2015 European Control Conference (ECC)
July 15-17, 2015. Linz, Austria

Least-Costly Experiment Design for Uni-Parametric Linear Models: An Analytical Approach

M.G. Potters, M. Forgione, X. Bombois, and P.M.J. Van den Hof

Abstract— **Least-costly experiment design has received ample attention over the past decades, and efficient numerical algorithms that can compute optimal excitation spectra for linear models have been found. The interpretation of such spectra, however, has received far less attention. We restrict ourselves to uni-parametric models, for which an analytical solution to the experiment design problem is derived. This solution enables us to address, among other things, the following questions: What determines the frequency and amplitude of the excitation signal? Does the optimal frequency depend on the location(s) that the parameter occupies in the transfer function? With the optimal signal, is a closed-loop identification experiment cheaper than an open-loop one?**

I. INTRODUCTION

The design of optimal identification experiments for control has received a lot of attention in recent years [3], [4], [5], [8]. In the least-costly paradigm, the problem is formulated as an optimization problem where the decision variable is the power spectrum Φ_r of the excitation signal. The optimal spectrum is the one that minimizes the cost of the identification experiment while guaranteeing that the uncertainty of the to-be-identified model is small enough to enable robust control design with satisfactory performance. This is for instance important in industries with high demands on product purity. Adopting the framework of [4] reduces the problem to one in which the cost of the identification has to be minimized, while ensuring that the inverse of the covariance matrix of the to-be-identified parameter vector, P_θ^{-1}, is larger than a positive definite matrix R_{adm}. This matrix can be computed based on user-imposed control specifications. The methodology has recently been applied in the European Autoprofit project [9].

The cost of an identification experiment is generally expressed as a linear function of the spectrum Φ_r, and the inverse of the covariance matrix is affine in Φ_r (in the case of a linear, explicitly-known controller). Therefore, the optimal experiment design problem is convex and can thus be solved using standard convex optimization techniques.

Recent theoretical and numerical developments have resulted in the ability to solve the optimization problem for linear systems with nonlinear or implicit controllers as well, see e.g. [5] and [8]. This has broadened the range of applications considerably.

Apart from some technicalities, this least-costly optimization problem is now solvable for essentially all linear systems parameterized by an arbitrary number of parameters and with any type of controller, through the use of convex optimization algorithms.

A remaining challenge from a theoretical point of view is, however, the interpretation of the optimal power spectra that result from these numerical algorithms. An analytical solution to the least-costly problem will enhance our understanding, and can be hard-coded in the numerical programs. Obviously, this ability will be limited to specific cases. We restrict attention in this paper to LTI systems where only one parameter has to be identified. (The parameter may be present at multiple locations in the transfer function.) This situation occurs in e.g. reservoir engineering [7] and in functional magnetic resonance imaging [2].

For this particular class of systems, we derive an analytical solution of the least-costly identification experiment design problem (formulated in the framework of [4]). We consider both the open-loop and closed-loop identification cases and we show that the solution of the least-costly identification experiment problem is a single sinusoid with a specific amplitude and frequency. A specific solution in open loop at a fixed frequency is given in [10]. The fact that the optimal solution is a sinusoid could have been derived from the result in [11, pages 35-36]. However, its frequency is there only given for a very special case due to normalization, and no interpretation is provided.

Our analytical solution for this special case allows one to analyse, understand, and check numerical solutions. As an example, we provide insight into the frequency and amplitude of the optimal excitation signal. It is also shown that when the to-be-identified parameter resides at just one location in the transfer function, the optimal frequency only depends, location-wise, on whether the parameter is present in the numerator or denominator. Furthermore, we give some insights for this particular case on the comparison between the open-loop and closed-loop settings (see [6], [1] for previous results on that topic).

The paper is organized as follows. In Section II, we

M.G. Potters is with Delft Center for Systems and Control, Delft University of Technology, 2628 CD Delft, The Netherlands m.g.potters@tudelft.nl

M. Forgione and X. Bombois are with Laboratoire Ampère UMR CNRS 5005, Ecole Centrale de Lyon, 36 avenue Guy de Collongue, 69134 Ecully Cedex, France {marco.forgione, xavier.bombois}@ec-lyon.fr

P.M.J. Van den Hof is with the Department of Electrical Engineering, Eindhoven University of Technology, Postbus 513, 5600 MB, Eindhoven, The Netherlands p.m.j.vandenhof@tue.nl

The research leading to these results has received funding from the European Union's Seventh Framework Programme (FP7/2007-2013) under grant agreement no 257059, The 'Autoprofit' project (www.fp7-autoprofit.eu).

introduce the notations and the least-costly optimization problem under consideration. Our analytical solutions are derived in Section III. We discuss and interpret our findings in Section IV, and provide a numerical study in Section V. Lastly, we draw conclusions in Section VI.

II. LEAST-COSTLY IDENTIFICATION EXPERIMENT

Consider a linear data-generating system consisting of a uni-parametric model $G_0 = G(z,\theta_0)$ with unknown parameter $\theta_0 \in \mathbb{R}$ and known noise model $H_0 = H(z)$. The parameter θ_0 may be present at an arbitrary number of positions in the transfer function $G(z,\theta_0)$. We will consider the identification of this unique parameter in both an open-loop and in a closed-loop setting. Since the open-loop setting is a special case of the closed-loop one, we first present the closed-loop identification setting. We thus suppose that the true system is operated in closed loop with a known linear controller $C(z)$. The output $y(t) \in \mathbb{R}$ and input $u(t) \in \mathbb{R}$ of the true system are thus given by (see Fig. 1)

$$u(t) = r(t) - C(z)y(t), \quad (1)$$
$$y(t) = G_0(z)u(t) + H(z)e(t). \quad (2)$$

Here, $e(t) \in \mathbb{R}$ is white noise with variance σ_e^2, and $G_0(z)$ and $H(z)$ are stable, discrete-time transfer functions. Furthermore, $H(z)$ is assumed to be monic and minimum-phase. The signal $r(t) \in \mathbb{R}$ is the excitation signal that is used to identify the true system. Suppose this excitation signal is applied from $t = 1$ to $t = N$ and that the corresponding input and output data are collected: $Z_N = \{u(t), y(t) \mid t = 1,\dots,N\}$. This data set can be used to determine an estimate of θ_0 in a full order model structure $\mathscr{M} = \{G(z,\theta)\}$: $\hat{\theta}_N = \arg\min_\theta V_{id}(\theta)$, with $V_{id}(\theta) = \frac{1}{N}\sum_{t=1}^{N} \varepsilon^2(t,\theta)$, where $\varepsilon(t,\theta) = H^{-1}(z)\left[y(t) - G(z,\theta)u(t)\right]$ is the prediction error.

With the assumption of a model structure containing the true system, the identified parameter vector $\hat{\theta}_N$ is asymptotically normally distributed around the true parameter vector θ_0. Hence, under certain conditions on the excitation signal (and the controller in closed loop), we have for $N \to \infty$ that $\sqrt{N}(\hat{\theta}_N - \theta_0) \to \mathscr{N}(0, P_\theta)$, with P_θ^{-1} a positive scalar given by $P_\theta^{-1} = \sigma_e^2 \bar{E}\left[(\psi(t,\theta_0))^2\right]$, that can be estimated from $\hat{\theta}_N$ and Z_N [6]. In this equation, the expectation operator $\bar{E}$ is defined as $\bar{E}f(t) \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} Ef(t)$, where E is the usual expectation operator. Lastly, $\psi(t,\theta) \equiv -\frac{\partial \varepsilon(t,\theta)}{\partial \theta}$, where $\varepsilon(t,\theta)$ is the prediction error defined previously.

It can be shown that P_θ^{-1} admits a frequency domain expression given by [6]

$$P_\theta^{-1} = \frac{N}{2\pi\sigma_e^2}\int_{-\pi}^{\pi} |F_r(e^{-i\omega},\theta_0)|^2 \Phi_r(\omega)\, d\omega + R_0, \quad (3)$$

where $R_0 = \frac{N}{2\pi}\int_{-\pi}^{\pi} |F_v(e^{-i\omega},\theta_0)|^2\, d\omega$, and $F_r(e^{-i\omega},\theta_0) = H^{-1}(e^{-i\omega})S(e^{-i\omega},\theta_0)\Lambda_G(e^{-i\omega},\theta_0)$, $F_v(e^{-i\omega},\theta_0) = -C(e^{-i\omega})S(e^{-i\omega},\theta_0)\Lambda_G(e^{-i\omega},\theta_0)$. In these expressions, $S(e^{-i\omega},\theta) = (1 + C(e^{-i\omega})G(e^{-i\omega},\theta))^{-1}$ is the sensitivity function of the closed-loop system,

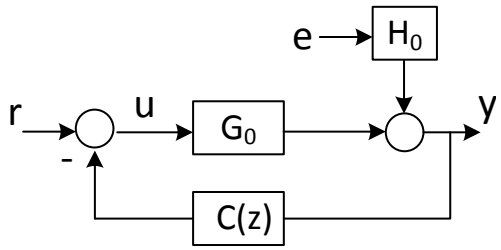

Fig. 1. Schematic overview of the true closed-loop system.

$\Lambda_G(e^{-i\omega}) = \frac{\partial G(e^{-i\omega},\theta)}{\partial \theta}\,|_{\theta_0}$, and $\Phi_r(\omega)$ is the power spectrum of the excitation signal $r(t)$ in (1). The spectrum is a positive, even function in ω.

With these notations, the least-costly experiment design problem can be defined mathematically as

$$\min_{\Phi_r(\omega)} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathscr{L}(e^{-i\omega})\Phi_r(\omega)\, d\omega \quad (4)$$
$$\text{subject to } P_\theta^{-1} \geq R_{adm},$$

with P_θ^{-1} as defined in (3), $\mathscr{L}(e^{-i\omega})$ an even and positive weighting function, and R_{adm} a scalar value calculated prior to the experiment, guaranteeing a certainty of the to-be-identified parameter that is large enough to enable robust control design. For more details on the computation of R_{adm}, see [4].

The optimization problem (4) depends on the true parameter value θ_0 through $\mathscr{L}(e^{-i\omega})$ and P_θ^{-1} (3). In other words, the optimal spectrum $\Phi_{r,opt}$ depends on θ_0, which we have yet to identify. This so-called chicken-and-egg problem is always present, but is generally circumvented by substituting θ_0 with a prior estimate.

The solution to (4), denoted by $\Phi_{r,opt}(\omega)$, is used to generate a time-domain excitation signal $r(t) = r_{opt}(t)$, which is used in the identification method explained previously. This completely defines the least-costly experiment.

A popular choice of the weighting function $\mathscr{L}(e^{-i\omega})$ in (4) reads

$$\mathscr{L}(e^{-i\omega}) = \left(\alpha|G_0(e^{-i\omega})|^2 + \beta\right)|S(e^{-i\omega},\theta_0)|^2, \quad (5)$$

which results in an objective function being the sum of the power of the perturbations induced by $r(t)$ on the input and the output. The parameters $\alpha, \beta \geq 0 \in \mathbb{R}$ are user-chosen constants, i.e. not tuning parameters [3].

Until now, we have considered the identification of θ_0 in a closed-loop setting. However, the identification can also be performed in an open-loop setting. The open-loop setting corresponds to the case where $C(z) = 0$ and thus $u(t) = r(t)$. The excitation signal $r(t)$ is thus directly applied to the input of the system (1)-(2). In the open-loop setting it thus follows that $R_0 = 0$ and the sensitivity $S = 1$, see (3). Equation (5)

for the open-loop case is thus

$$\mathscr{L}(e^{-i\omega}) = \alpha|G_0(e^{-i\omega})|^2 + \beta. \tag{6}$$

In the Introduction, we mentioned that the optimization problem for objective functions linear in $\Phi_r(\omega)$ can be cast into a convex problem, which can be solved numerically. However, the somewhat unsatisfactory aspect in this procedure is the inability to obtain a good understanding of the resulting excitation spectrum. Furthermore, as we shall see later, the numerical algorithm only approaches the optimal solution and is therefore still suboptimal.

In the following section, we adopt an analytical approach that provides insight into the solution of (4). For the sake of brevity, we will from now on omit the argument θ_0 in all terms in (3).

III. ANALYTICAL SOLUTION TO THE LEAST-COSTLY PROBLEM

We consider (4) with P_θ^{-1} as given in (3). The optimization problem thus reads:

$$\min_{\Phi_r(\omega)} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathscr{L}(e^{-i\omega})\Phi_r(\omega)d\omega \tag{7}$$

subject to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |F_r(e^{-i\omega})|^2\Phi_r(\omega)d\omega \geq K, \tag{8}$$

where $K = \frac{\sigma_e^2}{N}(R_{adm} - R_0)$. We assume that $R_{adm} > R_0$. Otherwise, the contribution of the noise to P_θ^{-1} is sufficient to satisfy the first constraint in (4) and the optimal excitation signal is simply $r(t) = 0, \forall t$.

We are now ready to derive the solution to the above problem. It is given for the closed-loop case, but the open-loop equivalent can be derived from it as a particular case.

Proposition 1: Consider the optimization problem (7)-(8). The spectrum solving this problem reads:

$$\Phi_{r,opt}(\omega) = \frac{\pi K}{|F_r(e^{-i\omega^\dagger})|^2}\sum_{k=\{-1,1\}}\delta(\omega - k\omega^\dagger) \tag{9}$$

with

$$\omega^\dagger = \arg\max_{\omega\in[0,\pi]} \frac{|F_r(e^{-i\omega})|^2}{\mathscr{L}(e^{-i\omega})}. \tag{10}$$

The excitation signal in the time-domain is thus given by

$$r_{opt}(t) = \frac{\sqrt{2K}}{|F_r(e^{-i\omega^\dagger})|}\cos(\omega^\dagger t + \phi),$$

for $\omega^\dagger \neq 0$ and $r_{opt}(t)/\sqrt{2}$ for $\omega^\dagger = 0$. Here ϕ is an arbitrary phase.

Proof: With $\omega^\dagger$ defined in (10), for all eventual solutions Φ_r to the optimization problem, the following inequality should hold:

$$K \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|F_r(e^{-i\omega})|^2}{\mathscr{L}(e^{-i\omega})}\mathscr{L}(e^{-i\omega})\Phi_r(\omega)\,d\omega \leq \tag{11}$$

$$\frac{1}{2\pi}\frac{|F_r(e^{-i\omega^\dagger})|^2}{\mathscr{L}(e^{-i\omega^\dagger})}\int_{-\pi}^{\pi}\mathscr{L}(e^{-i\omega})\Phi_r(\omega)\,d\omega,$$

where the first inequality is the constraint (8), while the latter is derived using (10). The last term is thus greater than or equal to K. Multiplying the inequality equation formed by the first and latter terms of (11) with $\frac{\mathscr{L}(e^{-i\omega^\dagger})}{|F_r(e^{-i\omega^\dagger})|^2}$ shows that the minimum of (7) subject to (8) is equal to $\frac{\mathscr{L}(e^{-i\omega^\dagger})}{|F_r(e^{-i\omega^\dagger})|^2}K$. With K defined in (8), it is obvious that this minimal cost:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathscr{L}(e^{-i\omega})\Phi_r(\omega)d\omega = \frac{\mathscr{L}(e^{-i\omega^\dagger})}{|F_r(e^{-i\omega^\dagger})|^2}K \tag{12}$$

is obtained for the spectrum $\Phi_r(\omega) = \Phi_{r,opt}(\omega)$ given in (9). ■

Corollary 1: The same procedure in Proposition 1 can be used to solve the dual problem, i.e., $\min_{\Phi_r}\det(P_\theta(\Phi_r(\omega)))$ such that $\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathscr{L}(e^{-i\omega})\Phi_r(\omega)\,d\omega \leq \gamma$. The optimal solution is then given by $\Phi_{r,opt}(\omega) = \pi\gamma\mathscr{L}(e^{-i\omega^\dagger})\sum_{k=\{-1,1\}}\delta(\omega - \omega^\dagger)$, with $\omega^\dagger$ as given in (10).

Proposition 2: Solution (9) is the unique minimizer of (7)-(8) if (10) has a unique global maximum.

Proof: We first assume (10) has a unique global maximum. Suppose another solution, $\hat{\Phi}_{r,opt}(\omega) = \Phi_{r,opt}(\omega) + \eta(\omega)$ exists. Then this solution should be even ($\hat{\Phi}_{r,opt}(\omega) = \hat{\Phi}_{r,opt}(-\omega)$) and positive ($\hat{\Phi}_{r,opt}(\omega) \geq 0\forall\omega$). This in turn means that $\eta(\omega) = \eta(-\omega)$, and that η must be positive everywhere except at $\omega = \omega^\dagger$, where it may be negative.

Substitution of $\Phi_r(\omega) = \hat{\Phi}_{r,opt}(\omega)$ into (12) yields

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathscr{L}(e^{-i\omega})(\Phi_{r,opt}(\omega) + \eta(\omega))d\omega = \frac{\mathscr{L}(e^{-i\omega^\dagger})}{|F_r(e^{-i\omega^\dagger})|^2}K + \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathscr{L}(e^{-i\omega})\eta(\omega)\,d\omega.$$

The minimum of (7) is given by (12). Hence, in order to obtain the same minimum, the condition

$$\int_{-\pi}^{\pi}\mathscr{L}(e^{-i\omega})\eta(\omega)\,d\omega = 0 \tag{13}$$

must hold. The function $\eta(\omega)$ may only be negative at the points $\omega = \pm\omega^\dagger$. Hence, this point must be given infinite weight in order to generate a negative contribution to the above integral. Therefore, the only possible solution for $\eta(\omega)$ is of the form

$$\eta(\omega) = f(\omega) + \frac{B}{2}\sum_{m=\{-1,1\}}\delta(\omega - m\omega^\dagger), \tag{14}$$

where $B < 0$ is a constant, and $f(\omega) \geq 0\forall\omega$ is an even function with a non-zero contribution to (13). Indeed, this solution satisfies all the conditions on $\eta(\omega)$ stated above. Both B and $f(\omega)$ are yet to be determined. Furthermore, we

do not allow $f(\omega) = -\frac{B}{2}\sum_{m=\{-1,1\}}\delta(\omega - m\omega^{\dagger})$, as this will result in the trivial solution $\eta(\omega) = 0\,\forall\omega$. Substitution of (14) into (13) fixes B, which is given by

$$B = -\frac{1}{\mathscr{L}(e^{-i\omega^{\dagger}})}\int_{-\pi}^{\pi}\mathscr{L}(e^{-i\omega})f(\omega)\,d\omega. \qquad (15)$$

Condition (13) is now satisfied. The remaining freedom, $f(\omega)$, is to be used such that the constraint (8) with the potential solution $\hat{\Phi}_{r,opt}$ is satisfied. Substitution of $\Phi_r(\omega) = \hat{\Phi}_{r,opt} = \Phi_{r,opt}(\omega) + \eta(\omega)$ in the l.h.s. of (8), and using (11), (12), and (15) give

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|F_r(e^{-i\omega})|^2\hat{\Phi}_{r,opt}(\omega)d\omega =$$
$$K + \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{|F_r(e^{-i\omega})|^2}{\mathscr{L}(e^{-i\omega})} - \frac{|F_r(e^{-i\omega^{\dagger}})|^2}{\mathscr{L}(e^{-i\omega^{\dagger}})}\right]\mathscr{L}(e^{-i\omega})f(\omega)\,d\omega.$$

By virtue of (10), $\frac{|F_r(e^{-i\omega})|^2}{\mathscr{L}(e^{-i\omega})} \leq \frac{|F_r(e^{-i\omega^{\dagger}})|^2}{\mathscr{L}(e^{-i\omega^{\dagger}})}$ for all $\omega \in [-\pi,\pi]$. It thus follows that the above equation is less than K for any nontrivial $f(\omega)$, hence the solution to (7)-(8) is unique. If (10) is not unique (i.e. multiple, identical maxima at different frequencies), then the solution is a superposition. ■

IV. INTERPRETATION

In the previous section we formally derived the optimal solution to problem (7)-(8). In this section we give heuristic arguments to show how the solution can also be derived, and provide insight in the expressions (9) and (10).

Consider first the optimization problem (4) in which $\mathscr{L}(e^{-i\omega}) = 1\,\forall\omega$. In other words, we want to minimize the power of $r(t)$ s.t. $P_{\theta}^{-1} \geq R_{adm}$. Since all functions in the problem are positive definite, the optimal solution $\Phi_{r,opt}$ occurs only when $P_{\theta}^{-1} = R_{adm}$. It is then clear that in order to minimise the objective function, we should minimize Φ_r. Thus, the excitation frequency $\omega = \omega^{\dagger}$ should be chosen such that $|F_r(e^{-i\omega})|^2$ is maximal. Consequently, the optimal excitation signal requires only a single frequency and hence its spectrum reads $\Phi_{r,opt}(\omega) = C\sum_{k=\{-1,1\}}\delta(\omega - \omega^{\dagger})$. We see that the objective function is minimised by minimising the amplitude C of the excitation signal, which happens when maximising $|F_r(e^{-i\omega})|$. The constant C is found through substitution of $\Phi_{r,opt}(\omega)$ in (8). We then find the solution to be given by (9) and (10) for $\mathscr{L}(e^{-i\omega}) = 1\,\forall\omega$.

Intuitively, maximising $|F_r(e^{-i\omega})|^2$ means that the prediction error $\varepsilon(t,\theta) = H^{-1}(z)[y(t) - G(z,\theta)u(t)]$ is most sensitivity to variations in θ, see Section II. This ensures that we can distinguish with high probability between the true parameter θ_0 and an arbitrary close parameter $\theta_0 + \delta\theta$, i.e. we get high accuracy in our estimate of θ_0. This is reflected in the expression of P_{θ}^{-1} (c.f. (3)).

We now consider the full problem (7)-(8) in which $\mathscr{L}(e^{-i\omega}) \neq 1\,\forall\omega$ is an even and positive function. It is then no longer true that we can minimize the objective function by minimising $\Phi_r(\omega)$ through maximizing $|F_r(e^{-i\omega})|$.

To obtain the expressions (9) and (10) in an intuitive manner, we change the closed-loop configuration in Fig. 1 by considering the excitation signal $\xi(t)$ that is filtered through $L^{-1}(z)$, resulting in the original excitation signal $r(t) = L^{-1}(z)\xi(t)$. We define $|L(e^{-i\omega})|^2 = \mathscr{L}(e^{-i\omega})$ and $\Phi_{\xi}(\omega)$ as the spectrum of $\xi(t)$.

It follows that the spectrum of $\xi(t)$ is related to the spectrum of $r(t)$ through $\Phi_{\xi}(\omega) = \mathscr{L}(e^{-i\omega})\Phi_r(\omega)$. Furthermore, we recalculate the expression of P_{θ}^{-1} as a function of $\Phi_{\xi}(\omega)$, obtained by substitution of $r(t) = L^{-1}(z)\xi(t)$ in (1)-(2) and following the calculations in Section II. As a result, the optimization problem (4) reduces to

$$\min_{\Phi_{\xi}(\omega)}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{\xi}(\omega)d\omega \qquad (16)$$

subject to

$$P_{\theta}^{-1} = \frac{N}{2\pi\sigma_e^2}\int_{-\pi}^{\pi}\frac{|F_r(e^{-i\omega})|^2}{\mathscr{L}(e^{-i\omega})}\Phi_{\xi}(\omega)\,d\omega + R_0 \geq R_{adm} \qquad (17)$$

We are now in the same position as we had when $\mathscr{L}(e^{-i\omega}) = 1\,\forall\omega$: minimising the power of the excitation signal $\xi(t)$ subject to $P_{\theta}^{-1} \geq R_{adm}$. Following the same reasoning at the beginning of this section, the optimal frequency should in this case be $\omega^{\dagger} = \arg\max_{\omega}\frac{|F_r(e^{-i\omega})|}{\mathscr{L}(e^{-i\omega})}$. This is equal to (10). Realizing that P_{θ}^{-1} should be equal to R_{adm} and knowing that the optimal spectrum is of form $\Phi_{\xi,opt}(\omega) = C\sum_{k=\{-1,1\}}\delta(\omega - \omega^{\dagger})$ we find amplitude C through substitution of $\Phi_{\xi,opt}$ in $P_{\theta}^{-1} = R_{adm}$. The spectrum for $r(t)$ is then easily retrieved through the relation $\Phi_{r,opt}(\omega) = \mathscr{L}^{-1}(e^{-i\omega^{\dagger}})\Phi_{\xi,opt}(\omega)$ and is equal to (9).

We finish this part with some examples. First, when choosing $\mathscr{L}(e^{-i\omega}) = 1\,\forall\omega$, each frequency is penalised equally. The optimal frequency then guarantees the accuracy R_{adm} with the smallest possible input power of the excitation signal. Secondly, for any $\mathscr{L}(e^{-i\omega})$, using the optimal spectrum allows the user to satisfy the certainty constraint $P_{\theta}^{-1} = R_{adm}$ with minimal cost. Equivalently, a suboptimal spectrum with the same experiment cost as the optimal spectrum requires a longer experiment (N must be larger) to achieve the same accuracy in the parameter (R_{adm}).

We move to another topic. Consider transfer functions for which the unknown parameter θ_0 may now occur only *once*: in either the denominator or numerator. Let these functions be given by $G_1(z,\theta_0) = \frac{B(z)}{1+A(z,\theta_0)}$ and $G_2(z,\theta_0) = \frac{B(z,\theta_0)}{1+A(z)}$, where A, B are polynomials in z^{-1}. Note that this is a restriction from the general case we have considered so far.

A first question that arises is: Will the optimal frequency (10) depend on the position of θ_0 in $G(z,\theta_0)$? For example, if we have $G(z,\theta_0) = \frac{a_0+a_1z+a_2z^2+\theta_0z^3}{1+b_1z+b_2z^2}$, we assume that all parameters are known except θ_0, which here appears in the numerator. First, observe from the expression of $\omega^{\dagger}$ that only the term $|\Lambda_G(e^{-i\omega})|^2$ could influence the optimal frequency position-wise, while all the others remain the same. We analyse what happens for G_1 and G_2. For $G_1(z,\theta_0)$, $|\Lambda_G(e^{-i\omega})|^2 = \frac{|B(e^{-i\omega})|^2}{|1+A(e^{-i\omega})|^4}$, showing that $\omega^{\dagger}$ depends on both the location of the poles and the zeros of $G_1(z,\theta_0)$, but

is independent of the position of θ_0. For G_2, $|\Lambda_G(\omega)|^2 = \frac{1}{|1+A(e^{-i\omega})|^2}$, showing that $\omega^\dagger$ only depends on the location of the poles of $G_2(z,\theta_0)$ but is invariant with respect to the position of θ_0 in the numerator.

Thus, given a transfer function of type $G_1(z,\theta_0)$ ($G_2(z,\theta_0)$), the optimal excitation frequency is invariant w.r.t. the parameter position in the denominator (nominator). For an arbitrary transfer function, i.e. for which the parameter may have multiple occurrence in the numerator or denominator, the optimal frequency will depend on the parameter position. Remark: this analysis only holds for discrete-time systems.

Consider again the general case of transfer functions $G(z,\theta_0)$, of which all parameters are known except θ_0, which may again appear at multiple positions in $A(z,\theta_0)$ and $B(z,\theta_0)$. We now study and compare the least-costly identification experiment problem in the open-loop and the closed-loop settings. We recall that the open-loop setting is characterized by $R_0 = 0$, $S(e^{-i\omega}) = 1\,\forall\omega$ and $F_r(e^{-i\omega}) = \Lambda_G(e^{-i\omega})/H(e^{-i\omega})$.

The experiment cost (7) of the optimal spectrum (9) is given by

$$J_{opt} = \frac{\sigma_e^2}{N|F_r(e^{-i\omega^\dagger})|^2}(R_{adm} - R_0)\mathscr{L}(e^{-i\omega^\dagger}). \qquad (18)$$

Suppose we factorize $\mathscr{L}(e^{-i\omega}) = \mathscr{L}_{in}(e^{-i\omega})|S(e^{-i\omega})|^2$, where $\mathscr{L}_{in}$ is identical for the open- and closed-loop cases, i.e., controller-independent. This family of functions includes the popular choice (5)-(6). Then it follows from (10) that the optimal excitation frequency is identical for the open-loop and in the closed-loop settings:

$$\omega^\dagger = \arg\max_{\omega\in[0,\pi]} \frac{|\Lambda_G(e^{-i\omega})|^2}{|H(e^{-i\omega})|^2|\mathscr{L}_{in}(e^{-i\omega})|^2}, \qquad (19)$$

and that the ratio of the open- and closed-loop experiment costs reads $f_{O/C} = \frac{J_{opt}^{OL}}{J_{opt}^{CL}} = \frac{1}{1-R_0/R_{adm}}$. This equation clearly reveals the advantage of the closed-loop: in closed-loop the noise also excites the system and contributes to the certainty in the parameter through the term R_0, resulting in a experimental cost that is always smaller compared to the open-loop case. Note that this does not necessarily hold for arbitrary $\mathscr{L}(e^{-i\omega})$.

In [6],[1], it has been shown that the closed-loop configuration allows to obtain the desired parameter accuracy R_{adm} with the smallest output variance (at least for BJ model structures). The output variance is particularly interesting. Indeed, if the output represents a quality measure of a product to be sold, it is important to reduce as much as possible its variance. It is clear that the closed-loop configuration has the advantage to reduce the influence of the process noise $H(z)e(t)$ on the output. In [1], it is moreover shown that this configuration allows to obtain the desired accuracy with an excitation $r(t)$ leading to less output perturbation than for the open-loop case. The result of the present paper allows us to compare exactly the obtained output variance for both configurations. For this purpose, it is logical to consider the least costly experiment design problem with the objective function (5). Recall that, for the open-loop case, $S = 1$ in (5). Since we focus on the output, we could choose $\beta = 0$ in (5). However, all what follows holds for arbitrary α and β.

Denote by r_{opt}^{OL} and r_{opt}^{CL} the optimal excitation sinusoids given by Proposition 1 for (5) in the open-loop and closed-loop configurations, respectively. The variances (powers) of the outputs $y^{OL}(t)$ and $y^{CL}(t)$ that are obtained using r_{opt}^{OL} and r_{opt}^{CL} in their respective configuration are given by:

$$\begin{aligned}\mathscr{P}_y^{OL/CL} &= |G_0(e^{-i\omega^\dagger})|^2 \frac{\sigma_e^2|H(e^{-i\omega^\dagger})|^2}{N|\Lambda_G(e^{-i\omega^\dagger})|^2}(R_{adm} - R_0) \\ &+ \frac{\sigma_e^2}{2\pi}\int_{-\pi}^{\pi}|H(e^{-i\omega})|^2|S(e^{-i\omega})|^2\,d\omega, \qquad (20)\end{aligned}$$

where the open-loop case is found by taking $R_0 = 0$ and $S = 1$. We see that the contribution of the excitation signal are very similar in both cases. However, the term R_0 makes that this contribution is strictly smaller in closed-loop than in open-loop. The contribution of the noise is of course smaller in closed loop (if the controller is functioning properly). It is also to be noted that the contribution of the optimal excitation sinusoid on the input of the system is also smaller in closed-loop than in open-loop in this case. If the cost is not given by (6), it is then not clear which configuration is better (this will be case-dependent).

V. ILLUSTRATION

The aim of this section is to verify our theoretical findings of Section III.

To this end, we consider the true system $G_0(e^{-i\omega}) = \frac{a_1(1-\theta_0)e^{-i\omega}}{1-\theta_0 e^{-i\omega}}$, and $H(e^{-i\omega}) = 1$, where $a_1 = 10$ is the steady-state gain, $\sigma_e^2 = 4.0$ is the variance of the noise $e(t)$, and $\theta_0 = 0.45$ is the unknown parameter that we need to identify with a least-costly experiment. We first consider the closed-loop setting with a PI controller $C(z)$ defined by the constants $K_p = 0.00853$ and $K_i = 0.0171$.

The optimization problem we consider is (7), with $\mathscr{L}(e^{-i\omega})$ given by (5). The parameters defining the optimization problem are: $N = 200$, $\alpha = 0.1$, $\beta = 0.7$, and $R_{adm} = 1000$. The term $R_0 = 3.0826$.

We first consider the theoretical predictions. The optimal excitation is given by (11) and is here equal to $r_{opt}(t) = 0.5661\cos(\omega^\dagger t + \phi)$ with an arbitrary phase shift ϕ and with $\omega^\dagger = 1.8903\ rad/s$. The frequency $\omega^\dagger$ can be determined by inspection of the frequency response of $|F_r(\omega)|^2/\mathscr{L}(\omega)$ or, alternatively, via straightforward but tedious algebraic manipulations leading to the expression $\omega^\dagger = \arccos\left[1 - \frac{(1-\theta_0)^2}{2\beta\theta_0}\sqrt{\beta(\beta+\alpha a_1^2)}\right]$. The cost of the identification corresponding to this identification is equal to $J_{opt} = 0.4580$.

We compared this optimal signal with the one obtained by solving the convex optimization problem. Although other parameterizations are possible, we use the following class of power spectra in this convex optimization [3]: $\Phi_r(\omega) =$

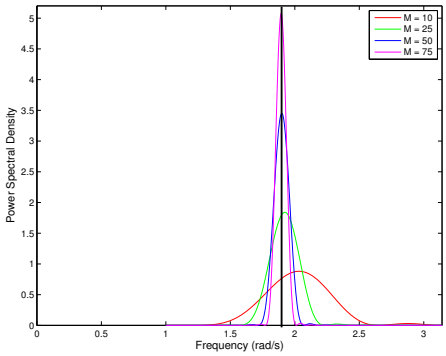

Fig. 2. Optimal excitation spectra for $M = 10$ (red), $M = 25$ (green), $M = 50$ (blue), and $M = 75$ (purple). The black vertical line is the analytical optimal spectrum. The parameter values for the simulations are $\theta_0 = 0.45$, $N = 200$, $a_1 = 10$, $\sigma_e^2 = 4.0$, and $R_{adm} = 1000$. Observe that when M is increased, the numerical spectra converge to the analytical solution.

$\sum_{k=-M}^{M} c_k e^{-i\omega k}$, where $c_{-k} = c_k$ are to-be-determined coefficients, and M the order. Note that, for finite M, this class does not contain all possible spectra and, in particular, does not contain the optimal spectrum (9).

The numerical solutions for $M = \{10, 25, 50, 75\}$ are depicted in Fig. 2. Observe that for small values of M, the spectra are relatively far from the optimal one: the Dirac pulse at $\omega^\dagger = 1.8903\ rad/s$. This discrepancy is also observed in the optimal identification cost that are respectively given by $J_{opt} = \{0.4601, 0.4584, 0.4581, 0.4580\}$ for these four values of M. We see that only for large values of M the cost converges to the optimal one, i.e., the one obtained with the sinusoid excitation. For $\dim(\theta_0) > 1$ the speed of convergence is likely to be slower.

Suppose we used the sinusoidal excitation signal $r(t) = 0.273\cos(0.5t)$. This signal has the same cost $J = J_{opt}$, but the certainty in the parameter is $P_\theta^{-1} = 414.12$, which is far less than the accuracy obtained with $r_{opt}(t)$. To obtain the same accuracy, the experiment length should be $N = 484$, which is more than double compared to the optimal case. A white noise excitation with variance $\sigma_r^2 = 0.09$ also yields a cost $J = J_{opt}$, but gives $P_\theta^{-1} = 601.34$ so that the accuracy constraint is not honoured. It does yield a better estimate than the non-optimal sinusoidal excitation as its spectrum also excites the optimal frequency.

We now compare the closed-loop setting and the open-loop setting. In the open-loop setting, the optimal excitation signal is $r_{opt}^{OL}(t) = 0.579\cos(\omega^\dagger t)$ with the same frequency $\omega^\dagger$ as in the closed-loop case, but with another amplitude. Since the controller reduces the effects of the disturbance on the output and given the relation (20), the optimal open-loop experiment leads to a signal y with a higher variance than the optimal closed-loop experiment, c.f. (20). Their values are $\mathscr{P}_{opt}^{CL} = 72.31$ versus $\mathscr{P}_{opt}^{OL} = 72.7$. The ratio of open- and closed-loop experiment cost is $f_{O/C} = 1.003$. However, if R_{adm} is chosen equal to $1.1R_0$ the differences between the open-loop and closed-loop costs are much larger: $f_{O/C} = 11$, showing that the open-loop setting is much more expensive.

As a last experiment, we change the objective/cost function to $\mathscr{L}(e^{-i\omega}) = 1\,\forall\omega$ (i.e. the experiment cost is the power of the excitation signal). As mentioned at the end of Section IV, it is not guaranteed that the closed-loop configuration is better than the open-loop one in this case. We in fact observe here that the cost of $r(t)$ is 19% higher in the closed-loop case.

VI. CONCLUSIONS

We have analytically calculated the optimal excitation spectrum that is required to identify a uni-parametric linear system in a least-costly manner, and shown that the solution is unique if (10) has a unique global maximum. This framework allows parameter estimation with a pre-specified accuracy in short experiment times, or high accuracy with the same experiment cost or length.

Our analytical solution addressed several fundamental questions about the shape of optimal signal and the selection of the optimal frequency. Using this solution, it has been shown that for $\mathscr{L}(e^{-i\omega}) \propto |S(e^{-i\omega})|^2$, the optimal excitation frequency is identical for a system being in an open or a closed loop, and that for this case a closed-loop experiment is always cheaper than an open-loop one. We have also derived explicit expressions for the output variance in open- and closed-loop systems and compared them. This is of interest in practice.

REFERENCES

[1] X. Bombois, B. Anderson, and G. Scorletti. Open-loop vs. closed-loop identification of box-jenkins systems in a least costly context. *European Control Conference*, pages 4510–4517, July 2007.

[2] X. Bombois, A.J. den Dekker, C.R. Rojas, H. Hjalmarsson, and P.M.J. Van den Hof. Optimal experiment design for hypothesis testing applied to functional magnetic resonance imaging. *18th IFAC World Congress*, pages 9953–9958, 2011.

[3] X. Bombois, G. Scorletti, M. Gevers, P.M.J. Van den Hof, and R. Hildebrand. Least costly identification experiment for control. *Automatica*, 42:1651–1662, 2006.

[4] H. Hjalmarsson. System identification of complex and structured systems. In *Proceedings of the European Control Conference 2009*, pages 3424–3452, Budapest, Hungary, 2009.

[5] C.A. Larsson, M. Annergren, and H. Hjalmarsson. On optimal input design in system identification for model predictive control. In *Proceedings of the 50th Conference on Decision and Control and European Control Conference (CDC-ECC)*, pages 805–810, Orlando, USA, 2011.

[6] L. Ljung. *Systems Identification: Theory for the User*. Prentice Hall, 1999.

[7] M. Mansoori, A. Dankers, and P.M.J. Van den Hof. Errors-in-variables identification in bilaterally coupled systems with application to oil well testing. In *Proceedings of the 19th IFAC World Congress*, Cape Town, South Africa, 2014.

[8] M.G. Potters, M. Forgione, X. Bombois, H. Hjalmarsson, P.E. Modén, M. Lundh, and P.M.J. Van den Hof. Optimal experiment design in closed loop with unknown, nonlinear or implicit controllers using stealth identification. *European Control Conference, Strasbourg*, 2014.

[9] The Autoprofit Project. www.fp7-autoprofit.eu.

[10] B. Wahlberg, M. Annergren, and C. Rojas. On optimal input signal design for identification of output error models. *50th IEEE Conference on Decision and Control, Orlando, FL, USA*, 2011.

[11] M. Zarrop. *Design for Dynamic System Identification*. Lecture Notes in Control and Inform. Sci. 21, Springer Verlag, Berlin, New-York, 1979.