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ORTHONORMAL BASIS FUNCTIONS IN TIME AND FREQUENCY DOMAIN: HAMBO TRANSFORM THEORY*

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Abstract. The class of finite impulse response (FIR), Laguerre, and Kautz functions can be generalized to a family of rational orthonormal basis functions for the Hardy space H_2 of stable linear dynamical systems. These basis functions are useful for constructing efficient parameterizations and coding of linear systems and signals, as required in, e.g., system identification, system approximation, and adaptive filtering. In this paper, the basis functions are derived from a transfer function perspective as well as in a state space setting. It is shown how this approach leads to alternative series expansions of systems and signals in time and frequency domain. The generalized basis functions induce signal and system transforms (Hambo transforms), which have proved to be useful analysis tools in various modelling problems. These transforms are analyzed in detail in this paper, and a large number of their properties are derived. Principally, it is shown how minimal state space realizations of the system transform can be obtained from minimal state space realizations of the original system and vice versa.

Key words. orthogonal basis functions, Hambo transform, cascade inner network, expansion coefficients

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1. Introduction.

Orthonormal bases and the transformations that are related to them are useful tools in many branches of science. Well-known examples are the trigonometric bases which induce the various Fourier transforms or the more recently developed orthonormal wavelet bases and their associated transforms. Within the field of systems and control theory, *rational* orthonormal bases play an important role. By approximating the impulse response of a linear time-invariant (LTI) system by a finite sum of exponentials, the problem of modelling and identification is considerably simplified. This comes down to using rational basis functions in the model structure.

Over the last years a general theory has been developed for the construction and analysis of generalized orthonormal rational basis functions for the class of stable linear systems, which extends the work on Laguerre filters by Wiener in the thirties [19]. The corresponding filters are parameterized in terms of prespecified poles, which makes it possible to incorporate a priori information about time constants in the model structure. The main applications are in system identification and adaptive signal processing, where the parameterization of models in terms of finite expansion coefficients is attractive because it is linear-in-the-parameters. This allows the use of simple linear regression estimation techniques to identify the system from observed data, thus avoiding nonconvex optimization problems. Orthonormality is associated with white

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noise input signals. However, the special shift structure of generalized orthonormal basis functions gives a certain Toeplitz structure for general quasi-stationary input signals, which can be utilized to construct efficient algorithms and to derive statistical performance results. The use of orthogonal basis functions has also resulted in intuitive expressions for the variance of estimated transfer functions and noise models. Here the basis functions and related reproducing kernels are used to analyze and simplify complicated variance expressions. See [46, 27, 28] for the most recent contributions. For the field of adaptive filtering, see, for instance, [2, 9, 21].

The application potentials of orthogonal basis functions go beyond the areas of system identification and adaptive signal processing. Many problems in circuit theory, signal processing, telecommunication, systems and control theory, estimation, and optimization theory benefit from an efficient representation or parameterization of particular classes of signals/systems. See, for instance, [31, 5] for applications in audio processing and [24, 23, 36] for the use of orthogonal basis functions in nonlinear modelling and estimation.

By exploiting prior knowledge of the object (signal/system) to be described, a decomposition of signals/systems in terms of flexibly chosen orthogonal (independent) components leads to efficient and robust estimation and prediction algorithms. Orthogonality is the key principle in linear estimation; see [16]. Orthogonal filters, which correspond to orthogonal rational functions, are of capital importance in filter design and robust filter implementation, as discussed in, e.g., [32].

In this paper a comprehensive account is given of the unitary transforms that result when considering series expansion representations of signals and systems in terms of a special class of generalized rational orthonormal basis functions, the so-called Hambo[1] functions. This transform generalizes the Z- and the Laguerre transforms and will be shown to have very intriguing structural properties. Preliminary results on this transform have appeared earlier in the analysis of system identification algorithms [39], in system approximation [13], and in minimal partial realization [37, 8]. In these papers, the transform results were shown to be instrumental in the statistical analysis of system identification and in solving partial realization problems. The present paper is the first to give a comprehensive account of the development and the properties of the considered transform, including analysis and algorithms in state space form.

The technique of transformation, or, equivalently, the choice of an alternative domain of representation, has been used successfully for the solution of a wide range of problems in various scientific areas; cf. Laplace and Fourier transformations in the fields of system and control theory or signal processing. It is expected that the transformation which is proposed in this paper and that has the powerful property that it can be adapted to the dynamics of a specific problem will open new possibilities for the solution of a broad class of problems.

The remainder of the paper is constructed as follows. First, in section 2, the considered basis functions will be specified and reviewed. After considering series expansion expressions in section 3, the related signal and system transforms are presented in section 4. In section 5, the constituting expressions for calculating the transforms are presented. Additional properties are discussed in section 6, while in section 7 some extensions are briefly indicated.

[1]The word Hambo originated as an acronym for Hankel matrix based orthogonality. In the remainder of the paper, these Hambo functions will also be referred to as generalized basis functions.

Notation.

$A^T, \overline{A}, A^*$ Transpose, respectively, complex conjugate and complex conjugate transpose of the matrix A.

$\mathbb{T}$ Unit circle.

$L_2^{p\times m}(\mathbb{T})$ Hilbert space of complex matrix functions of dimension $p\times m$ that are square integrable on the unit circle. The superscript $p \times m$ will be suppressed if $p = m = 1$.

$H_2^{p\times m}$ Hardy space of all functions which are analytic in the exterior of the unit disc such that[2]

$$\lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \text{Trace}(f(re^{i\omega})f(re^{i\omega})^*)d\omega < \infty.$$

$RH_2^{p\times m}$ Subspace of *rational* transfer functions of $H_2^{p\times m}$.

$H_2^{\perp}$ The orthogonal complement of H_2 in L_2.

$H_{2-}^{p\times m}$ The same as $H_2^{p\times m}$, with the restriction that the functions must be zero at infinity (i.e., $f_0 = 0$).

$RH_{2-}^{p\times m}$ Subspace of *rational* transfer functions of $H_{2-}^{p\times m}$.

$\ell_2^n(J)$ The space of square summable vector sequences, of vector dimension n, where J denotes the index set of the sequence. The superscript n will be omitted if $n = 1$.

$\langle F, G\rangle$ Inner product of F and G in $L_2^{p\times m}(\mathbb{T})$:

$$\frac{1}{2\pi i} \int_0^{2\pi} \text{Trace}\{F^T(e^{i\omega})\overline{G(e^{i\omega})}\}d\omega.$$

$\langle x, y\rangle$ Inner product of x and y in $\ell_2^n(J)$: $\sum_{k\in J} x^T(k)\overline{y(k)}$.

$[\![x, y]\!]$ ℓ_2 Matrix "inner product" $\sum_{k\in J} x(k)y^T(k)$, with $x \in \ell_2^{n\times p}(J)$, $y \in \ell_2^{m\times p}(J)$.

$[\![X, Y]\!]$ L_2 Matrix "inner product" $\frac{1}{2\pi i} \oint X(z)Y^*(1/z)\frac{dz}{z}$, with $X \in L_2^{n\times p}(\mathbb{T})$, $Y \in L_2^{m\times p}(\mathbb{T})$.[3]

$\mathbf{P}_X$ Orthogonal projection onto the subspace X.

$\mathbf{e}_i$ ith canonical Euclidean basis (column) vector.

q shift operator; for $x \in \ell_2$, $n \in \mathbb{Z}$: $(q^n x)(t) = x(t + n)$.

In this paper, ℓ_2 signals will be generally denoted by small characters, whereas capitals will be used for their Z-transforms, i.e., $x(t)$, respectively, $X(z)$. Expansion coefficients of a signal in a nonstandard basis are characterized with the $\breve{}$ symbol, as in $x(t) = \sum_k \breve{x}(k)f_k(t)$. By abuse of notation, systems and operators will generally be denoted with arguments; for instance, $x(t), G(z)$ will denote elements of ℓ_2, respectively, H_2.

Unless otherwise mentioned, the notion of orthonormality will be used with respect to the ℓ_2 or L_2 inner products, as defined above.

2. Basis construction. In this section, we will present the basis functions under consideration, first in transfer function form, followed by an interpretation in a state space setting.

[2]Here H_2 is identified with the subspace of L_2 with vanishing negative Fourier coefficients. More precisely, for $F \in H_2, F(z) = f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots$, and $\sum_{k=0}^{\infty} |f(k)|^2 < \infty$.

[3]Here $Y^*(1/z) = \sum_k y(k)^* z^{-k}$.

2.1. Transfer function approach. The main idea of constructing rational orthonormal basis functions is to generate a set of orthonormal functions that have exponential decay. A straightforward approach to this problem is to orthonormalize the set of functions

$$F_{i,j}(z) = \frac{1}{(z-a_i)^j}, \quad i \in \mathbb{N}, \quad 1 \le j \le m_i, \tag{2.1}$$

where the poles a_i can be any complex number with $|a_i| < 1$, such that $a_i \neq a_k$, $i \neq k$, and where m_i is the multiplicity of pole a_i. Obviously any rational function in H_{2-} can be described as a weighted sum of these functions if the poles a_i are chosen appropriately.

PROPOSITION 2.1. *Application of the Gram–Schmidt procedure to the sequence of functions, given by* (2.1), *yields the orthonormal functions*

$$\Phi_k(z) = \frac{\sqrt{1-|\xi_k|^2}}{z-\xi_k} \prod_{j=1}^{k-1} \frac{1-\bar{\xi}_j z}{z-\xi_j}, \quad k \in \mathbb{N}, \tag{2.2}$$

where $\xi_{N_i+l} = a_i$, $1 \le l \le m_i$, *with* $N_i = \sum_{j=1}^{i-1} m_j$.

According to [45], this sequence of orthonormal functions was originally derived in the 1920s by Takenaka [38] and Malmquist [20] and will henceforth be referred to as the Takenaka–Malmquist functions. In the 1950s, the continuous-time version of these functions was derived by Kautz [18] in the context of network synthesis. They emerged again in the work of Ninness and Gustafsson [26] in the context of system identification. See also [4]. Orthonormality of these functions can easily be established using residue calculus. A more fundamental question is whether the orthonormal set is complete in H_{2-}. The following result, already given in [38] and [20], gives necessary and sufficient conditions for completeness.

PROPOSITION 2.2. *Let* $\{\xi_k\}_{k\in\mathbb{N}}$ *be such that* $|\xi_k| < 1$ *for all* $k \in \mathbb{N}$. *The set of Takenaka–Malmquist functions* $\{\Phi_k(z)\}_{k\in\mathbb{N}}$, *as given in* (2.2), *is complete in* H_{2-} *if and only if*

$$\sum_{k=1}^{\infty} (1-|\xi_k|) = \infty. \tag{2.3}$$

In other words, if the sequence of poles does not converge to the unit circle "too fast," then the set of Takenaka–Malmquist functions constitutes an orthonormal basis for H_{2-}. Until the early 1990s, only special cases of these functions have been used extensively, especially in the context of system identification and signal processing. Of these special cases, the pulse and Laguerre functions are the best known examples. Consider the case where for all $k, \xi_k = a \in \mathbb{R}$, with $|a| < 1$. The corresponding basis functions are the discrete Laguerre functions

$$\Phi_k(z) = \frac{\sqrt{1-a^2}}{z-a} \left[\frac{1-az}{z-a}\right]^{k-1} \tag{2.4}$$

that reduce to the pulse functions $\Phi_k(z) = z^{-k}$ for $a = 0$.

A second special case that is discussed in detail in this paper considers the situation where all poles are taken in a repetitive manner from a finite set $\{\xi_1, \xi_2, \dots, \xi_{n_b}\}$, such that $\xi_{k\cdot n_b+j} = \xi_j$, where $k \in \mathbb{N}$ and $j = 1, \dots, n_b$. When the poles appear in complex conjugate pairs, this results in the class of so-called generalized orthonormal basis functions, or Hambo functions [13]. For ease of notation, we introduce the inner (stable all-pass) function $G_b(z) = \prod_{i=1}^{n_b}[\frac{1-\bar{\xi}_i z}{z-\xi_i}]$. Now since $\xi_{n_b+1} = \xi_1$, it follows that $\Phi_{n_b+1}(z) = \frac{\sqrt{1-|\xi_1|^2}}{z-\xi_1}G_b(z) = \Phi_1(z)G_b(z)$, and it is easy to see that an equivalent relation holds for the next functions, $\Phi_{n_b+j}(z) = \Phi_j(z)G_b(z), j \in \mathbb{N}$. From these relations it is straightforward to derive the so-called generalized shift property:

$$\Phi_{k\cdot n_b+j}(z) = \Phi_j(z)G_b^{k-1}(z), \quad k \in \mathbb{N}, \quad j = 1, \dots, n_b.$$

For convenience of notation, these functions are often grouped into vector functions

$$V_k(z) = \begin{bmatrix} \Phi_{(k-1)\cdot n_b+1}(z) & \Phi_{(k-1)\cdot n_b+2}(z) & \cdots & \Phi_{k\cdot n_b}(z) \end{bmatrix}^T, \tag{2.5}$$

in which case the shift property comes down to $V_k(z) = V_1(z)G_b^{k-1}(z)$. This shift property will be of paramount importance in the remainder of this paper.

In the context of system approximation and identification, it is often desired that the system responses are real-valued, and for that reason it will be advantageous to restrict the basis functions to being real-valued as well. Ninness and Gustafsson [26] showed that if the poles appear in complex conjugate pole pairs, all basis functions can be made real-valued by a simple unitary transformation of the set of basis functions.

2.2. State space interpretation. An alternative way to interpret or derive these basis functions employs state space models. Consider a (single input) stable state space model

$$x(t+1) = Ax(t) + Bu(t). \tag{2.6}$$

The function $V(z) = [zI - A]^{-1} B$ is the transfer function from the input $u(t)$ to the states $x(t)$. Now assume that the input signal $u(t)$ is a zero mean white noise process with variance 1, i.e., $\mathbb{E}\{u(t)u(t+k)\} = \delta_k$. The state covariance matrix $P = \mathbb{E}\{x(t)x^T(t)\}$ satisfies the Lyapunov equation $P = APA^T + BB^T$. P also equals the so-called controllability Gramian of the state space model. The reason why we are interested in the state covariance matrix is that

$$P = \frac{1}{2\pi i}\oint_{\mathbb{T}} V(z)V^T(1/z)\frac{dz}{z} = [\![V, V]\!]. \tag{2.7}$$

The basic idea now is to find a new state space realization for which the state covariance equals the identity matrix, $P = I$. The corresponding input to state transfer functions will then be orthonormal and will span the same space as the original functions, as only linear transformations are considered. A state space realization for which $P = I$ is called input balanced [22].

In order to extend this resulting finite set of orthonormal functions, we consider the class of square inner functions, i.e., stable transfer functions $G_b(z)$ that satisfy

$$G_b(z)G_b^T\left(\frac{1}{z}\right) = I.$$

It was shown in [33] that square inner functions can be realized by so-called orthogonal state space realizations; i.e., they satisfy $G_b(z) = D + C(zI - A)^{-1}B$, where

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}^T = I. \tag{2.8}$$

From this orthogonality property, it directly follows that the controllability Gramian P and the observability Gramian Q, which are defined as $P = APA^T + BB^T$ and $Q = A^TQA + C^TC$, satisfy $P = Q = I$, and so realizations with this property are balanced in the sense of [22]. Thus it follows that the input-to-state functions (i.e., the elements of $V(z) = [zI - A]^{-1}B$) are mutually orthonormal with respect to the H_{2-} inner product (assuming $G_b(z)$ is scalar).

Example 2.3. We consider first and second order inner functions.

1. Let $G_b(z) = \frac{1-az}{z-a}$, with $|a| < 1$. Then $\{a, \sqrt{1-a^2}, \sqrt{1-a^2}, -a\}$ is a balanced realization for G_b, and the input to state transfer is $\frac{\sqrt{1-a^2}}{z-a}$, the first Laguerre function with pole in a.
2. Let $G_b(z) = \frac{-cz^2+b(c-1)z+1}{z^2+b(c-1)z-c}$ with some real-valued b, c satisfying $|c|, |b| < 1$. A balanced realization (see, e.g., [39]) results in $V(z) = \frac{\sqrt{1-c^2}}{z^2+b(c-1)z-c}[(z-b)\cdot\sqrt{(1-b^2)}]^T$, which represents the first two functions of the so-called 2-parameter Kautz construction.

On the other hand, when given an arbitrary pair (A, B) with controllability Gramian $P = I$, it is easy to show that there exist matrices (C, D) such that the transfer function $G(z) = D + C(zI - A)^{-1}B$ is an inner function [12]. Note that this realization is automatically balanced.

Hence, when the state space approach is used to create orthonormal functions, these functions can be considered as the input-to-state functions of a balanced realization of an inner function.

A second result from [33] as indicated in [3] is that for two inner functions $G_i(z) \in H_2$ $(i = 1, 2)$, with corresponding balanced realizations (A_i, B_i, C_i, D_i), the product $G_2(z)G_1(z)$ has a balanced realization (A, B, C, D) with

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2C_1 & A_2 & B_2D_1 \\ \hline D_2C_1 & C_2 & D_2D_1 \end{array}\right]. \tag{2.9}$$

For any input signal $u(t)$, the state sequence $x(t)$ related to this realization can be decomposed by $x(t) = [x_1(t) \quad x_2(t)]^T$, where $x_1(t)$ is the state trajectory related to the realization of $G_1(z)$ separately:

$$x_1(t) = [qI - A_1]^{-1}B_1u(t) \qquad \text{and} \qquad x_2(t) = [qI - A_2]^{-1}B_2G_1(q)u(t).$$

Here q denotes the shift operator, as defined in our notation.

In other words, there exists a recursive structure, where concatenating inner functions provide an increasing number of state functions that are orthogonal to each other with respect to the standard ℓ_2 inner product, i.e., $\sum_t x_k^T(t)\overline{x_j(t)} = \delta_{kj}$ or, equivalently, $\frac{1}{2\pi i}\int_0^{2\pi} X_k^T(e^{i\omega})\overline{X_j(e^{i\omega})}d\omega = \delta_{kj}$. This leads to the following construction.

PROPOSITION 2.4. *Given a sequence of inner functions* $G_i(z), i = 1, 2, \dots,$ *each with balanced realization* (A_i, B_i, C_i, D_i)*, the collection of functions* $\{X_i(z), i = 1, 2, \dots\}$ *with*

$$X_1(z) = [zI - A_1]^{-1}B_1, \qquad X_i(z) = [zI - A_i]^{-1}B_iG_1(z)G_2(z)\cdots G_{i-1}(z),$$

is mutually orthonormal.

With this property and the balanced realizations of Example 2.3, it is straightforward to rederive the Takenaka–Malmquist functions (2.2) as well as the Laguerre functions and the Hambo functions (see (2.5)). Both approaches lead to the same class of functions. Hence the completeness condition (2.3) is valid for both approaches. The special case of Proposition 2.4, where all $G_i(z)$ are equal to the same second order inner function with a complex conjugate pole pair (see Example 2.3 2) is known in the literature as the 2-parameter Kautz construction [42, 14, 26].

3. Related bases and series expansions. Since the Takenaka–Malmquist functions constitute a basis for H_{2-}, a basis for the related space $\ell_2(\mathbb{N})$ follows by considering the inverse Z-transform, which is isomorphic. With $\{\phi_k(t)\}$ the impulse response (Fourier coefficients) of $\Phi_k(z)$, according to $\Phi_k(z) := \sum_{t=1}^{\infty} \phi_k(t) z^{-t}$, the functions $\{\phi_k(t)\}$ will constitute an orthonormal basis for $\ell_2(\mathbb{N})$. Note that these basis functions exhibit the property that they can incorporate system dynamics in a very general way. One can construct inner functions from any given set of poles, and thus the resulting basis can incorporate dynamics of any complexity, combining, e.g., both fast and slow dynamics in damped and resonant modes. Considering the Takenaka–Malmquist basis functions, for any system $H(z) \in H_{2-}$ or signal $y(t) \in \ell_2(\mathbb{N})$, there exist unique series expansions:

$$H(z) = \sum_{k=1}^{\infty} \langle H, \Phi_k \rangle \Phi_k(z), \tag{3.1a}$$

$$y(t) = \sum_{k=1}^{\infty} \langle y, \phi_k \rangle \phi_k(t). \tag{3.1b}$$

In the remainder of this paper, attention will be focused on the Hambo functions, as introduced in section 2.1, i.e., the subclass of Takenaka–Malmquist functions where the basis poles are taken in a repetitive manner from a finite set $\{\xi_1, \ldots, \xi_{n_b}\}$. When these poles $\{\xi_i\}_{i=1}^{n_b}$ are stable, i.e., $|\xi_i| < 1$, it follows from Proposition 2.2 that the set of Hambo functions constitutes a basis for H_{2-}. In what follows, we will also assume that the basis poles appear in complex conjugate pairs only. Furthermore, we will primarily consider the real-rational form of these functions that results from the application of Proposition 2.4, using a real-valued state space realization of the inner function

$$G_b(z) = \prod_{i=1}^{n_b} \frac{1 - \bar{\xi}_i z}{z - \xi_i}. \tag{3.2}$$

DEFINITION 3.1. *Let $G_b(z)$ be a real-rational inner function, with real-valued minimal balanced realization (A_b, B_b, C_b, D_b). Let for $k \in \mathbb{N}$ the vector functions $V_k(z)$ be defined as $V_k(z) = [zI - A_b]^{-1} B_b G_b^{k-1}(z)$. Then the collection of all scalar elements of the vectors $V_k(z)$, $\Phi_{k,i}(z) = e_i^T V_k(z)$, $k \in \mathbb{N}$, $1 \le i \le n_b$, is referred to as a Hambo basis of H_{2-}. The corresponding vectors with basis functions for $\ell_2(\mathbb{N})$ will be denoted by $\{v_k(t)\}$.*

It is straightforward to recognize the shift structure in the functions $v_k(t)$:

$$v_{k+1}(t) = G_b(q) \cdot v_k(t), \quad k = 1, 2, \ldots, \tag{3.3a}$$

$$v_1(t) = A_b^{t-1} B_b. \tag{3.3b}$$

For the class of Hambo functions, based on an inner function $G_b(z)$, the series expansions (3.1) can be rewritten such that the vector structure is maintained:

$$H(z) = \sum_{k=1}^{\infty} \breve{h}^T(k) V_k(z), \quad \breve{h}(k) = [\![V_k, H]\!], \tag{3.4a}$$

$$y(t) = \sum_{k=1}^{\infty} \breve{y}^T(k) v_k(t), \quad \breve{y}(k) = [\![v_k, y]\!]. \tag{3.4b}$$

The vector coefficient sequence $\breve{y} = \{\breve{y}(k)\}_{k\in\mathbb{N}}$ in (3.4) is called the Hambo signal transform of y. This transform will play a fundamental role in this paper. A formal definition will be given in section 4. The next proposition shows that the Parseval identity holds for this transform.

PROPOSITION 3.2 (Parseval's identity). *For any pair $x(t), y(t) \in \ell_2(\mathbb{N})$ and corresponding expansion coefficient sequences $\breve{x}, \breve{y}$, taken with respect to the basis vectors $\{v_k(t)\}_{k\in\mathbb{N}}$ as in* (3.4), *it holds that $\langle x, y\rangle = \langle \breve{x}, \breve{y}\rangle$.*

Proof. $\langle x, y\rangle = [\![\sum_k \breve{x}^T(k) v_k, \sum_{k'} \breve{y}^T(k') v_{k'}]\!] = \sum_k \sum_{k'} \breve{x}^T(k) [\![v_k, v_{k'}]\!] \breve{y}(k') = \sum_k \breve{x}^T(k) \breve{y}(k)$. ∎

A dual orthonormal basis of $\ell_2^{n_b}(\mathbb{N})$. One consequence of Proposition 3.2 is that an orthonormal basis of $\ell_2^{n_b}(\mathbb{N})$ can be obtained by taking the signal transform of the standard orthonormal basis functions of $\ell_2(\mathbb{N})$: $\delta(t-k),\ k > 0$. The resulting basis functions w_l are given by

$$w_l(k) = [\![v_k(t), \delta(t-l)]\!] = \sum_{t=1}^{\infty} v_k(t)\delta(t-l) = v_k(l). \tag{3.5}$$

Therefore, we can state the following.

PROPOSITION 3.3 (dual orthonormal basis). *Consider the basis function vectors $v_k(t)$ with $k \in \mathbb{N}$, as defined in Definition* 3.1. *The vector functions $w_t(k) \in \ell_2^{n_b},\ t \in \mathbb{N}$, defined by $w_t(k) = v_k(t)$, constitute an orthonormal basis of the space $\ell_2^{n_b}(\mathbb{N})$.*

It turns out that—as is the case with $v_k(t)$ (see (3.3))— these functions $w_k(t)$ can be calculated using a shift structure.

PROPOSITION 3.4. *Let $G_b(z)$ be a scalar inner function with McMillan degree $n_b > 0$, having a minimal balanced realization (A_b, B_b, C_b, D_b). Consider $v_k(t), w_k(t)$ as before, and let $N(z) = A_b + B_b[zI - D_b]^{-1}C_b$. Then*

$$w_{k+1}(t) = N(q) \cdot w_k(t), \quad k = 1, 2, \ldots, \tag{3.6a}$$

$$w_1(t) = B_b D_b^{t-1}, \tag{3.6b}$$

where the shift operator q operates on the time sequence w_k, i.e., $(qw_k)(t) = w_k(t+1)$.

Proof. The proof uses the balanced state space realization $(A_{k+1}, B_{k+1}, C_{k+1}, D_{k+1})$ of $G_b^{k+1}(z)$ (see (2.9)), where

$$A_{k+1} = \begin{bmatrix} A_b & 0 & \cdots & \cdot & 0 \\ B_bC_b & A_b & 0 & \cdot & 0 \\ B_bD_bC_b & B_bC_b & \cdot & \cdot & 0 \\ \vdots & \vdots & \cdot & \ddots & 0 \\ B_bD_b^{k-1}C_b & B_bD_b^{k-2}C_b & \cdots & B_bC_b & A_b \end{bmatrix},$$

$$B_{k+1} = \begin{bmatrix} B_b \\ B_bD_b \\ B_bD_b^2 \\ \vdots \\ B_bD_b^k \end{bmatrix}. \tag{3.7}$$

It is straightforward that $\left[v_1^T(t)\ \cdots\ v_{k+1}^T(t)\right]^T = A_{k+1}^{t-1}B_{k+1}$, and hence $w_1(t) = B_b D_b^{t-1}$. For $t \geq 1$

$$\begin{aligned} w_{k+1}(t+1) = v_{t+1}(k+1) &= A_b v_{t+1}(k) + B_b C_b v_t(k) + \cdots + B_b D_b^{k-1} C_b v_1(k) \\ &= A_b w_k(t+1) + \sum_{i=1}^{t} B_b D_b^{i-1} C_b w_k(t+1-i), \end{aligned}$$

which proves the result. □

We will denote the Z-transform of the functions $w_k(t)$ by $W_k(z) := \sum_{t=1}^{\infty} w_k(t) z^{-t}$, while as a direct result of Proposition 3.4 it holds that $W_k(z) = N^{k-1}(z) \cdot W_1(z)$, with $W_1(z) := (zI - D_b)^{-1} B_b$. Note the duality between the functions $G_b(z)$ and $N(z)$, which are simply related by ordering the state space realizations in reverse.

As a consequence, for any strictly proper system $\breve{H}(z) \in H_{2-}^{n_b}$ or signal $\breve{y}(t) \in \ell_2^{n_b}[1, \infty)$, there exist unique series expansions:

$$\breve{H}(z) = \sum_{k=1}^{\infty} h(k) W_k(z), \quad h(k) = \langle \breve{H}, W_k \rangle, \tag{3.8}$$

$$\breve{y}(t) = \sum_{k=1}^{\infty} y(k) w_k(t), \quad y(k) = \langle \breve{y}, w_k \rangle. \tag{3.9}$$

In fact, these are exactly the inverses of the expansions given by (3.4).

Extension to L_2. The bases for H_{2-} that we introduced can be extended to $L_2(\mathbb{T})$, i.e., to include $(H_{2-})^{\perp}$ (see, e.g., [1]). First observe that given a basis $\{F_k(z)\}$ for H_{2-}, $\{z^{-1}F_k(\frac{1}{z})\}$ is a basis for $(H_{2-})^{\perp}$. In fact, given two bases for H_{2-}, say, $\{F_k(z)\}$ and $\{G_k(z)\}$, the set of functions $\{F_k(z), z^{-1}G_k(\frac{1}{z}), k = 1, 2, \dots\}$ is a basis for $H_{2-} \cup (H_{2-})^{\perp} = L_2(\mathbb{T})$. Using an inner function $G_b(z)$ with balanced realization (A_b, B_b, C_b, D_B), the Hambo functions have been defined as $\{V_1(z)G_b(z)^{k-1}, k \in \mathbb{N}\}$, where $V_1(z) = [zI - A_b]^{-1}B$. Another Hambo basis is created by $\{U_1(z)G_b(z)^{k-1}, k \in \mathbb{N}\}$, where $U_1(z) = [zI - A_b^T]^{-1}C_b^T$. In line with the forgoing, it follows that $\{z^{-1}U_1(\frac{1}{z})G_b^{k-1}(\frac{1}{z}), k \in \mathbb{N}\}$ is a basis for $H_{2-}^{\perp}$. Now an interesting observation is given by the following lemma.

Lemma 3.5. *Let $G_b(z), V_1(z), U_1(z)$ be defined as above. Then $U_1(z)$ and $V_1(z)$ are related by $z^{-1}U_1(\frac{1}{z}) = V_1(z)G_b(\frac{1}{z})$.*

Proof. Using (2.9), it is easy to show that $C_b^T G_b(z) = \left(I - zA_b^T\right)[zI - A_b]^{-1}B_b$. Substituting this relation in the expression $U_1(\frac{1}{z})G_b(z)$ yields $U_1(\frac{1}{z})G_b(z) = z[I - zA_b^T]^{-1}\left(I - zA_b^T\right)[zI - A_b]^{-1}B_b = z[zI - A_b]^{-1}B_b = zV_1(z)$. □

Corollary 3.6. *Let $G_b(z)$ and $V_1(z)$ be defined as above. The set $\{V_1(z)G_b^k(z), k \in J\}$ defines a basis, respectively, for H_{2-} if $J = \mathbb{N}$, for $H_{2-}^{\perp}$ if $J = \mathbb{Z} \setminus \mathbb{N}$, and for $L_2(\mathbb{T})$ if $J = \mathbb{Z}$.*

Analogously the dual Hambo basis of $H_{2-}^{n_b}$ can be complemented with a set of basis functions of $H_{2-}^{n_b\perp}$ such that a basis of $L_2^{n_b}(\mathbb{T})$ is obtained. A dual basis of $H_{2-}^{n_b\perp}$ is given by the functions $W_{-t}(z) = \left(N^T(\frac{1}{z})\right)^t W_0(z)$, $t > 0$, with $W_0(z)$ given by $C_b^T z^{-1}\ (z^{-1}\ I - D_b^T)^{-1}$. The vector $W_0(z)$ can be related to $W_1(z)$ (the first basis element of the dual basis of $H_{2-}^{n_b}$) as follows.

Lemma 3.7. *With $N(z)$ a square inner function with orthogonal realization (D_b, C_b, B_b, A_b) and $W_1(z) = B_b(zI - D_b)^{-1}$, it holds that $W_0(z) = C_b^T z^{-1}\ (z^{-1}\ I - D_b^T)^{-1} = N^T(\frac{1}{z})W_1(z)$.*

Proof. The proof is similar to that of Lemma 3.5. It is straightforward to show that $N(z)W_0(z) = W_1(z)$, using the fact that $N(z)$ is inner. ◻

As a consequence, the inner function $N(z)$ generates a basis of $L_2^{n_b}(\mathbb{T})$ in the same way that G_b generates a basis of $L_2(\mathbb{T})$. We use that $N^T(\frac{1}{z})$ is the inverse of $N(z)$.

PROPOSITION 3.8. *The set of vector functions $\{W_k(z), k \in J\}$, with $W_k \in L_2^{n_b}(\mathbb{T})$, defined as $W_k(z) = N(z)^{k-1}B_b(zI - D_b)^{-1}$, constitutes an orthonormal basis of $H_{2-}^{n_b}$ if $J = \mathbb{N}$, of $H_{2-}^{n_b \perp}$ if $J = \mathbb{Z} \setminus \mathbb{N}$, and of $L_2^{n_b}(\mathbb{T})$ if $J = \mathbb{Z}$.*

4. Signal and operator transforms.

In this section, the fundamentals of the transform theory that underlies expansions in the generalized basis are given. It is an extension of the work that was started in [13, 39] and can be viewed as a generalization of the Laguerre transform theory for signals and systems that was developed in [30] and [29].

4.1. Signals.

In the previous section, it was shown how ℓ_2 signals can be expanded in terms of general rational orthonormal basis functions that are generated by an inner function $G_b(z)$ in balanced state space form.

It will turn out to be expedient to have a definition of the Hambo signal transform that also applies to multivariable signals. Also, we will need a definition that not only applies to the Hambo basis of $\ell_2(\mathbb{N})$ but also to the Hambo bases of $\ell_2(\mathbb{Z} \setminus \mathbb{N})$ and $\ell_2(\mathbb{Z})$, as discussed in section 3. Therefore, the definitions in this section will be given for $\ell_2(J)$ signals, where J is either $\mathbb{N}, \mathbb{Z}$ or $\mathbb{Z} \setminus \mathbb{N}$.

Consider a vector signal $x(t) \in \ell_2^n(J)$ such that $x(t) = \begin{bmatrix} x_1(t) \; x_2(t) \; \cdots \; x_n(t) \end{bmatrix}^T$. Each scalar signal $x_i(t)$ can be expanded in the corresponding Hambo basis, yielding the expansion sequences $\breve{x}_i(k)$ which are elements of $\ell_2^{n_b}(J)$. Hence it holds that

$$x(t) = \sum_{k=1}^{\infty} \begin{bmatrix} \breve{x}_1(k) & \breve{x}_2(k) & \cdots & \breve{x}_n(k) \end{bmatrix}^T v_k(t) = \sum_{k \in J} \breve{x}^T(k) v_k(t). \tag{4.1}$$

DEFINITION 4.1 (multivariable Hambo signal transform). *Given a signal $x(t) \in \ell_2^n(J)$, its Hambo signal transform is defined as the matrix sequence $\{\breve{x}(k)\}_{k \in J}$, with $\breve{x}(k) \in \mathbb{R}^{n_b \times n}$ given by*

$$\breve{x}(k) = [\![v_k, x]\!]. \tag{4.2}$$

Furthermore, we define the λ-domain representation of the Hambo signal transform as

$$\breve{X}(\lambda) = \sum_{k \in J} \breve{x}(k) \lambda^{-k}.$$

Note that $\breve{X}(\lambda)$ is simply the Z-transform of $\breve{x}(k)$ with Z replaced by λ to avoid confusion. As $\breve{X}(\lambda)$ is just a representation of the Hambo signal transform $\breve{x}(k)$ in an alternative domain, it is also commonly called the Hambo signal transform [13].

For purposes of calculation, we will also need a definition for the Hambo transform of a signal $y(t) \in \ell_2^{1 \times n_b}$. This is defined through Definition 4.1 by using $x(t) = y^T(t)$ and defining

$$\breve{Y}(\lambda) := \breve{X}^T(\lambda).$$

With the multivariable signal transform as defined above, the following isomorphic relation holds.

PROPOSITION 4.2 (multivariable Hambo signal transform isomorphism). *With $X(z) \in L_2^{n_x}(\mathbb{T})$ and $Y(z) \in L_2^{n_y}(\mathbb{T})$, it holds that $[\![X, Y]\!] = [\![\breve{X}^T, \breve{Y}^T]\!]$.*

Proof. The (i, j) element of $[\![X, Y]\!]$ is equal to $\langle X_i, Y_j\rangle$. By the isomorphism of the Hambo signal transform for scalar signals, it holds that this is equal to $\langle \breve{X}_i, \breve{Y}_j\rangle$. Then, with $\breve{X}(\lambda)$ and $\breve{Y}(\lambda)$ as defined before, it follows that $[\![X, Y]\!]=\frac{1}{2\pi}\int_0^{2\pi} \breve{X}^T(e^{i\omega})\overline{\breve{Y}(e^{i\omega})}d\omega = [\![\breve{X}(\lambda)^T, \breve{Y}^T(\lambda)]\!]$. ∎

4.2. Systems. A system $G(z) \in L_2^n(\mathbb{T})$ is uniquely described by its impulse response $\{g(k)\} \in \ell_2^n$. We will use this property to define the *Hambo signal transform of a system* as the Hambo signal transform of the impulse response of the system.

DEFINITION 4.3. *Consider a system $G(z)\in L_2^n(\mathbb{T})$ and a Hambo basis $\{V_k(z)\}_{k\in\mathbb{Z}}$. The Hambo signal transform of $G(z)$, denoted as $\breve{G}(\lambda)$, is defined as*

$$\breve{G}(\lambda) = \sum_{k=-\infty}^{\infty} \breve{g}(k)\lambda^{-k}, \quad \textit{where} \quad \breve{g}(k) = [\![V_k, G]\!].$$

Example 4.4. Consider the Hambo signal transform of the basis function vector $G(z) = V_j(z)$. Obviously, in this simple case, the expansion vector coefficients are given by $\breve{g}(k) = \delta(k - j)I$. Hence it holds that the Hambo signal transform of $V_j(z)$ is equal to $\lambda^{-j}I$.

Another transform of the system $G(z)$ that is closely related to the signal transform but essentially different is the so-called *Hambo operator transform*, which describes the relationship between the signal transforms of the input and output signals of a *scalar* stable and causal system.

DEFINITION 4.5 (Hambo operator transform). *Consider a system $G(z) \in H_2$ and a Hambo basis $\{V_k(z)\}_{k\in\mathbb{N}}$, associated with the inner function $G_b(z)$. We define the* Hambo operator transform *of $G(z)$, denoted by $\widetilde{G}(\lambda)$, as*

$$\widetilde{G}(\lambda) = \sum_{\tau=0}^{\infty} M_\tau\lambda^{-\tau}, \tag{4.3}$$

$$\textit{where} \quad M_\tau = [\![V_1(z)G_b^\tau(z), V_1(z)G(z)]\!]. \tag{4.4}$$

PROPOSITION 4.6. *Consider signals $u(t), y(t) \in \ell_2(\mathbb{N})$ and a system $G(z) \in H_2$ such that $y(t) = G(q)u(t)$. With $\widetilde{G}(\lambda)$ the Hambo operator transform of $G(z)$, it holds that $\breve{Y}(\lambda) = \widetilde{G}(\lambda)\breve{U}(\lambda)$.*

Proof. Let $\breve{u}(k), \breve{y}(k)$ be the expansion coefficients of $u(t)$ and $y(t)$. $\breve{y}(k)$ can be expressed as $\breve{y}(k) = [\![V_k, G\sum_{j=1}^{\infty}\breve{u}^T(j)V_j]\!] = \sum_{j=1}^{\infty}[\![V_k, V_jG]\!]\breve{u}(j) = \sum_{j=1}^{\infty}[\![V_1G_b^{k-1}, V_1G_b^{j-1}G]\!]\breve{u}(j)$. Consider the inner product term for the case where $j \leq k$. Use is made of the fact that the adjoint of $G_b(z)$ by its inner property is equal to $G_b^{-1}(z)$. Hence $[\![V_1G_b^{k-1}, V_1G_b^{j-1}G]\!] = [\![V_1G_b^{k-j}, V_1G]\!]$. Now consider the inner product term for the case where $j > k$. Then, with the same argument, one finds that it holds that $[\![V_1G_b^{k-1}, V_1G_b^{j-1}G]\!] = [\![V_1, V_1G_b^{j-k}G]\!]$. This latter expression is equal to zero, which follows from the fact that the elements of the transfer function $V_1(z)$ constitute an orthonormal set which exactly spans the orthogonal complement in H_2 of the shift-invariant subspace $G_b(z)H_2$. The right-hand side argument of the inner product is an element of that subspace. Applying the signal transform of Definition 4.1 to $\breve{y}(k)$

(with $J = \mathbb{N}$) reveals that it holds that

$$\breve{Y}(\lambda) = \left(\sum_{\tau=0}^{\infty} M_\tau \lambda^{-\tau} \right) \breve{U}(\lambda). \qquad \square \tag{4.5}$$

The parameters M_τ are matrices of dimension $n_b \times n_b$. They can be viewed as the Markov parameters of the multivariable transfer function $\widetilde{G}(\lambda)$. The expansion coefficients $\{\breve{g}(k)\}$ and the Markov parameters $\{M_\tau\}$, as given by Definitions 4.3 and 4.5, are closely connected through a linear relation; see [37, 8, 7] for details.

The Hambo operator transform of the system $G_b(z)$ has a particularly simple form. It holds for all $U \in H_{2-}$ that

$$G_b(z)U(z) = \sum_{k=1}^{\infty} \breve{u}^T(k)V_k(z)G_b(z) = \sum_{k=1}^{\infty} \breve{u}^T(k)V_{k+1}(z).$$

Hence, with $Y(z) = G_b(z)U(z) = \sum_{k=1}^{\infty} \breve{y}^T(k)V_k(z)$, it follows that $\breve{y}(k) = \breve{u}(k-1)$ for $k > 1$ and $\breve{y}(1) = 0$. Therefore, it holds that $M_1 = I$ and $M_\tau = 0$ for all $\tau \neq 1$, and consequently

$$\widetilde{G_b}(\lambda) = \lambda^{-1} I. \tag{4.6}$$

We can hence conclude that a multiplication with $G_b(z)$ in the Z-domain corresponds to applying a canonical shift in the λ-domain.

Although the Hambo operator transform is defined only for SISO systems, there is a simple multivariable case in which it can also be used. We will need it in the next section.

PROPOSITION 4.7. *Consider a signal $u(t) \in \ell_2^m(J)$ and an SISO system $G(z) \in H_2$. Let $y(t) \in \ell_2^m(J)$ be given by $y(t) = G(q) \cdot I\ u(t)$. Then it holds that $\breve{Y}(\lambda) = \widetilde{G}(\lambda)\breve{U}(\lambda)$.*

Proof. Denoting the elements of $U(z)$ and $Y(z)$ as $U_i(z)$ and $Y_i(z)$ according to $U(z) = \begin{bmatrix} U_1(z)\ U_2(z)\ \cdots\ U_m(z) \end{bmatrix}^T$ and $Y(z) = \begin{bmatrix} Y_1(z)\ Y_2(z)\ \cdots\ Y_m(z) \end{bmatrix}^T$, we have that $Y_i(z) = G(z)U_i(z)$ for $1 \leq i \leq m$. Then the Hambo signal transform of $Y_i(z)$ satisfies, by definition of the Hambo operator transform, $\breve{Y}_i(\lambda) = \widetilde{G}(\lambda)\breve{U}_i(\lambda)$. The result then follows from the fact that

$$\breve{Y}(\lambda) = \begin{bmatrix} \breve{Y}_1(\lambda) & \breve{Y}_2(\lambda) & \cdots & \breve{Y}_m(\lambda) \end{bmatrix} = \widetilde{G}(\lambda)\breve{U}(\lambda). \qquad \square$$

5. Operator transform expressions. As shown, the Hambo operator transform of a system $G(z) \in H_2$ is a causal LTI system. Furthermore, the transform of a rational transfer function is again rational. We will now derive expressions by which the operator transform can actually be computed. First it is shown that an expression for $\widetilde{G}(\lambda)$ is obtained by making a variable substitution in the Laurent expansion of $G(z)$. Next it is shown how a state space realization of $\widetilde{G}(\lambda)$ can be derived on the basis of a state space realization of $G(z)$.

5.1. Variable substitution property. The Hambo operator transform, as defined in Definition 4.5, can be obtained from the original transfer function $G(z) \in H_2$ by applying a variable substitution in its Laurent expansion, which is given by

$$G(z) = \sum_{\tau=0}^{\infty} g(\tau) z^{-\tau}. \tag{5.1}$$

This variable substitution consists of a replacement of the shift operation z^{-1} by the causal linear time-invariant operator $N(\lambda)$.

PROPOSITION 5.1 (variable substitution property [39]). *Let $N(\lambda)$ be as in Proposition 3.4. Then the Hambo operator transform $\widetilde{G}(\lambda)$ of a given system $G(z) \in H_2$ is equal to*

$$\widetilde{G}(\lambda) = \sum_{\tau=0}^{\infty} g(\tau) N^{\tau}(\lambda). \tag{5.2}$$

With slight abuse of notation, (5.2) is sometimes stated as $\widetilde{G}(\lambda) = G(z)|_{z^{-1}=N(\lambda)}$. An immediate consequence of Proposition 5.1 is that the operator transform of the canonical shift z^{-1} is equal to $N(\lambda)$. This means that a shift in the time domain corresponds to the application of the operator $N(\lambda)$ in the signal transform domain. Another immediate consequence of this proposition is that $N(\lambda)$ and $\widetilde{G}(\lambda)$ are commuting operators. A third consequence of Proposition 5.1 is the following relation between the Hambo signal transform and the Hambo operator transform.

COROLLARY 5.2. *The Hambo signal transform $\breve{G}(\lambda)$ and Hambo operator transform $\widetilde{G}(\lambda)$ of a given system $G(z) \in H_{2-}$ are related through $\breve{G}(\lambda) = \widetilde{G}(\lambda)W_0(\lambda)$, with $W_0(\lambda) \in H_2^{n_b\,\perp}$ equal to $C_b^T \frac{1}{\lambda} (\frac{1}{\lambda} I - D_b^T)^{-1}$, in accordance with Proposition* 3.8.

Proof. As the functions $\{W_t(\lambda)\}_{t\in\mathbb{N}}$ constitute the dual Hambo basis, $\breve{G}(\lambda)$ satisfies $\breve{G}(\lambda) = \sum_{t=1}^{\infty} g(t)W_t(\lambda)$, with $g(t)$ the impulse response coefficients of $G(z)$. By Proposition 3.4 and the fact that $N(\lambda)$ is inner, we can write $\breve{G}(\lambda) = \sum_{t=1}^{\infty} g(t)N(\lambda)^t \cdot N^T(\frac{1}{\lambda})W_1(\lambda)$. By Lemma 3.7 and Proposition 5.1, it then follows that $\breve{G}(\lambda) = \sum_{t=1}^{\infty} g(t)N(\lambda)^t W_0(\lambda) = \widetilde{G}(\lambda)W_0(\lambda)$. ◻

It was shown in [13] that, inversely, $G(z)$ can also be obtained from $\widetilde{G}(\lambda)$ by means of a variable substitution:

$$G(z) = zV_1^T(z)\, \widetilde{G}(\lambda)W_1(\lambda)\lambda\Big|_{\lambda^{-1}=G_b(z)}. \tag{5.3}$$

Using the multivariable signal transform Definition 4.1 one can establish an isomorphic relation that involves the Hambo operator transform.

PROPOSITION 5.3 (Hambo operator transform isomorphism). *Consider the Hambo basis of $L_2(\mathbb{T})$, generated by an inner function $G_b(z)$. Hence we have that $V_k(z) = V_1(z)G_b(z)^{k-1}$ and $W_k(\lambda) = N(\lambda)^{k-1}W_1(\lambda)$. Then for all $G_1(z), G_2(z) \in H_2$, $k \in \mathbb{Z}$,*

$$[\![V_k G_1, V_k G_2]\!] = [\![\widetilde{G}_1^T, \widetilde{G}_2^T]\!], \tag{5.4}$$

$$\textit{and} \qquad \langle G_1, G_2\rangle = \langle \widetilde{G}_1 W_k, \widetilde{G}_2 W_k\rangle. \tag{5.5}$$

Proof. We will prove both assertions for the case $k = 1$. The other cases follow immediately from the inner property of $G_b(z)$, and $N(\lambda)$. By Proposition 4.2, it holds that $[\![V_1G_1, V_1G_2]\!] = [\![(V_1\breve{G}_1)^T, (V_1\breve{G}_2)^T]\!]$. The elements of the vector $V_1(z)G_k(z)$, $k = 1, 2$, are equal to $G_k(z)\Phi_{1,i}(z)$, $1 \le i \le n_b$, where $\Phi_{1,i}(z)$ are the first n_b scalar basis functions. The Hambo signal transform of $G_k(z)\Phi_{1,i}(z)$ is, by definition of the operator transform, equal to $\widetilde{G}_k(\lambda)\breve{\Phi}_{1,i}(\lambda) = \widetilde{G}_k(\lambda)e_i^T\lambda^{-1}$. By Definition 4.1, it then follows that $(V_1\breve{G}_k) = \widetilde{G}_k(\lambda)\lambda^{-1}$. Hence $[\![V_1G_1, V_1G_2]\!] = [\![\widetilde{G}_1^T(\lambda)\lambda^{-1}, \widetilde{G}_2^T(\lambda)\lambda^{-1}]\!] = [\![\widetilde{G}_1^T, \widetilde{G}_2^T]\!]$. The second assertion is proved as follows. It holds that $\langle G_1, G_2\rangle = \langle G_1z^{-1}, G_2z^{-1}\rangle = \langle (G_1\breve{z}^{-1}), (G_2\breve{z}^{-1})\rangle$. The last equality follows from the isomorphism of the signal transform. Using the fact that $W_1(\lambda)$ is the Hambo signal transform of z^{-1} and by definition of the Hambo operator transform, the result follows. ◻

5.2. Hankel operator representations. The Hankel operator associated with an LTI system $G(z)$ can be represented in a number of ways, depending on the (orthonormal) coordinate systems that are used for the input and output signal spaces. The Hankel operator of a scalar system maps from $\ell_2(-\infty, 0]$ to $\ell_2[1, \infty)$. Usually, the canonical bases of these spaces are employed to represent the input and output signals. In that case, the Hankel operator can be represented as a Hankel matrix $\mathbf{H}$ that contains the Markov parameters $g(k), k > 0$, of $G(z)$, as $\mathbf{H}_{i,j} = g(i+j-1)$. Now define $\mathbf{y} = \begin{bmatrix} y(1) & y(2) & \cdots \end{bmatrix}^T$, $\mathbf{u} = \begin{bmatrix} u(0) & u(-1) & \cdots \end{bmatrix}^T$. Then it holds that

$$\mathbf{y} = \mathbf{H}\mathbf{u}. \tag{5.6}$$

Alternative representations of the Hankel operator would be obtained if one were to use other orthonormal bases for the representation of the input and output signals. A particularly interesting case occurs when we use a Hambo basis for the output space $\ell_2[1, \infty)$ and the complementary Hambo basis for $\ell_2(-\infty, 0]$ for the input space. Consider the expansion of the output signal $y(t) \in \ell_2[1, \infty)$ and the input signal $u(t) \in \ell_2(-\infty, 0]$ in terms of a Hambo basis. We then obtain the coefficients $\breve{y}(k) = [\![y, v_k]\!]^T$ with $k \in \mathbb{N}$ and $\breve{u}(k) = [\![u, v_k]\!]^T$ with $k \in \mathbb{Z} \backslash \mathbb{N}$. We collect these coefficients in column vectors $\breve{\mathbf{y}}, \breve{\mathbf{u}}$ defined as

$$\breve{\mathbf{y}}^T = \begin{bmatrix} \breve{y}^T(1) & \breve{y}^T(2) & \breve{y}^T(3) & \cdots \end{bmatrix}, \tag{5.7}$$

$$\breve{\mathbf{u}}^T = \begin{bmatrix} \breve{u}^T(0) & \breve{u}^T(-1) & \breve{u}^T(-2) & \cdots \end{bmatrix}. \tag{5.8}$$

Defining the block row vectors $\mathbf{v}_k$ with $k \in \mathbb{Z}$ as

$$\mathbf{v}_k = \begin{cases} \begin{bmatrix} v_k(1) & v_k(2) & v_k(3) & \cdots \end{bmatrix}, & k \geq 1, \\ \begin{bmatrix} v_k(0) & v_k(-1) & v_k(-2) & \cdots \end{bmatrix}, & k < 1, \end{cases}$$

and defining $V_f = \begin{bmatrix} \mathbf{v}_1^T & \mathbf{v}_2^T & \cdots \end{bmatrix}^T$ and $V_p = \begin{bmatrix} \mathbf{v}_0^T & \mathbf{v}_{-1}^T & \cdots \end{bmatrix}^T$, we can write

$$\breve{\mathbf{y}} = \mathbf{V}_f \mathbf{y} \quad \text{and} \quad \breve{\mathbf{u}} = \mathbf{V}_p \mathbf{u}. \tag{5.9}$$

It is clear that the infinite dimensional matrices $\mathbf{V}_f$ and $\mathbf{V}_p$ are unitary (orthogonal) matrices as their rows are orthogonal vectors. It hence follows that we can also write $\mathbf{y} = \mathbf{V}_f^T \breve{\mathbf{y}}$ and $\mathbf{u} = \mathbf{V}_p^T \breve{\mathbf{u}}$. Substituting this in (5.6) gives the relation $\mathbf{V}_f^T \breve{\mathbf{y}} = \mathbf{H} \mathbf{V}_p^T \breve{\mathbf{u}}$. Again using the fact that $\mathbf{V}_f$ is orthogonal, this can be rephrased as $\breve{\mathbf{y}} = \widetilde{\mathbf{H}} \breve{\mathbf{u}}$, with $\widetilde{\mathbf{H}} = \mathbf{V}_f \mathbf{H} \, \mathbf{V}_p^T$. The matrix operator $\widetilde{\mathbf{H}}$ is an alternative representation of the Hankel operator of $G(z)$. If we partition the matrix $\widetilde{\mathbf{H}}$ in blocks of dimension $n_b \times n_b$ corresponding to the partitioning of $\breve{\mathbf{u}}$ and $\breve{\mathbf{y}}$, then we find that the (i, j) block element equals $\widetilde{\mathbf{H}}_{(i,j)} = \mathbf{v}_i \mathbf{H} \, \mathbf{v}_{-j+1}^T$, with $\mathbf{v}_k$ the vector representations of the basis functions v_k as defined above. It is then clear that $\widetilde{\mathbf{H}}_{(i,j)}$ is equal to the matrix inner product between $V_i(z)$ and the Z-transform expression for the vector $\mathbf{H}\mathbf{v}_{-j+1}^T$. This leads to the following proposition.

PROPOSITION 5.4. *Let $\widetilde{\mathbf{H}}$ be the matrix representation of the Hankel operator of a system $G(z) \in H_2$, in terms of a Hambo basis associated with an inner function $G_b(z)$, such that $\breve{\mathbf{y}} = \widetilde{\mathbf{H}} \breve{\mathbf{u}}$, where $\breve{\mathbf{y}}$ and $\breve{\mathbf{u}}$ are as defined by* (5.7) *and* (5.8). *Let $\widetilde{\mathbf{H}}$ be partitioned in blocks of dimension $n_b \times n_b$, and let $\widetilde{\mathbf{H}}_{(i,j)}$ denote the (i, j)th block* *element. Then it holds that* $\widetilde{\mathbf{H}}_{(i,j)} = M_{i+j-1}$*, where* M_k *represents the* k*th Markov parameter of* $\widetilde{G}(\lambda)$*, as defined in* (4.4).

Proof. By definition of the Hankel map, the term $\mathbf{H}\mathbf{v}_{-j+1}^T$ is the output of the system $G(z)$ in response to the input $v_{-j+1} \in \ell_2^{n_b}(-\infty, 0]$, restricted to the space of future signals $\ell_2^{n_b}[1,\infty)$. In Z-transform notation, this output can be expressed as $\mathbf{P}_{H_{2-}^{n_b}} G(z) V_{-j+1}^T(z) = \mathbf{P}_{H_{2-}^{n_b}} G(z) G_b^{-j} V_1^T(z)$. The last equality follows from the fact that $V_k(z) = G_b^{k-1}(z) V_1(z)$ for all $k \in \mathbb{Z}$. It then follows that $\widetilde{\mathbf{H}}_{(i,j)} = [\![V_i, \mathbf{P}_{H_{2-}^{n_b}} G G_b{}^{-j} V_1]\!] = [\![G_b^{i-1} V_1, G G_b^{-j} V_1]\!]$. Because $G_b(z)$ is inner, this expression simplifies to $\widetilde{\mathbf{H}}_{(i,j)} = [\![G_b^{i+j-1} V_1, G V_1]\!]$, which is equal to M_{i+j-1}, as was established earlier; see (4.4). ∎

Proposition 5.4 shows that $\widetilde{\mathbf{H}}$ has a block Hankel form, which coincides with the standard block Hankel matrix representation of the Hambo operator transform $\widetilde{G}(\lambda)$. One consequence of this observation is that Hankel singular values and the McMillan degree are invariant under Hambo operator transformation.

5.3. State space expressions for the Hambo operator transform and its inverse. In this section, we will derive the expressions by which a minimal realization of the Hambo operator transform can be obtained from a minimal state space realization of the original system and vice versa. The derivation is based on the isomorphic relation that exists between such state space realizations. We will first establish this relation. Consider the (block) Hankel matrix representation $\mathbf{H}$ of the Hankel operator of an LTI system $G(z)$. It is a well-known result from realization theory that *any* full rank decomposition $\mathbf{H} = \mathbf{\Gamma}\mathbf{\Delta}$ corresponds to a minimal realization of $G(z)$ [15, 17]. That is, there exists a minimal realization (A, B, C, D) of $G(z)$ such that $\mathbf{\Gamma} = \begin{bmatrix} C^T & (CA)^T & (CA^2)^T & \cdots \end{bmatrix}^T$ and $\mathbf{\Delta} = \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix}$. We define the transfer functions $\Gamma(z) \in H_{2-}^n$ and $\Delta(z) \in H_{2-}^n{}^{\perp}$ as

$$\Gamma(z) = \sum_{k=1}^{\infty} CA^{k-1} z^{-k} = C(zI - A)^{-1}, \qquad \Delta(z) = \sum_{k=0}^{\infty} A^k B z^k = z^{-1}\,(z^{-1}\,I - A)^{-1} B.$$

The following lemma establishes an important relation between these functions and their counterparts in the transform domain.

LEMMA 5.5. *Consider a system* $G(z) \in RH_2$ *with minimal realization* (A, B, C, D)*. Let* $\Gamma(z)$ *and* $\Delta(z)$ *be defined as* $\Gamma(z) = C(zI - A)^{-1}$*,* $\Delta(z) = z^{-1}\,(z^{-1}\,I - A)^{-1} B$*. Then the Hambo operator transform* $\widetilde{G}(\lambda)$ *of* $G(z)$ *has a minimal state space realization* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ *such that it holds that*

$$\tilde{C}(\lambda I - \tilde{A})^{-1} = \breve{\Gamma}^T(\lambda) \;\; and \;\; \lambda^{-1}\,(\lambda^{-1}\,I - \tilde{A})^{-1} \tilde{B} = \breve{\Delta}^T(\lambda), \tag{5.10}$$

where $\breve{\Gamma}(\lambda)$ *and* $\breve{\Delta}(\lambda)$ *are the (multivariable) Hambo signal transforms of* $\Gamma(z)$*, respectively* $\Delta(z)$*, as defined in Definition* 4.3.

Conversely, any Hambo operator transform $\widetilde{G}(\lambda)$ *with minimal state space realization* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ *has a preimage* $G(z)$ *with minimal realization* (A, B, C, D) *such that* (5.10) *holds.*

Proof. From the analysis in the previous section, it follows that, given a full rank factorization $\mathbf{H} = \mathbf{\Gamma}\mathbf{\Delta}$, a full rank factorization of $\widetilde{\mathbf{H}}$ can be obtained according to $\widetilde{\mathbf{H}} = (\mathbf{V}_f \mathbf{\Gamma})(\mathbf{\Delta} \mathbf{V}_p^T)$. Denote the minimal state space realization of $\widetilde{G}(\lambda)$ that corresponds to this realization by $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$. We then denote $(\mathbf{V}_f\mathbf{\Gamma})_{(k)} = \tilde{C}\tilde{A}^{k-1}$, $k \geq 1$, and $(\mathbf{\Delta V}_p^T)_{(-k)} = \tilde{A}^k\tilde{B}$, $k \geq 0$. It holds that $(\mathbf{V}_f\mathbf{\Gamma})_{(k)} = \mathbf{v}_k\mathbf{\Gamma}$, $k \geq 1$, and $(\mathbf{\Delta V}_p^T)_{(k)} = \mathbf{\Delta v}_k^T$, $k < 1$. With $\Gamma(z)$ and $\Delta(z)$ as defined above, we then see that

$$(\mathbf{V}_f\mathbf{\Gamma})_{(k)} = [\![\Gamma^T(z), V_k(z)]\!], \quad k \geq 1, \quad \text{and} \quad (\mathbf{\Delta V}_p^T)_{(k)} = [\![\Delta(z), V_k(z)]\!], \quad k < 1,$$

where the last equation holds under the assumption that the realization of $\Delta(z)$ is real. This shows, using (4.2), that $\{(\mathbf{V}_f\mathbf{\Gamma})_{(k)}\}$ and $\{(\mathbf{\Delta V}_p^T)_{(k)}\}$ constitute the multivariable Hambo signal transforms of $\Gamma^T(z)$ and $\Delta(z)$, respectively. Since any minimal realization of $G(z)$ corresponds to a full rank factorization of $\mathbf{H}$, the first part of the lemma is proven. The last statement of the lemma follows from the fact that the Hambo signal transform is a bijective map. ∎

Lemma 5.5 is a very powerful result as it permits us to derive very compact expressions for computing the Hambo operator transform and its inverse, using the isomorphism relation for the multivariable Hambo signal transform given in Proposition 4.2.

Suppose that the realizations (A, B, C, D) and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are linked to each other via the Hambo signal transform as described in Lemma 5.5. Let us denote the controllability Gramians associated with these realizations as X_c and $\tilde{X}_c$ and the observability Gramians as X_o and $\tilde{X}_o$, respectively. Then, by the Hambo signal transform isomorphism, it holds for the functions $\Gamma(z)$ and $\Delta(z)$ that

$$X_o = [\![\Gamma^T(z), \Gamma^T(z)]\!] = [\![\breve{\Gamma}^T(\lambda), \breve{\Gamma}^T(\lambda)]\!] = \tilde{X}_o, \tag{5.11}$$

$$X_c = [\![\Delta(z), \Delta(z)]\!] = [\![\breve{\Delta}(\lambda), \breve{\Delta}(\lambda)]\!] = \tilde{X}_c. \tag{5.12}$$

Using the Hambo signal transform isomorphism, we can now establish a matrix inner product expression for the realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, as follows.

PROPOSITION 5.6. *With $\Gamma(z)$ and $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ as defined in Lemma 5.5 and X_o the controllability Gramian of this realization, it holds that*

$$\begin{bmatrix} X_o & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \left[\!\!\left[\begin{bmatrix} \Gamma^T(z)G_b(z) \\ V_1(z) \end{bmatrix}, \begin{bmatrix} \Gamma^T(z) \\ V_1(z)G(z) \end{bmatrix} \right]\!\!\right]. \tag{5.13}$$

Proof. The system $\widetilde{G}^T(\lambda)$ is described by the equation

$$\begin{bmatrix} X(\lambda)\lambda \\ Y(\lambda) \end{bmatrix} = \begin{bmatrix} \tilde{A}^T & \tilde{C}^T \\ \tilde{B}^T & \tilde{D}^T \end{bmatrix} \begin{bmatrix} X(\lambda) \\ U(\lambda) \end{bmatrix}.$$

It holds that

$$\left[\!\!\left[\begin{bmatrix} X(\lambda) \\ U(\lambda) \end{bmatrix}, \begin{bmatrix} X(\lambda)\lambda \\ Y(\lambda) \end{bmatrix} \right]\!\!\right] = \left[\!\!\left[\begin{bmatrix} X(\lambda) \\ U(\lambda) \end{bmatrix}, \begin{bmatrix} X(\lambda) \\ U(\lambda) \end{bmatrix} \right]\!\!\right] \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}.$$

Let the input $u(t)$ be equal to $e_i\delta(t)$, with e_i the ith Euclidean basis vector of $\mathbb{R}^{n_b}$. Then $X(\lambda) = [\lambda I - \tilde{A}^T]^{-1}\tilde{C}^T e_i = \breve{\Gamma}(\lambda)e_i$, and by Lemma 5.5 this last equation can be written as

$$\left[\!\!\left[\begin{bmatrix} \breve{\Gamma}(\lambda)e_i \\ e_i \end{bmatrix}, \begin{bmatrix} \breve{\Gamma}(\lambda)e_i\lambda \\ \widetilde{G}^T(\lambda)e_i \end{bmatrix} \right]\!\!\right] = \left[\!\!\left[\begin{bmatrix} \breve{\Gamma}(\lambda)e_i \\ e_i \end{bmatrix}, \begin{bmatrix} \breve{\Gamma}(\lambda)e_i \\ e_i \end{bmatrix} \right]\!\!\right] \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}.$$

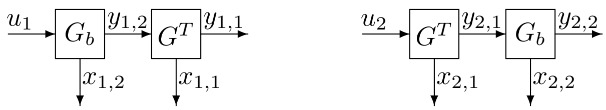

FIG. 5.1. *Systems for proof of Corollary* 5.7.

Because this holds for all i such that $1 \le i \le n_b$, we can also write (after summation of the latter equation over all $i = 1, \ldots, n_b$)

$$\left[\!\left[\begin{bmatrix} \breve{\Gamma}(\lambda) \\ I \end{bmatrix}, \begin{bmatrix} \breve{\Gamma}(\lambda)\lambda \\ \widetilde{G}^T(\lambda) \end{bmatrix} \right]\!\right] = \left[\!\left[\begin{bmatrix} \breve{\Gamma}(\lambda) \\ I \end{bmatrix}, \begin{bmatrix} \breve{\Gamma}(\lambda) \\ I \end{bmatrix} \right]\!\right] \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} = \begin{bmatrix} \tilde{X}_o & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}.$$

The term on the left-hand side of this equation equals

$$\left[\!\left[\begin{bmatrix} \breve{\Gamma}(\lambda)\lambda^{-1} \\ I\lambda^{-1} \end{bmatrix}, \begin{bmatrix} \breve{\Gamma}(\lambda) \\ \widetilde{G}^T(\lambda)\lambda^{-1} \end{bmatrix} \right]\!\right].$$

We observe that $\lambda^{-1}I$ is equal to the Hambo operator transform of $G_b(z)$ (see (4.6)). Further, $\lambda^{-1}I$ is the Hambo signal transform of $V_1(z)$, as was demonstrated in Example 4.4. From Proposition 4.7 it then follows that $\lambda^{-1}I\, \breve{\Gamma}^T(\lambda)$ is the Hambo signal transform of $G_b(z) \cdot I\Gamma^T(z)$. Similarly, $\lambda^{-1}I\widetilde{G}(\lambda)$ is the signal transform of $V_1(z)G(z)$. Using the Hambo signal transform isomorphism (Proposition 4.2), it therefore holds that

$$\left[\!\left[\begin{bmatrix} \breve{\Gamma}(\lambda)\lambda^{-1} \\ I\lambda^{-1} \end{bmatrix}, \begin{bmatrix} \breve{\Gamma}(\lambda) \\ \widetilde{G}^T(\lambda)\lambda^{-1} \end{bmatrix} \right]\!\right] = \left[\!\left[\begin{bmatrix} \Gamma^T(z)G_b(z) \\ V_1(z) \end{bmatrix}, \begin{bmatrix} \Gamma^T(z) \\ V_1(z)G(z) \end{bmatrix} \right]\!\right]. \qquad \square$$

Obviously, a dual formulation of this proposition that uses expressions involving $\Delta(z)$ and X_c is possible.

Proposition 5.6 can also be formulated in the form of a Sylvester equation.

COROLLARY 5.7. *Consider a system $G(z) \in RH_2$, with minimal realization (A, B, C, D) and observability Gramian X_o. Then $\widetilde{G}(\lambda)$ has a minimal realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ that satisfies the following Sylvester equation:*

$$\begin{aligned} &\begin{bmatrix} A^T & C^T C_b \\ 0 & A_b \end{bmatrix} \begin{bmatrix} X_o\tilde{A} & X_o\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} A & BB_b^T \\ 0 & A_b^T \end{bmatrix} + \begin{bmatrix} C^T D_b \\ B_b \end{bmatrix} \begin{bmatrix} C & DB_b^T \end{bmatrix} \\ &\qquad = \begin{bmatrix} X_o\tilde{A} & X_o\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}. \end{aligned} \tag{5.14}$$

Proof. The Sylvester equation is obtained by formulating (5.13) in the time domain using straightforward state space realizations of the transfer functions that appear in the inner product. Consider the systems shown in Figure 5.1. State equations of these systems are

$$\begin{bmatrix} x_{1,1}(t+1) \\ x_{1,2}(t+1) \end{bmatrix} = \begin{bmatrix} A^T & C^T C_b \\ 0 & A_b \end{bmatrix} \begin{bmatrix} x_{1,1}(t) \\ x_{1,2}(t) \end{bmatrix} + \begin{bmatrix} C^T D_b \\ B_b \end{bmatrix} u(t)$$

and

$$\begin{bmatrix} x_{2,1}(t+1) \\ x_{2,2}(t+1) \end{bmatrix} = \begin{bmatrix} A^T & 0 \\ B_b B^T & A_b \end{bmatrix} \begin{bmatrix} x_{2,1}(t) \\ x_{2,2}(t) \end{bmatrix} + \begin{bmatrix} C^T \\ B_b D^T \end{bmatrix} u(t),$$

respectively. The solution of (5.13) is then equal to

$$\left[\!\left[\begin{bmatrix} x_{1,1} \\ x_{1,2} \end{bmatrix}, \begin{bmatrix} x_{2,1} \\ x_{2,2} \end{bmatrix}\right]\!\right],$$

which results in (5.14). □

Existence of a solution to this Sylvester equation (5.14) is guaranteed if the systems in the inner product expression in (5.13) are stable. This is true by assumption for $\Gamma(z)$ and $G(z)$ and by definition for $G_b(z)$ and $V_1(z)$.

Note that (5.14) can be simplified further in the case where $X_o = I$, i.e., when the realization (A, B, C, D) is output balanced.

Using the Hambo signal transform isomorphism, it is equally simple to derive a matrix inner product expression for the realization (A, B, C, D) that involves $\breve{\Gamma}(\lambda)$.

PROPOSITION 5.8. *With* $\Gamma(z)$ *and* (A, B, C, D) *as defined in Lemma* 5.5*, it holds that*

$$\begin{bmatrix} \tilde{X}_o & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\!\left[\begin{bmatrix} \breve{\Gamma}^T(\lambda)N^T(\lambda) \\ W_1^T(\lambda) \end{bmatrix}, \begin{bmatrix} \breve{\Gamma}^T(\lambda) \\ W_1^T(\lambda)\widetilde{G}^T(\lambda) \end{bmatrix}\right]\!\right]. \tag{5.15}$$

Proof. The system $G^T(z)$ is described by the state equation

$$\begin{bmatrix} X(z)z \\ Y(z) \end{bmatrix} = \begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} X(z) \\ U(z) \end{bmatrix}.$$

It holds that

$$\left[\!\left[\begin{bmatrix} X(z) \\ U(z) \end{bmatrix}, \begin{bmatrix} X(z)z \\ Y(z) \end{bmatrix}\right]\!\right] = \left[\!\left[\begin{bmatrix} X(z) \\ U(z) \end{bmatrix}, \begin{bmatrix} X(z) \\ U(z) \end{bmatrix}\right]\!\right] \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Let the input $u(t)$ be equal to $\delta(t)$. Then this last equation can be written as

$$\left[\!\left[\begin{bmatrix} \Gamma^T(z) \\ 1 \end{bmatrix}, \begin{bmatrix} \Gamma^T(z)z \\ G^T(z) \end{bmatrix}\right]\!\right] = \left[\!\left[\begin{bmatrix} \Gamma(z) \\ 1 \end{bmatrix}, \begin{bmatrix} \Gamma(z) \\ 1 \end{bmatrix}\right]\!\right] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} X_o & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

The term on the left-hand side of this equation equals

$$\left[\!\left[\begin{bmatrix} \Gamma^T(z)z^{-1} \\ z^{-1} \end{bmatrix}, \begin{bmatrix} \Gamma^T(z) \\ G^T(z)z^{-1} \end{bmatrix}\right]\!\right].$$

We observe that z^{-1} is equal to the inverse Hambo operator transform of $N(\lambda)$ (as follows from Proposition 5.1). At the same time, z^{-1} is the inverse Hambo signal transform of $W_1(\lambda)$. Then it follows from Proposition 4.7 that $z^{-1}I\ \Gamma(z)$ is the Hambo inverse signal transform of $N(\lambda)\breve{\Gamma}(\lambda)$. Similarly, by definition of the Hambo operator transform, $G(z)z^{-1}$ is then the inverse signal transform of $\widetilde{G}(\lambda)W_1(\lambda)$. Using the Hambo signal transform isomorphism (Proposition 4.2), it therefore holds that

$$\left[\!\left[\begin{bmatrix} \Gamma^T(z)z^{-1} \\ z^{-1} \end{bmatrix}, \begin{bmatrix} \Gamma^T(z) \\ G^T(z)z^{-1} \end{bmatrix}\right]\!\right] = \left[\!\left[\begin{bmatrix} \breve{\Gamma}^T(\lambda)N^T(\lambda) \\ W_1^T(\lambda) \end{bmatrix}, \begin{bmatrix} \breve{\Gamma}^T(\lambda) \\ W_1^T(\lambda)\widetilde{G}^T(\lambda) \end{bmatrix}\right]\!\right]. \qquad \square$$

Again a dual formulation of this proposition is possible that uses expressions involving $\Delta(z)$ and X_c. Expression (5.15) can also be put in Sylvester equation form.

COROLLARY 5.9. *Consider a Hambo transform* $\widetilde{G}(\lambda)$ *of a system* $G(z) \in RH_2$*, with minimal state space realization* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ *and observability Gramian* $\tilde{X}_o$*. Then*

FIG. 5.2. *Systems for proof of Corollary* 5.9.

$G(z)$ has a minimal state space realization (A, B, C, D) that satisfies the following Sylvester equation:

$$\begin{bmatrix} \tilde{A}^T & \tilde{C}^T C_b^T \\ 0 & D_b^T \end{bmatrix} \begin{bmatrix} \tilde{X}_o A & \tilde{X}_o B \\ C & D \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{B} B_b \\ 0 & D_b \end{bmatrix} + \begin{bmatrix} \tilde{C}^T A_b^T \\ B_b^T \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} B_b \end{bmatrix} = \begin{bmatrix} \tilde{X}_o A & \tilde{X}_o B \\ C & D \end{bmatrix}. \tag{5.16}$$

Proof. The proof is similar to that of Corollary 5.7. Consider the systems shown in Figure 5.2. State equations of these systems are

$$\begin{bmatrix} x_{3,1}(t+1) \\ x_{3,2}(t+1) \end{bmatrix} = \begin{bmatrix} \tilde{A}^T & \tilde{C}^T C_b^T \\ 0 & D_b^T \end{bmatrix} \begin{bmatrix} x_{3,1}(t) \\ x_{3,2}(t) \end{bmatrix} + \begin{bmatrix} \tilde{C}^T A_b^T \\ B_b^T \end{bmatrix} u(t)$$

and

$$\begin{bmatrix} x_{4,1}(t+1) \\ x_{4,2}(t+1) \end{bmatrix} = \begin{bmatrix} \tilde{A}^T & 0 \\ B_b^T \tilde{B}^T & D_b^T \end{bmatrix} \begin{bmatrix} x_{4,1}(t) \\ x_{4,2}(t) \end{bmatrix} + \begin{bmatrix} \tilde{C}^T \\ B_b^T \tilde{D}^T \end{bmatrix} u(t),$$

respectively. The solution of (5.15) is then equal to

$$\left[\!\left[\begin{bmatrix} x_{3,1} \\ x_{3,2} \end{bmatrix}, \begin{bmatrix} x_{4,1} \\ x_{4,2} \end{bmatrix} \right]\!\right],$$

which results in (5.16). ∎

Existence of a solution to this Sylvester equation (5.16) is guaranteed if the systems in the inner product expression in (5.15) are stable. That this is true for $\breve{\Gamma}(\lambda)$ follows from the assumption that $G(z)$ is stable and the fact that $\Gamma^T(z)$ is stable. Consequently $\widetilde{G}(\lambda)$ is also stable. $W_1(\lambda)$ and $N^T(\lambda)$ are stable by definition.

Equation (5.16) can again be simplified further when $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is output balanced.

Note that formulas (5.14) and (5.16) look very similar. Also note that the formulas are reciprocal: using a realization (A, B, C, D) in (5.14) results in a realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$, which, when used in (5.16), yields the original (A, B, C, D) again. This follows from the fact that the functions $\Gamma^T(z)$ and $\breve{\Gamma}(\lambda)$ correspond uniquely through the Hambo signal transform.

As stated, similar results as those given by Corollaries 5.7 and 5.9 can be given using a controllability approach. We state the results here without proof. Details can be found in [7].

COROLLARY 5.10 (Hambo system transform—controllability form [7]). *Consider a system $G(z) \in RH_2$ with minimal state space realization (A, B, C, D) and controllability Gramian X_c. Then its Hambo transform $\widetilde{G}(\lambda)$ has a minimal state* *space realization* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ *with controllability Gramian* $\tilde{X}_c = X_c$ *that satisfies the following Sylvester equation:*

$$\begin{bmatrix} A & 0 \\ C_b^T & A_b^T \end{bmatrix} \begin{bmatrix} \tilde{A}X_c & \tilde{B} \\ \tilde{C}X_c & \tilde{D} \end{bmatrix} \begin{bmatrix} A^T & 0 \\ B_b B^T & A_b \end{bmatrix} + \begin{bmatrix} B \\ C_b^T D \end{bmatrix} \begin{bmatrix} D_b B^T & C_b \end{bmatrix} = \begin{bmatrix} \tilde{A}X_c & \tilde{B} \\ \tilde{C}X_c & \tilde{D} \end{bmatrix}. \tag{5.17}$$

COROLLARY 5.11 (inverse Hambo system transform—controllability form [7]). *Consider a Hambo transform* $\widetilde{G}$ *of a system* $G(z) \in RH_2$ *with minimal state space realization* $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ *and controllability Gramian* $\tilde{X}_c$. *Then* $G(z)$ *has a minimal state space realization* (A, B, C, D) *with controllability Gramian* $X_c = \tilde{X}_c$ *that satisfies the following Sylvester equation:*

$$\begin{aligned} &\begin{bmatrix} \tilde{A} & 0 \\ C_b\tilde{C} & D_b \end{bmatrix} \begin{bmatrix} A\tilde{X}_c & B \\ C\tilde{X}_c & D \end{bmatrix} \begin{bmatrix} \tilde{A}^T & 0 \\ B_b^T \tilde{B}^T & D_b^T \end{bmatrix} + \begin{bmatrix} \tilde{B} \\ C_b\tilde{D} \end{bmatrix} \begin{bmatrix} A_b^T \tilde{B}^T & C_b^T \end{bmatrix} \\ &\quad = \begin{bmatrix} A\tilde{X}_c & B \\ C\tilde{X}_c & D \end{bmatrix}. \end{aligned} \tag{5.18}$$

There are various formulas that can be derived in this context. For instance, it is straightforward to derive a generic formula for $\tilde{D}$ that is a direct result of substituting $\lambda = \infty$ in (5.2):

$$\tilde{D} = \sum_{k=0}^{\infty} g(k) A_b^k. \tag{5.19}$$

An equivalent relation (see [7]) can be derived for $\tilde{A}$ as defined by (5.14), (5.17) when we define $g_b(k)$ as the impulse response sequence of $G_b(z)$:

$$\tilde{A} = \sum_{k=0}^{\infty} g_b(k) A^k. \tag{5.20}$$

This expression can be verified as follows. Define $F = \sum_{k=0}^{\infty} g_b(k) A^k$, and consider the expression $A^T X_o F A$, where X_o is the observability Gramian of the realization (A, B, C, D). It follows that

$$\begin{aligned} A^T X_o F A &= \sum_{k=0}^{\infty} g_b(k)(A^T X_o A) A^k = \sum_{k=0}^{\infty} g_b(k)(X_o - C^T C) A^k \\ &= X_o F - C^T C \cdot D_b - \sum_{k=1}^{\infty} C^T C_b A_b^{k-1} B_b C A^k \\ &= X_o F - C^T D_b C - C^T C Y A, \quad \text{where} \quad A_b Y A + B_b C = Y. \end{aligned}$$

Evaluation of the terms in (5.14) yields that it must hold that $Y = \tilde{C}$ and $F = \tilde{A}$, as defined by (5.14). Analogously, evaluation of $A F X_c A^T$ shows that $F = \tilde{A}$, as defined by (5.17).

6. Properties of Hambo transforms. We proceed with demonstrating a number of interesting properties of Hambo transforms that ensue from the theory developed in the preceding sections. These properties are of interest because they are instrumental to the application of the basis function theory in the context of system modelling [39, 8, 7].

6.1. Calculation rules. The Hambo operator transform obeys the following rules:

$$\text{(6.1)} \quad \text{if} \quad H(z) = (\alpha G_1(z) + \beta G_2(z)), \quad \text{then} \quad \widetilde{H}(\lambda) = \alpha \widetilde{G_1}(\lambda) + \beta \widetilde{G_2}(\lambda),$$

$$\text{(6.2)} \quad \widetilde{G_1 G_2}(\lambda) = \widetilde{G_1}(\lambda)\widetilde{G_2}(\lambda) = \widetilde{G_2}(\lambda)\widetilde{G_1}(\lambda),$$

$$\text{(6.3)} \quad \widetilde{G^{-1}}(\lambda) = \left(\widetilde{G}(\lambda)\right)^{-1},$$

where $G_1(z), G_2(z), G(z), G^{-1}(z) \in H_2$, and $\alpha, \beta \in \mathbb{R}$.

Proof. (6.1): The proof follows trivially from the definition of the Hambo operator transform and the linearity of the Hambo signal transform.

(6.2): Let $Y(z) = G_1(z)G_2(z)U(z)$. Define $X(z) = G_2(z)U(z)$. By definition of the operator transform, it holds that $\breve{Y}(\lambda) = \widetilde{G_1}(\lambda)\breve{X}(\lambda) = \widetilde{G_1}(\lambda)\widetilde{G_2}(\lambda)\breve{U}(\lambda)$. Since this holds for all $U(z), Y(z)$, (6.2) follows. The second equality follows from the fact that the scalar systems $G_1(z)$ and $G_2(z)$ commute.

(6.3): Assuming that $G^{-1}(z) \in H_2$, we have by definition of the Hambo transform that $\breve{U}(\lambda) = \widetilde{G^{-1}}(\lambda)\breve{Y}(\lambda)$. We also know that $\breve{Y}(\lambda) = \widetilde{G}(\lambda)\breve{U}(\lambda)$. Hence $(\widetilde{G}(\lambda))^{-1} = \widetilde{G^{-1}}(\lambda)$. ∎

On the basis of these properties, it holds, for instance, that if $H(z) = (G(z)(1 + G(z))^{-1})$, then $\widetilde{H}(\lambda) = \widetilde{G}(\lambda)(I + \widetilde{G}(\lambda))^{-1} = (I + \widetilde{G}(\lambda))^{-1}\widetilde{G}(\lambda)$, assuming that $(1 + G(z))^{-1} \in H_2$.

These properties thus imply that parallel and series interconnections of systems remain unchanged under Hambo operator transformation. Feedback interconnections also remain unchanged under the condition that the inverse taken is also in H_2. It follows immediately that the same goes for linear fractional transformations (LFT), where we assume a pointwise definition of the operator transform for multivariable systems, i.e.,

$$\widetilde{\begin{bmatrix} G_{11}(z) & G_{12}(z) \\ G_{21}(z) & G_{22}(z) \end{bmatrix}} = \begin{bmatrix} \widetilde{G}_{11}(z) & \widetilde{G}_{12}(z) \\ \widetilde{G}_{21}(z) & \widetilde{G}_{22}(z) \end{bmatrix}.$$

6.2. Pole locations. It was established in section 5.2 that the McMillan degree of a Hambo operator transform is equal to the McMillan degree of the original system. Hence the number of poles of $\widetilde{G}(\lambda)$ is equal to that of $G(z)$. The locations of the poles of $\widetilde{G}(\lambda)$ are determined as follows.

PROPOSITION 6.1. *Consider a system $G(z) \in RH_2$ and a Hambo basis generated by an inner function $G_b(z)$. If $G(z)$ has a pole at $z = a_i$, its Hambo operator transform $\widetilde{G}(\lambda)$ will have a pole at $\mu_i = G_b(\frac{1}{a_i}) = G_b^{-1}(a_i)$.*

Proof. This assertion can be proved on the basis of (5.20). That is, if $G(z)$ has a state space realization (A, B, C, D), $\widetilde{G}(\lambda)$ will have a state space realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ with

$$\text{(6.4)} \quad \tilde{A} = \sum_{k=0}^{\infty} g_b(k) A^k,$$

where $g_b(k)$ represents the impulse response sequence of $G_b(z)$. Consider any eigenvalue a_i of A and a corresponding eigenvector $x_i \in \mathbb{C}^n$, $Ax_i = x_i a_i$. If we multiply (6.4) from the right with x_i, we find $\tilde{A}x_i = x_i \sum_{k=0}^{\infty} g_b(k) a_i^k = x_i G_b(\frac{1}{a_i}) = x_i G_b^{-1}(a_i)$. Therefore, $\tilde{A}$ has eigenvalue $G_b(\frac{1}{a_i})$ with corresponding eigenvector x_i. ∎

COROLLARY 6.2. *The Hambo operator transform $\widetilde{G}(\lambda)$ of a system $G(z) \in H_2$ is stable.*

Proof. By the maximum modulus theorem [34] it holds that for an inner function $G_b(z)$, $|G_b(z)| < 1$ outside the unit disk. Hence $|G_b(\frac{1}{a})| < 1$ for all $a < 1$. Consequently, $\widetilde{G}(\lambda)$ is stable if $G(z)$ is stable. ∎

Corollary 5.2 showed that for $G(z) \in H_{2-}$, $\breve{G}(\lambda) = \widetilde{G}(\lambda)W_0(\lambda)$. Since this must be an element of $H_2^{n_b}$, it therefore must hold that the unstable pole of W_0 which lies at $\frac{1}{D_b}$ is cancelled by a zero at $\frac{1}{D_b}$ of $\det \widetilde{G}(\lambda)$, and hence we can immediately conclude that the poles of $\breve{G}(\lambda)$ constitute a subset of the poles of $\widetilde{G}(\lambda)$.

COROLLARY 6.3. *Let $G(z) \in H_{2-}$ have McMillan degree n with poles at a_i with $1 \leq i \leq n$. Then the Hambo signal transform $\breve{G}(\lambda)$ is stable, and its poles form a subset of $\{\mu_i\}_{1\leq i\leq n}$ with $\mu_i = G_b(\frac{1}{a_i})$. Hence the McMillan degree of $\breve{G}(\lambda)$ is smaller than or equal to n.*

On the basis of Corollary 6.3, one can make the following statement about the convergence rate of an expansion in terms of Hambo basis functions [13].

PROPOSITION 6.4. *Let a Hambo basis function expansion of $G(z) \in H_{2-}$ be given by $G(z) = \sum_{k\in\mathbb{N}} \breve{g}^T(k)V_k(z) = \sum_{k\in\mathbb{N}}\sum_{i=1}^{n_b} \breve{g}(k)_i\Phi_{k,i}(z)$. Further, let $G(z)$ have McMillan degree n and poles $a_i, 1 \leq i \leq n$. Then with μ defined as $\mu = \max_{1\leq i\leq n} |G_b(\frac{1}{a_i})|$, it holds that there exists a positive constant $c \in \mathbb{R}$ such that $\max_{1\leq i\leq n_b} |\breve{g}(k)_i| \leq c\mu^{k-1}$.*

This is simply a result of the well-known fact that the convergence of an impulse response sequence is dominated by the pole with the largest modulus. If the poles of $G(z)$ are a subset of the poles $\xi_j, 1 \leq j \leq n_b$, of $G_b(z)$, then it holds that $G_b(\frac{1}{a_i}) = 0$ for $1 \leq i \leq n$. Hence it follows that in this case $\breve{g}(k) = 0$ for all $k > 1$, and the basis function expansion converges to zero in one step. This illustrates the mechanism that the convergence becomes very fast when the poles in the basis generating inner function lie close to the poles of $G(z)$.

6.3. Eigenstructure of Hambo operator transforms. In this section, we analyze some of the structural properties of Hambo operator transforms. A direct relation between the eigenvalues of a Hambo operator transform $\widetilde{G}(\lambda)$ and its pre-image $G(z)$ is established. It is further shown how $\widetilde{G}(\lambda)$, evaluated on the unit circle, can be diagonalized by means of a similarity transformation with an orthogonal matrix, thus revealing information about the singular values of the Hambo operator transform. We first observe the following result which was previously shown to hold in [43, 44].

LEMMA 6.5. *Given a Hambo basis generating inner function $G_b(z)$ and its corresponding dual basis generating inner function $N(\lambda)$, for $z \neq 0$*

$$zV_1^T(z)\; N(\lambda)|_{\lambda^{-1}=G_b(z)} = V_1^T(z). \tag{6.5}$$

Proof. The proof follows by direct evaluation of $N(G_b(z))V_1(z)$ using $G_b(z) = C_b(zI - A_b)^{-1}B_b + D_b$, making the assumption that $z \notin \sigma(A_b)$:

$$\begin{aligned} N(G_b(z))V_1(z) &= \left(A_b + B_b(G_b(z) - D_b)^{-1}C_b\right)(zI - A_b)^{-1}B_b \\ &= A_b(zI - A_b)^{-1}B_b + B_b\left(C_b(zI - A_b)^{-1}B_b\right)^{-1} C_b(zI - A_b)^{-1}B_b \\ &= A_b(zI - A_b)^{-1}B_b + B_b = V_1(z)z. \end{aligned}$$

By the inner property of $N(\lambda)$ and $G_b(z)$, this latter equation can be rephrased as (6.5). Since A_b has only a finite number of eigenvalues, continuity shows that the result is valid for all $z \in \mathbb{C}$. ∎

We see that for $z \neq 0$, $V_1^T(z)$ is a left eigenvector of $N(\lambda)|_{\lambda^{-1}=G_b(z)}$, with z^{-1} the corresponding eigenvalue. This has the following consequence.

PROPOSITION 6.6. *Consider a Hambo basis generated by the inner function $G_b(z)$ and a transfer function $G(z) \in H_2$. Then the Hambo operator transform $\widetilde{G}(\lambda)$ satisfies*

$$V_1^T(z)\,\widetilde{G}(\lambda)\Big|_{\lambda^{-1}=G_b(z)} = G(z)V_1^T(z) \tag{6.6}$$

for all $z \neq 0$.

Proof. It follows by direct substitution of Lemma 6.5 in Proposition 5.1 that

$$V_1^T(z)\,\widetilde{G}(\lambda)\Big|_{\lambda^{-1}=G_b(z)} = \sum_{\tau=0}^{\infty} g(\tau)V_1^T(z)\,N(\lambda)^{\tau}|_{\lambda^{-1}=G_b(z)} = \sum_{\tau=0}^{\infty} g(\tau)z^{-\tau}V_1^T(z). \qquad ∎$$

Consider a certain fixed value of λ denoted as λ_0. Because $G_b(z)$ is an inner function of McMillan degree n_b, the equation ${\lambda_0}^{-1} = G_b(z)$ will have n_b solutions which we will denote as z_i. Defining the matrix $X(\{z_i\})$ as $X(\{z_i\}) = \left[V_1(z_1)\ V_1(z_2)\ \cdots\ V_1(z_{n_b})\right]$, one can write, using Proposition 6.6, $X^T(\{z_i\})\widetilde{G}(\lambda_0) = \operatorname{diag} G(z_i)X^T(\{z_i\})$. If the solutions z_i to ${\lambda_0}^{-1} = G_b(z)$ are distinct, it holds that $V_1^T(z_i)V_1(\frac{1}{z_j}) = 0$ for $z_i \neq z_j$. This follows directly from the following result, which is known as the *Christoffel–Darboux* formula [6, 4] for the Hambo basis. It gives an expression for the reproducing kernel of the subspace spanned by the functions $\Phi_{1,i}(z), 1 \leq i \leq n_b$, which is equal to $K(z, z') = V_1^T(z')V_1(\frac{1}{z})$.

LEMMA 6.7 (Christoffel–Darboux formula). *Consider a Hambo basis generating inner function $G_b(z)$. It holds for all $z_1, z_2 \in \mathbb{C}$, $z_1 \neq z_2$, that*

$$V_1^T(z_1)V_1\left(\frac{1}{z_2}\right) = \frac{G_b(z_1)G_b(\frac{1}{z_2}) - 1}{1 - \frac{z_1}{z_2}}. \tag{6.7}$$

Proof. The proof follows from the properties of the orthogonal realization (A_b, B_b, C_b, D_b). Using that $zV_1(z) = A_bV_1(z) + B_b$, we have that

$$\begin{aligned} z_1V_1^T(z_1)V_1\left(\frac{1}{z_2}\right)\,\frac{1}{z_2} &= (V_1^T(z_1)A_b^T + B_b^T)\left(A_bV_1\left(\frac{1}{z_2}\right) + B_b\right) \\ &= V_1^T(z_1)A_b^TA_bV_1\left(\frac{1}{z_2}\right) + V_1^T(z_1)A_b^TB_b + B_b^TA_bV_1\left(\frac{1}{z_2}\right) + B_b^TB_b. \end{aligned}$$

Substituting $A_b^TA_b = I - C_b^TC_b$, $A_b^TB_b = -C_b^TD_b$, and $B_b^TB_b = 1 - D_b^TD_b$ results in

$$z_1V_1^T(z_1)V_1\left(\frac{1}{z_2}\right)\,\frac{1}{z_2} = V_1^T(z_1)V_1\left(\frac{1}{z_2}\right) - G_b(z_1)G_b\left(\frac{1}{z_2}\right) + 1,$$

which can be rephrased as (6.7). ∎

We now have that, if the solutions z_i to $\lambda_0^{-1} = G_b(z)$ are distinct, it holds that

$$X^T(\{z_i\})\widetilde{G}(\lambda_0)X\left(\left\{\frac{1}{z_i}\right\}\right) = \operatorname{diag} G(z_i)\operatorname{diag} V_1^T(z_i)V_1\left(\frac{1}{z_i}\right).$$

The case where $|\lambda_0| = 1$ is a simple but important situation for which it holds that the solutions z_i to $\lambda_0^{-1} = G_b(z)$ are all distinct. This follows directly from the fact that any scalar inner function with McMillan degree n_b can be written as a Blaschke product $G_b(z) = \pm \prod_{k=1}^{n_b} \frac{1-\xi_k^*}{z-\xi}$, and thus the map $e^{i\omega} \to G_b(e^{i\omega})$ will go through the unit circle n_b times as ω goes from 0 to 2π, and hence there are n_b different solutions $0 \le \omega_1 < \omega_2 < \cdots < \omega_{n_b} < 2\pi$ with $G_b(e^{i\omega_k}) = 1$.

A further consequence of the observation that $V_1^T(z)V_1(\frac{1}{z}) > 0$ if $|z| = 1$ is that for $|\lambda_0| = 1$, the matrix $X(\{\frac{1}{z_i}\})(\text{diag}\sqrt{V_1^T(z_i)V_1(\frac{1}{z_i})})^{-1}$ is an orthogonal matrix. This brings us the following diagonal decomposition of $\widetilde{G}(\lambda_0)$.

Proposition 6.8. *Let z_i, with $1 \le i \le n_b$, be the solutions to $\lambda_0^{-1} = G_b(z_i)$ with $|\lambda_0| = 1$. Then, defining $R = \text{diag}\sqrt{V_1^T(z_i)V_1(\frac{1}{z_i})}$, it holds that*

$$R^{-1}X^T(\{z_i\})\widetilde{G}(\lambda_0)X\left(\left\{\frac{1}{z_i}\right\}\right)R^{-1} = \text{diag}\, G(z_i).$$

$X(\{\frac{1}{z_i}\})R^{-1}$ is an orthogonal matrix. Hence the singular values of $\widetilde{G}(\lambda_0)$ are equal to $|G(z_i)|$.

This proposition also shows that $\widetilde{G}(\lambda)$ is Hermitian when $|\lambda| = 1$.

6.4. Norm invariance under Hambo operator transformation. It was shown before that the Hambo transforms of scalar stable finite dimensional LTI systems are again stable finite dimensional LTI systems, albeit that they have input/output dimension $n_b \times n_b$. For the particular case of the Hambo operator transform, it was further shown that the McMillan degree, Hankel singular values, and ℓ_2-gain are also invariant under Hambo operator transformation. This leads to the following observations.

Corollary 6.9. *The Hankel and H_∞-norms of a system $G(z) \in H_\infty$ are invariant under Hambo operator transformation.*

The assertion for the Hankel norm follows from invariance of the Hankel singular values. Invariance of the H_∞-norm follows from the fact that the H_∞-norm is equal to the ℓ_2-gain. Alternatively, it follows from Proposition 6.8, which shows that $\sup_{\omega\in[0,2\pi)} \bar{\sigma}(\widetilde{G}(e^{i\omega})) = \sup_{\omega\in[0,2\pi)} |G(e^{i\omega})|$. Given the definition of the Hambo operator transform, it is not surprising that these norms are invariant as they are both norms that are induced by the ℓ_2-norm for signals, which is invariant under Hambo signal transformation as follows, e.g., from Proposition 4.2. It is important to take notice of the fact that the H_2-norm is *not* invariant under Hambo *operator* transformation. On the basis of Proposition 5.3, we can, however, conclude the following.

Corollary 6.10. *The H_2-norm of the Hambo operator transform of $G(z) \in H_2$ satisfies*

$$\|\widetilde{G}\|_2 = \|V_k G\|_2 \quad \forall k \in \mathbb{Z}.$$

Proof. The proof follows by taking the trace of both sides of (5.4) with $G_1(z) = G_2(z) = G(z)$. ∎

7. Extensions and derivatives. In this section, we briefly discuss some closely related subjects in the context of the Hambo transform theory.

Time-varying transforms. In [7] a more generalized transform theory is developed, where the transforms are directly based on the Takenaka–Malmquist functions, as discussed in section 2. The main difference with the Hambo transforms is that the transforms for the generalized case turn out to be scalar time-varying operators instead of multivariable time-invariant systems.

Multivariable systems. In this paper, the Hambo operator transform has been restricted to the class of scalar systems. An important issue here is that for scalar systems the transformed system turns out to be an element of $H_2^{n_b \times n_b}$. While it is straightforward (see, e.g., [25, 7]) to define Hambo transforms for multivariable $p \times m$ systems, the transform will blow up to dimensions $pn_b \times mn_b$. An alternative method which does not increase the input/output dimension, using a time-varying transformation, is discussed in [7].

Unstable systems. This paper primarily considers stable systems. It is not difficult to extend the transformation formulas of section 5.3 to unstable systems as well. In fact, the same formulas are valid with the exception of systems that contain poles that are reciprocals of basis poles. The problem in the latter case is that the resulting transform may be a noncausal system. This is explained by the following example for the Laguerre basis functions.

Let a be the (stable) pole of the Laguerre basis functions (2.4), and let $G(z) \in H_2^{\perp}$ be given by $G(z) = \frac{1}{z^{-1}-a}$. The Hambo transform of $G(z)$ can be calculated with (5.2), using that $N(\lambda) = \frac{1+a\lambda}{\lambda+a}$. This results in a noncausal $\widetilde{G}(\lambda) = \frac{\lambda-a}{1+a^2}$. So, while the Hambo transform is still well defined, the state space formulas cannot be used as is.

Realization. In [37, 8, 7], the problems of exact and partial realization in terms of Hambo functions have been solved. This concerns the situation where a sequence of expansion coefficients $\{\breve{g}(k), k = 1, \dots, N\}$ is given and a system $G(z)$ of minimal degree is sought such that the first N expansion coefficients of $G(z)$ coincide with the given set. Such a situation typically arises in an identification setting, as described in [39]. In fact, the state space relations described in section 5.3 are a direct spin-off of this research.

Frequency warping. The variable substitution of (5.3) is sometimes referred to as a *frequency transformation*, as it maps $\mathbb{T}$ to $\mathbb{T}$. With $z = e^{i\omega}$ and $\lambda = e^{i\vartheta}$, it holds that this transformation, defined as $\vartheta = \beta(\omega)$, constitutes a continuously differentiable nondecreasing (hence bijective) mapping from $\omega \in [0, 2\pi)$ to $\vartheta \in [0, 2n_b\pi)$. The properties of this β mapping, and in particular its inverse β^{-1}, are analyzed in [35], where it is used in a frequency domain approach to Hambo basis function modelling. A discrete set of equidistantly distributed frequency points in the ϑ domain is mapped by β^{-1} to a nonequidistantly distributed set of frequency points in the ω domain. This frequency distortion, or "warping" property, is exploited in [41] for the case $n_b = 1$ to enable the application of the fast Fourier transform (FFT) algorithm to nonuniformly spaced samples of a discrete time Fourier transform (DTFT).

(Future) applications. The theory on Hambo transforms proved to be a powerful tool in the derivation of variance expressions for identification in terms of orthogonal basis functions [39]. Furthermore, as stated before, this theory has been instrumental in the derivation of approximate realization algorithms that are based on expansions in orthonormal basis functions. In [7] it is shown that these algorithms can also be used to solve certain classes of interpolation problems. Other promising future directions for use of the transform theory are, for instance, the application of system identification in the transform domain, extending the results of [40, 11, 10], and control design in the transform domain, utilizing the property that any linear system can be transformed into a system with all poles located at the origin.

8. Conclusions. In this paper, we have analyzed a signals and systems transform that is induced by the Hambo functions. These functions, which are a special case of the Takenaka–Malmquist functions, are induced by the balanced states of scalar inner (stable all-pass) functions and encompass the classical pulse, Laguerre, and Kautz functions. The induced signals and systems transforms generalize the Z-transform and the Laguerre transform to a multidimensional representation. The transforms have been analyzed in detail, providing insight into their structural properties. Explicit and efficient algorithms have been provided that enable the calculation of minimal state space realizations of the operator transform and its inverse.

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