring constant K and the masses m_1 and m_2 are taken as; $K = 1,\ m_1 = 0.5,\ m_2 = 2$. We present two sets of solutions, the first case showing the differences between the closed-loop responses of different predictive controllers (Mean, MV, WC and BWC), in which we skipped the CC-MPC, since the results are almost comparable with the other moment based formulations. The second study demonstrates the aggressiveness of MV-MPC for different λ_0 values. In both of the simulations we have used the same state and input weighting matrices, such as

$$Q = 5I_4, \quad R = I_1, \quad Q_f = 0.$$

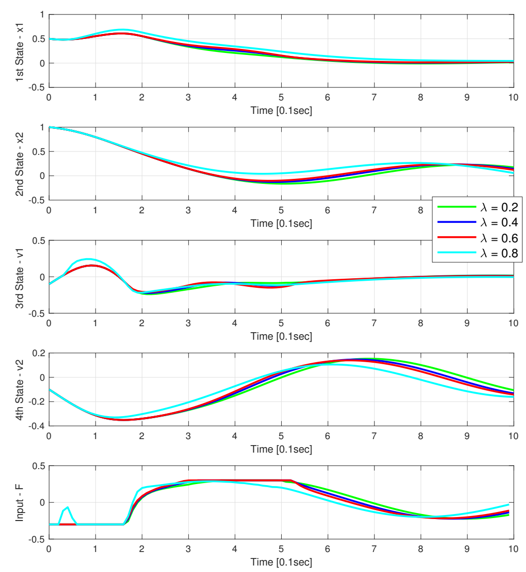
For the first simulation study, we make use of following state and input constraints,

$$\mathbb{X}_1 = \{x | -1 \leq x_1 \leq 1.5, -1 \leq x_2 \leq 1.5, -0.5 \leq x_3 \leq 0.5, -0.5 \leq x_4 \leq 0.5.\},$$
$$\mathbb{U}_1 = \{u | -0.5 \leq u \leq 0.5\},$$

while we initialized the state at $x_0^\top = \begin{bmatrix} 0.5 & 1 & -0.1 & 0.1 \end{bmatrix}$. For comparison purposes, we force the disturbance variables to zero, while the MV-MPC controller assumes that the input disturbance has a standard deviation of 0.15, the WC-MPC and BWC-MPC robustifies the closed-loop system for all of the disturbance realizations between $w_k^{WC} \in \begin{bmatrix} -0.45 & 0.45 \end{bmatrix}$, and $w_k^{BWC} \in \begin{bmatrix} -0.15 & 0.15 \end{bmatrix}$, respectively. The simulated trajectories are visualized in Figure 11a. The deterministically robust MPC controllers (WC-MPC and BWC-MPC) are almost inactive during the whole simulation, hence leading to much larger settling times in comparison to the Mean-MPC or the MV-MPC. The second simulation study



(a) closed-loop responses for different MPC approaches.



(b) Differences within the closed-loop responses of MV-MPC for different λ_0 values.

Fig. 11: Simulation results for a two mass-spring system.

demonstrates the effect of tuning parameter λ_0 for the MV-MPC construction. For this case we reshape the constraints as; $\mathbb{X}_2 = \{x | -1.1 \leq x_1 \leq 1.1, -1.1 \leq x_2 \leq 1.1, -0.4 \leq x_3 \leq 0.4, -0.4 \leq x_4 \leq 0.4\}$, $\mathbb{U}_2 = \{u | -0.3 \leq u \leq 0.3\}$, while the λ_0 value assumes values from the set $\{0.2,\ 0.4,\ 0.6,\ 0.8\}$. The constraints are tightened, via the parameter λ_{ij} which is set to 1 for all of the constraints. The simulated trajectories are visualized in Figure 11b. For the first three values of λ_0, we observe that the responses are similar to each other, although as the parameter λ_0 increases, the control action is growing in magnitude, and hence causing slight improvements in the state trajectories. The critical change occurs for the case of $\lambda_0 = 0.8$. In this case the optimal input signal differs from the other cases at the initial phase of the simulation, leading to slightly faster response due to the larger effect of variance term in the MPC problem, see also the discussion related to Figure 1b.

D. Debutanizer System and Budgets for Uncertainty

Lastly we provide a comparison study, comparing the achievable closed-loop performances for different uncertainty sets, on a debutanizer system. The simulation example, taken from [102], considers a 2-input 2-output MIMO system, with the transfer function given as

$$G(s) = \begin{bmatrix} \frac{-0.2949}{64.02s^2+61.66s+1} & \frac{0.1310}{854.75s^2+88.03s+1} \\ \frac{0.1287}{168.25s^2+15s+1} & \frac{-0.1434}{32s^2+17.76s+1} \end{bmatrix}, \qquad Y(s) = G(s)(U(s)+W(s)).$$

We first represent this transfer function in state-space from and then apply discretization with a sampling time of 1.8 minutes. We pose hard input and output constraints on a zero initial condition such as $\mathbb{Y} = \{y | -1 \leq y_1 \leq 1, -1 \leq y_2 \leq 1, \}$, $\mathbb{U} = \{u | -15 \leq u_1 \leq 15, -15 \leq u_2 \leq 15\}$. The goal of the process is to reach set point $y_i = 1,\ i = 1,2$, while due to the constraint, the RMPC controllers back-off the transient trajectories and the final operating point, see also the discussion related to Figure 1a. The cost function is taken as a quadratic form of shifted nominal outputs (towards the desired operating point) and inputs, with the weighting matrices taken as $Q = 10^7 I_2, \quad R = 0.1 I_2$ and the prediction horizon set to $N_p = 10$. We compare the responses for different selection of uncertainty sets, such as the disturbance set is set to $\mathscr{W}^{WC} = \{-50, 50\}$, and for the BWC cases we scale this set with ten different values, i.e., $\mathscr{W}^{\alpha BWC} = \alpha \cdot \{-50, 50\}, \alpha = 0, 1, \ldots, 9$, as visualized in Figure 12. As can be seen from the second output trajectories, the WC-MPC and BWC-MPC with budget level of $0.9\mathscr{W}$ hit the tightened constraints during the transition between $t \in [75, 90]$, while for the other uncertainty sets, the MPC controllers never reach to the tightened constraint levels for the second output. Regarding the first output, we can observe that 0.1-BWC MPC and nominal MPC are acting exactly same with with each other, hence meaning that Mean MPC is already providing robustified operation for additive uncertainties belonging to the set $\mathscr{W}^{0.1BWC} = [-5, 5]$. Furthermore, one can clearly observe the pessimistic results of WC or BWC MPCs with a large assumed set of uncertain effects, since the settling value for the first output deteriorates as the α increases in α-BWC MPC constructions.

The simulation examples studying the different robust MPC constructions introduced in previous sections for the debutanizer system have confirmed the fact that the performance of deterministically robust MPC approaches (WC-MPC and BWC-MPC) is overly conservative and responds cautiously

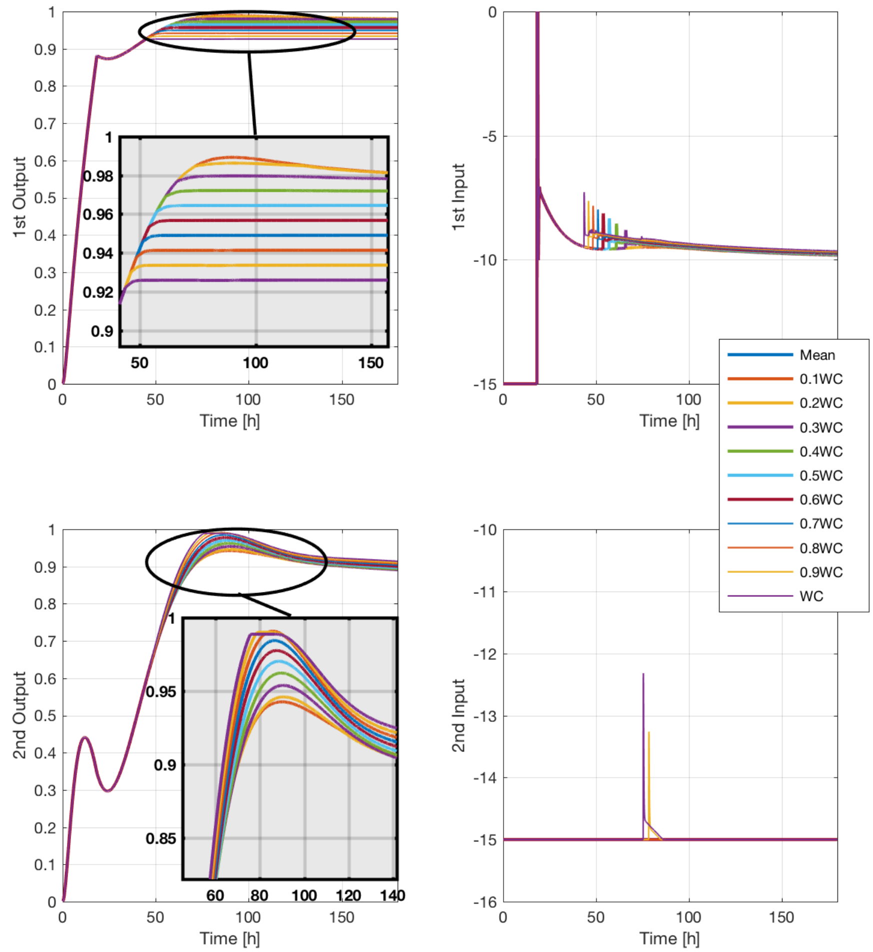


Fig. 12: Differences within the closed-loop responses of Mean, WC and different BWC-MPC controllers for different $\bar{\Delta}$ sets.

resulting in larger settling times in comparison to SMPC approaches. SPMC methods, on the other hand, have a tendency of providing aggressive control actions leading to constraint violations in some cases.

VI. Conclusions

In this work, we present and compare the methods used in the robust and stochastic MPC literature regarding the optimal operation under uncertainty. The vast literature and terminological or conceptual differences between the different branches of robust and stochastic MPC necessitates extensive time and effort for an understanding of current RMPC or SMPC algorithms. This induces a divergence between industrial and academic perspectives on guaranteeing robust operation with MPC controllers. In this paper we discuss various MPC paradigms from the closed-loop pessimism and required computational power perspectives. We present connections between the robust optimization and RMPC, while similar connections between the (coherent) risk theory and SMPC are also mentioned. We state and classify contributions from the RMPC and SMPC literature, according to different approaches and the properties associated with these approaches. Furthermore, we apply these methods to four different examples namely, a simple batch reaction, a CSTR example, a two mass system coupled with a spring and lastly a MIMO debutanizer system, to compare the computational requirements and closed-loop performance.

The research resulting in this paper indicates that for uncertain MPC problems controlling uncertain dynamics, the practitioner should concentrate on the following topics before proceeding with any desired solution method;

Modeling the process dynamics: Already the nominal prediction model complexity radically effects the quality of future predictions versus the computational complexity trade-off. Additionally, the resulting closed-loop pessimism due to uncertainty, price paid for robustness, depends also on how the uncertainty is evolving in the dynamics. Since the large-scale rigorous models provide better predictions, one would expect to use larger and more complex models in RMPC and SMPC algorithms over time. However these models are demanding severe computational requirements for industrial processes. The trade-off lies in between the improvement in the realistic prediction quality, generally corresponding to a more complex model, and the resulting computational complexity for incorporating uncertain effects.

Modeling the uncertainty set: It is extremely important to choose an uncertainty model which resembles only the true process unknowns. Many of the uncertainty models lead to unrealistic state predictions while increasing the required computational power and the resulting in pessimism in control. To circumvent this problem, the general tendency is to employ feedback gains to the controller, *pre-compensate* the process, which can be a challenging task for large-scale nonlinear systems. Currently we still miss detailed results where the optimal inputs start to be dominated by the unknowns of the dynamics. The hierarchical decomposition of control goals, cancelling the uncertain effects via (robust) feedback control techniques, while using MPC essentially as a trajectory optimizer is not necessarily valid for many processes, since in many cases uncertain effects can be helpful for the dynamical flow. Scenario based MPC might be a key approach related to this aspect, since one can effectively incorporate only the realistic uncertain realizations, bypassing the unnecessary conservatism.

Modeling the cost and constraints and quantification of uncertainty: The decision on how the cost and the constraint functions are treated effects the resulting computational power requirement and also the closed-loop performance. To improve the response, one should re-evaluate the allocated uncertainty budget for each constraint. If constraint violations are practically allowable, one should prefer chance or moment constraints rather than the deterministic guarantees. Similar issues are also valid for cost functions. It is not straightforward which cost function to minimize (or maximize) for the best performance. The connections between the expected cost function and the nominal cost needs further discussion, while there might be better candidates from the theory of risk, thus justifying the treatment of uncertainty in the MPC problems according to a risk metric.

In conclusion the dynamics of the process, the uncertainty attached to it and (risk-based) adjustment of the specifications (cost-constraint functions) are crucial in effecting the closed-loop performance, and hence should be tailored according to the final goal of the MPC problem. The theoretical dilemma between the predicted uncertain effect in the future and the causal optimal control to reduce adverse effects of it is inherently leading to a process/specification based construction, hence further separating practical applicability of developed methods.

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