REALIZATION ALGORITHMS FOR EXPANSIONS IN GENERALIZED ORTHONORMAL BASIS FUNCTIONS

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Abstract: In this paper a realization theory and associated algorithms are presented for the construction of minimal realizations on the basis of a sequence of expansion coefficients in a generalized orthonormal basis. Both the exact and the partial realization problem are addressed and solved, leading to extended versions of the classical Ho-Kalman algorithm.In the construction of the realization algorithms, fruitful use is made of a system analysis in the transform domain, being induced by the choice of basis functions. The resulting algorithms can also be applied in approximate realization and in system approximation.

Keywords: Orthogonal basis functions; discrete-time systems; system approximation; realization; identification.

1. INTRODUCTION

In recent years renewed attention has been paid to the development and use of orthonormal basis functions in system theory, and particularly in system identification. Considering linear system descriptions in terms of orthogonal basis functions expansions, linearly parameterized models can result by restricting the models to finite expansions. In this work the choice of basis functions is known to be rather crucial for determining the length of the expansion that is needed for arriving at accurate system descriptions. The flexibility that is available in the basis construction mechanism can essentially contribute to a fast convergence of the series expansion for a specific system.

For Laguerre functions (Wahlberg, 1991), this flexibility rests in the choice of a single pole location, while two pole locations can be fixed in the two-parameter Kautz functions (Kautz, 1954; Wahlberg, 1994). Generalized versions of these approaches have been developed by Heuberger et al. (1995) using repeated blocks of all-pass sections of user-chosen order, and have been analyzed for system identification purposes in Van den Hof et al. (1995). Ninness and Gustafsson (1997) have presented and analyzed an alternative structure where the need for repetition of all-pass sections has been removed.

Here we address the following problem: Consider a linear time-invariant system in the form of a series expansion

$$P(z) = \sum_{k=1}^{\infty} c_k F_k(z)$$

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where $\{F_k(z)\}_{k=1,2,\cdots}$ is a sequence of (orthonormal) basis functions in $\mathcal{H}_2$. The problem is to construct a minimal state space realization (A, B, C, D) of P on the basis of the sequence of coefficients $\{c_k\}_{k=1,\cdots,N}$.
For $N = \infty$ and $F_k(z) = z^{-k}$ the problem reduces to a minimal (exact) realization problem, as solved by the celebrated Ho-Kalman algorithm (Ho and Kalman, 1966). Using the same basis functions, and considering finite N, the corresponding minimal *partial* realization algorithm is analyzed and solved in Tether (1970), including the formulation of conditions under which the problem has a unique solution. In this latter situation, the required algorithm for obtaining a solution is basically the same Ho-Kalman algorithm as applied in the infinite data case.
Szabó and Bokor (1997) have extended the exact realization theory to the situation of Hambo basis functions, for the case of infinite data ($N = \infty$). In this paper it will be shown how these results connect to system descriptions in the related transform domain, and additionally the partial realization problem ($N < \infty$) will be addressed and solved, and consequences will be indicated also for the approximative case.
The paper is structured as follows: in section 2 the theory on generalized orthonormal basis functions is summarized. In sections 3 and 4 some results are formulated that are instrumental in handling the realization problems. The exact realization problem is addressed in section 5 and the partial problem in section 6. Some comments on the approximate realization problem conclude the paper.
Unless otherwise stated, all systems in this paper are scalar discrete time systems. Multivariable extensions are straightforward. In the discrete time domain the space $\mathcal{H}_2$ can either be defined to include only strictly proper systems or to include proper systems as well. In this paper we will use the latter definition. For brevity reasons all proofs are omitted, see (Heuberger et al., 1998).

2. ORTHONORMAL BASIS FUNCTIONS

In this section the main ingredients of the theory on orthonormal basis functions will be briefly reviewed. For details see Heuberger et al. (1995), Van den Hof et al. (1995), Heuberger and Van den Hof (1996), Ninness and Gustafsson (1997). The basis functions are constructed from state trajectories related to balanced realizations of square inner functions (i.e. stable all-pass systems). A transfer function $G_b \in \mathcal{H}_2$ is inner if it satisfies $|G_b(e^{i\omega})| = 1$ for all $\omega \in [-\pi, \pi]$. Given a balanced realization $\{A, B, C, D\}$ of an inner function $G_b(z)$, it is straightforward to show that the elements $\phi_1(z), \cdots, \phi_{n_b}(z)$ of the n_b dimensional vector function

$$V_1(z) = [zI - A]^{-1}B \in \mathcal{H}_2^{n_b} \qquad (1)$$

are mutually orthonormal in the standard $\mathcal{H}_2$ inner product sense. Furthermore consecutive multiplication of these functions with $G_b(z)$ results in an orthonormal set, given by the components $\{\phi_{(k-1)n_b+1}, \cdots, \phi_{kn_b}\}$ of the n_b dimensional vector functions

$$V_k(z) = [zI - A]^{-1}BG_b^{k-1}(z) \qquad k \in \mathbb{N}$$

and it can be shown that the set of all these functions $\{\phi_i(z)\}_{i=1}^{\infty}$ constitutes a basis for the strictly proper part of $\mathcal{H}_2$.
Directly resulting from the basis for strictly proper stable systems in $\mathcal{H}_2$, a basis $\{v_k(t)\}$ for the related Hilbert space of $\ell_2[1, \infty)$-signals follows, by considering the inverse z-transforms of the $\mathcal{H}_2$-signals. These functions are the impulse responses of the functions $V_k(z)$ with the property:

$$v_1(t) = A^{t-1}B \qquad (2)$$

$$v_{k+1}(t) = (G_b(q) \cdot I)v_k(t). \qquad (3)$$

To avoid confusion these functions will also be denoted by

$$v_k(t)^{\{A,B,C,D\}} \qquad (4)$$

Considering this general class of basis functions, for any strictly proper system $H(z) \in \mathcal{H}_2$ there exists a unique series expansion:

$$H(z) = \sum_{k=1}^{\infty} L_k^T V_k(z) \qquad L_k \in \mathbb{R}^{n_b \times 1}. \quad (5)$$

Specific choices of $G_b(z)$ lead to well known classical basis functions, such as the standard pulse basis $V_k(z) = z^{-k}$, the Laguerre basis and the two-parameter Kautz functions (Kautz, 1954; Wahlberg, 1994), see Heuberger et al. (1995).
These basis functions give rise to a general theory on system transformations induced by these so-called Hambo basis functions, such that a system $P(z)$ admits an alternative description, denoted by $\widetilde{P}(\lambda)$ in the transform domain. For details see (Heuberger and Van den Hof, 1996). This function $\widetilde{P}(\lambda)$ can be expressed through variable transformations, using the $n_b \times n_b$ inner function

$$N(z) := A + B(z - D)^{-1}C \qquad (6)$$

with McMillan degree 1 and balanced realization (D, C, B, A). If we write $P(z) = \sum_{k=1}^{\infty} p_k z^{-k}$, then

$$\widetilde{P}(\lambda) = \sum_{k=1}^{\infty} p_k N^k(\lambda)$$

or differently denoted: $\widetilde{P}(\lambda) = P(z)|_{z^{-1}=N(\lambda)}$.
The mapping T: $\mathcal{H}_2 \rightarrow \mathcal{H}_2^{n_b \times n_b}$ defined by $\mathsf{T}(P) := \widetilde{P}(\lambda)$ is referred to as the *Hambo system-transform.*
From a signal processing point of view this alternative representation of systems fits into the H-matrix representation framework as discussed by Audley and Rugh (1973).

Given this background material the problem addressed in this paper can be formulated as follows:

Problem 1. Given a system $G(z)$ and an orthonormal basis $\{V_k(z)\}_{k=1}^{\infty}$, such that $G(z) = \sum_{k=1}^{\infty} L_k^T V_k(z)$ construct an algorithm to derive a minimal state space realization of $G(z)$, based on the expansion sequence $\{L_1, \cdots, L_N\}$.

This problem will be approached through the Hambo system transform, revealing the relation between the expansion coefficients $\{L_k\}$ and the Markov parameters $\{M_k\}$ of the transformed system $\widetilde{G}(\lambda)$ and thus creating the Hankel matrix of the transformed system, denoted by $\widetilde{H}$. Furthermore the relation between $\widetilde{H}$ and the Hankel matrix of $G(z)$, denoted by H, will be established. Equivalent relations are given for the shifted system $\overleftarrow{G}(z) := zG(z)$ and its Hambo transform $\overleftarrow{\widetilde{G}}(z)$, with Hankel matrices $\overleftarrow{H}$ respectively $\overleftarrow{\widetilde{H}}$. In scheme:

$$\boxed{\{L_k\} \begin{array}{llll} \nearrow \{M_k\} & \longrightarrow & \widetilde{H} \sim & H \\ \searrow \{\overleftarrow{M}_k\} & \longrightarrow & \overleftarrow{\widetilde{H}} \sim & \overleftarrow{H} \end{array}}$$

The solution of the problem is based on an extension of the Ho-Kalman algorithm and the solution of the classical minimal partial realization problem, given by Tether (1970).

3. THE HAMBO TRANSFORM OF BASIS FUNCTIONS

The key property of the basis functions, that will be used in this paper, is the fact that the Hambo transform of the elements of $V_1(z)$ can be calculated explicitly, as stated in the following proposition:

Proposition 2. Let the vector valued function $V_1(z) = \left[\, \phi_1(z)\ \phi_2(z)\ \cdots\ \phi_{n_b}(z) \,\right]^T$ be given by (1). Then for $1 \leq k \leq n_b$ there exist matrices $P_k, Q_k \in \mathbb{R}^{n_b \times n_b}$ such that the Hambo system transform of $\phi_k(z)$ satisfies

$$\widetilde{\phi_k}(\lambda) = P_k + Q_k \lambda^{-1} \qquad 1 \leq k \leq n_b.$$

where the matrices $\{P_k, Q_k\}$ are the solution of the following set of Sylvester equations:

$$\begin{aligned} AP_kA^T + Ae_kB^T &= P_k \\ AQ_kA^T + BCe_kB^T + BCP_k^TA^T &= Q_k \end{aligned}$$

and where e_k is the k^{th} Euclidean unit vector.

To facilitate the notation in the following sections we introduce a compact form for the set of all matrices $\{P_k, Q_k\}$:

$$\mathcal{P} := \left[\, P_1^T\ P_2^T\ \cdots\ P_{n_b}^T \,\right]^T \in \mathbb{R}^{n_b \times n_b^2} \qquad (7)$$

$$\mathcal{Q} := \left[\, Q_1^T\ Q_2^T\ \cdots\ Q_{n_b}^T \,\right]^T \in \mathbb{R}^{n_b \times n_b^2} \qquad (8)$$

4. HANKEL MATRICES IN THE TRANSFORM DOMAIN

In this section it will be shown how the material of the previous sections allows the calculation of the Hambo system transform, given an expansion in terms of orthonormal basis functions. The system transform will be expressed in terms of Markov parameters, connecting directly to the Hankel matrix of the system transform. Hence, given an expansion $G(z) = \sum_{k=1}^{\infty} L_k^T V_k(z)$ we will show how this can be translated into an expansion $\widetilde{G}(\lambda) = \sum_{k=0}^{\infty} M_k \lambda^{-k}$. The result is given in terms of the notation (7,8), and $\otimes$ denoting Kronecker product.

Proposition 3. $\widetilde{G}(\lambda) = \sum_{k=0}^{\infty} M_k \lambda^{-k}$, *where*

$$M_0 = [L_1^T \otimes I]\mathcal{P} \qquad (9)$$

$$M_k = [L_{k+1}^T \otimes I]\mathcal{P} + [L_k^T \otimes I]\mathcal{Q} \qquad (10)$$

In the course of the paper we will need a similar expression for the Hambo transform of the *shifted* function $\overleftarrow{G}(z) := zG(z)$:

Proposition 4. $\overleftarrow{\widetilde{G}}(\lambda) = \sum_{k=0}^{\infty} \overleftarrow{M}_k\, \lambda^{-k}$, *where*

$$\overleftarrow{M}_0 = (L_1^T B)I + [(L_1^T A) \otimes I]\mathcal{P} \qquad (11)$$

$$\begin{aligned} \overleftarrow{M}_k &= (L_{k+1}^T B)I + [(L_{k+1}^T A) \otimes I]\mathcal{P} \\ &\quad + [(L_k^T A) \otimes I]\mathcal{Q} \end{aligned} \qquad (12)$$

The latter two Propositions show how the set of expansion coefficients can be efficiently transformed into Markov parameters of the Hambo system transform of the (shifted) transfer function, which immediately results in the Hankel matrices $\widetilde{H}$ and $\overleftarrow{\widetilde{H}}$.
This section will be concluded with establishing the connection between the Hankel matrices in the transform domain $(\widetilde{H}, \overleftarrow{\widetilde{H}})$ and the corresponding Hankel matrices in the standard domain $(H, \overleftarrow{H})$.

This connection turns out to be determined by a set of transformation matrices, that are derived from the all-pass function $G_b(z)$.

Proposition 5. Given a system $G(z)$ and its Hambo transform $\widetilde{G}(\lambda)$ with Hankel matrices H respectively $\widetilde{H}$, then there exist unitary matrices T_1, T_2 such that $H = T_1^{-1}\widetilde{H}T_2 = T_1^T\widetilde{H}T_2$, with

$$(T_1)_{ij} = v_i(j)^{\{A,B,C,D\}} \tag{13}$$

$$(T_2)_{ij} = v_i(j)^{\{A^T,C^T,B^T,D^T\}} \tag{14}$$

where $v_i(j)^{\{F,G,H,J\}}$ is defined by eq. (2-4).

This property shows that the Hankel matrices in both domains have equal rank and that any full rank decomposition of $\widetilde{H}$ immediately results in a full rank decomposition of H. This property will be of use in the generalization of the Ho-Kalman realization algorithm, discussed next.

5. EXACT REALIZATION

In this section the generalization of the Ho-Kalman algorithm will be discussed, generalized in terms of the GOBF basis. It is assumed that the Hankel matrices involved are fully known, i.e. the matrices are of infinite dimension, or equivalently all Markov parameters $\{G_i\}_{i=1}^{\infty}$ are known. This problem has a solution if and only if there exist integers N', N such that $\text{rank}(H_{N',N}) = \text{rank}(H_{N'+i,N+j}) = n_0$ for all $i, j = 0, 1, 2, \cdots$, where

$$H_{r,c} = \begin{bmatrix} G_1 & G_2 & \cdots & G_c \\ G_2 & G_3 & \cdots & G_{c+1} \\ \vdots & \vdots & & \vdots \\ G_r & G_{r+1} & \cdots & G_{r+c-1} \end{bmatrix} \tag{15}$$

and in this case n_0 is the minimal state space dimension. Under the assumption that the unknown system is finite dimensional, a minimal state space realization can be determined using the Ho-Kalman algorithm:

Algorithm 1. (Ho-Kalman). Given infinite Hankel matrices $H, \overleftarrow{H}$, with elements $H_{ij} = G_{i+j-1}$ and $\overleftarrow{H}_{ij} = G_{i+j}$, where $\{G_k\}$ is the set of Markov parameters of a finite dimensional system $G(z)$, the following steps result in a minimal state space realization $\{A_g, B_g, C_g\}$ of $G(z)$:

(1) Let $H = \Gamma \cdot \Delta$ be a full rank decomposition , i.e. rank(Γ)=rank(Δ)=rank(H).
(2) Then $\overleftarrow{H}$ obeys the relation $\overleftarrow{H} = \Gamma \cdot A_g \cdot \Delta$.
(3) Hence $A_g = \Gamma^+ \cdot \overleftarrow{H} \cdot \Delta^+$, with $(\cdot)^+$ indicating the Moore-Penrose pseudo-inverse.
(4) Furthermore B_g and C_g are created from
$$B_g = \Delta \begin{bmatrix} I_m \\ 0 \end{bmatrix} \text{ and } C_g = \begin{bmatrix} I_p & 0 \end{bmatrix} \Gamma.$$

The extended minimal realization problem addressed here is given in Problem 1, with $N = \infty$. For the solution of this problem explicit use is made of the transformation property $H = T_1^T\widetilde{H}T_2$ and $\overleftarrow{H} = T_1^T\overleftarrow{\widetilde{H}}T_2$, given by Proposition 5. Substituting these relations in Algorithm 1 results in a realization algorithm based on the Markov parameters in $\widetilde{H}$ and $\overleftarrow{\widetilde{H}}$. The missing link is then the relation between the expansion coefficients $\{L_k\}$ and these Hankel matrices. That relation is established by Propositions 3 and 4.
This approach leads to the following algorithm:

Algorithm 2. (Generalized Ho-Kalman). Given a series expansion $G(z) = \sum_{k=0}^{\infty} L_k^T V_k(z)$, the following steps result in a minimal state space realization $\{A_{\tilde{g}}, B_{\tilde{g}}, C_{\tilde{g}}\}$ of $G(z)$:

(1) Calculate with Propositions 3 and 4 the Markov parameters $\{M_k\}, \{\overleftarrow{M}_k\}$ of the Hambo system transforms $\widetilde{G}(\lambda), \overleftarrow{\widetilde{G}}(\lambda)$.
(2) Create Hankel matrices $\widetilde{H}, \overleftarrow{\widetilde{H}}$, with elements $\widetilde{H}_{ij} = M_{i+j-1}$ and $\overleftarrow{\widetilde{H}}_{ij} = \overleftarrow{M}_{i+j-1}$.
(3) Let $\widetilde{H} = \widetilde{\Gamma} \cdot \widetilde{\Delta}$ be a full rank decomposition of $\widetilde{H}$.
(4) Then $\overleftarrow{\widetilde{H}}$ obeys the relation $\overleftarrow{\widetilde{H}} = \widetilde{\Gamma} \cdot A_{\tilde{g}} \cdot \widetilde{\Delta}$
(5) Hence $A_{\tilde{g}}$ obeys $A_{\tilde{g}} = \widetilde{\Gamma}^+ \cdot \overleftarrow{\widetilde{H}} \cdot \widetilde{\Delta}^+$
(6) Furthermore $B_{\tilde{g}}$ and $C_{\tilde{g}}$ are created from
$$B_{\tilde{g}} = \widetilde{\Delta} T_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } C_{\tilde{g}} = \begin{bmatrix} 1 & 0 \end{bmatrix} T_1^T \widetilde{\Gamma}$$

An important feature here is the fact that only one row c.q. column of the transformation matrices T_1^T, T_2 is required to calculate the state space realization. In general the calculation of $B_{\tilde{g}}$ and $C_{\tilde{g}}$ involves infinite dimensional matrix calculations. In the special case that $\widetilde{G}(\lambda)$ is a finite impulse response model, this will be reduced to finite operations. This occurs when the underlying system $G(z)$ has only a finite number of non-zero expansion coefficients: $G(z) = \sum_{k=1}^{N} L_k^T V_k(z)$. For numerical examples of this application see (Szabó and Bokor, 1997).

6. PARTIAL REALIZATION

In this section the generalization of the so-called partial realization problem is described and a solution to this problem is presented. The partial realization problem (Tether, 1970) deals with the case of limited information, i.e. only a finite number of Markov-parameters is known. The problem is the construction of a finite dimensional minimal state space realization that fits these Markov parameters. The *minimal* partial realization problem aims at finding such a realization with a minimal McMillan degree over all possible realizations. The following lemma gives conditions for the existence of a unique minimal partial realization.

Lemma 6. (Tether, 1970). Let $\{Y_1, \cdots, Y_{N_0}\}$ *be an arbitrary sequence of* $p \times m$ *matrices and let* $H_{i,j}, i + j \leq N_0$*, be a corresponding block Hankel matrix. Then a minimal partial realization given by the Ho-Kalman algorithm is unique (modulo similarity transformations) if and only if there exist positive integers* N', N *such that*

(1) $N' + N = N_0$
(2) $rank(H_{N',N}) = rank(H_{N'+1,N}) = rank(H_{N',N+1})$*.*

Generalizing this property to the GOBF case is more involved than in the exact realization case, because proposition 5 is only valid for infinite Hankel matrices and not for finite matrices. The key idea to overcome this problem is to extend these finite matrices to infinite dimensions, using the Ho-Kalman algorithm *in the transform domain.* This will result in (minimal) realizations of the underlying systems $\widetilde{G}(z)$ and $\widetilde{\overleftarrow{G}}(z)$, which are guaranteed to be unique under rank conditions given by lemma 6. Since a minimal realization for $\widetilde{G}(z), \widetilde{\overleftarrow{G}}(z)$ automatically gives a full rank decomposition for the associated infinite Hankel matrices in terms of the product of the observability and controllability matrices

$$\widetilde{H} = \widetilde{\Gamma} \cdot \widetilde{\Delta} \qquad \widetilde{\overleftarrow{H}} = \widetilde{\overleftarrow{\Gamma}} \cdot \widetilde{\overleftarrow{\Delta}}$$

we can apply Algorithm 1 to obtain

$$\widetilde{\overleftarrow{H}} = \widetilde{\overleftarrow{\Gamma}} \cdot \widetilde{\overleftarrow{\Delta}} = \widetilde{\Gamma} \cdot A_g \cdot \widetilde{\Delta} \rightarrow A_{\tilde{g}} = \widetilde{\Gamma}^{+} \widetilde{\overleftarrow{\Gamma}} \widetilde{\overleftarrow{\Delta}} \widetilde{\Delta}^{+},$$

which expression can be calculated using Sylvester and Lyapunov equations. Along the same line of reasoning the matrices B_g and C_g are derived. This leads to the following algorithm:

Algorithm 3. (Generalized Minimal Partial Realization). Let $\{L_k\}_{k=1}^{N_0}$ *be the first* N_0 *expansion coefficients of a scalar system* $G(z)$ *and let* $\{M_k, \overleftarrow{M}_k\}_{k=0}^{N_0-1}$ *be the Markov parameters of the Hambo system transform of* $G(z)$ *respectively its shift* $\overleftarrow{G}(z)$*, as defined by Propositions 3 and 4. Assume that both* $\{\overleftarrow{M}_k\}_{k=1}^{N_0-1}$ *and* $\{M_k\}_{k=1}^{N_0-1}$ *satisfy the conditions of Lemma 6. Then a unique minimal state space realization* $\{A_{\tilde{g}}, B_{\tilde{g}}, C_{\tilde{g}}\}$ *of G(z) is obtained as follows:*

(1) Use the Ho-Kalman algorithm to create minimal state space realizations $\{\widetilde{A}, \widetilde{B}, \widetilde{C}\}$ *and* $\{\widetilde{\overleftarrow{A}}, \widetilde{\overleftarrow{B}}, \widetilde{\overleftarrow{C}}\}$*, such that for* $1 \leq k \leq N_0$

$$\overleftarrow{M}_k = \widetilde{\overleftarrow{C}} \widetilde{\overleftarrow{A}}^{k-1} \widetilde{\overleftarrow{B}} \qquad M_k = \widetilde{C} \widetilde{A}^{k-1} \widetilde{B}$$

(2) Observe that the infinite Hankel matrices $\widetilde{H}, \widetilde{\overleftarrow{H}}$*, constituted by these realizations have full rank decompositions*

$$\widetilde{H} = \widetilde{\Gamma} \cdot \widetilde{\Delta} := \begin{bmatrix} \widetilde{C} \\ \widetilde{C}\widetilde{A} \\ \vdots \end{bmatrix} \begin{bmatrix} \widetilde{B} & \widetilde{A}\widetilde{B} & \cdots \end{bmatrix}$$

$$\widetilde{\overleftarrow{H}} = \widetilde{\overleftarrow{\Gamma}} \cdot \widetilde{\overleftarrow{\Delta}} := \begin{bmatrix} \widetilde{\overleftarrow{C}} \\ \widetilde{\overleftarrow{C}}\widetilde{\overleftarrow{A}} \\ \vdots \end{bmatrix} \begin{bmatrix} \widetilde{\overleftarrow{B}} & \widetilde{\overleftarrow{A}}\widetilde{\overleftarrow{B}} & \cdots \end{bmatrix}$$

(3) Apply the generalized Ho-Kalman algorithm 2 (step 5 and 6) to derive

$$\begin{aligned} A_{\tilde{g}} &= \widetilde{\Gamma}^{+} \widetilde{\overleftarrow{\Gamma}} \widetilde{\overleftarrow{\Delta}} \widetilde{\Delta}^{+} \\ &= (\widetilde{\Gamma}^T \widetilde{\Gamma})^{-1} (\widetilde{\Gamma}^T \widetilde{\overleftarrow{\Gamma}}) (\widetilde{\overleftarrow{\Delta}} \widetilde{\Delta}^T) (\widetilde{\Delta} \widetilde{\Delta}^T)^{-1} \\ &= R_1^{-1} R_2 S_2 S_1^{-1} \\ B_{\tilde{g}} &= \widetilde{\Delta} T_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = R_3 \\ C_{\tilde{g}} &= \begin{bmatrix} 1 & 0 \end{bmatrix} T_1^T \widetilde{\Gamma} = R_4 \end{aligned}$$

using the Lyapunov/Sylvester equations:

$$\begin{aligned} R_1 &= \widetilde{A}^T R_1 \widetilde{A} + \widetilde{C}^T \tilde{C} \\ R_2 &= \widetilde{A}^T R_2 \widetilde{\overleftarrow{A}} + \widetilde{C}^T \widetilde{\overleftarrow{C}} \\ S_1 &= \widetilde{A} S_1 \widetilde{A}^T + \widetilde{B} \widetilde{B}^T \\ S_2 &= \widetilde{\overleftarrow{A}} S_2 \widetilde{A}^T + \widetilde{\overleftarrow{B}} \widetilde{B}^T \\ R_3 &= \widetilde{A}^T R_3 D + \widetilde{B} C^T \\ R_4 &= D^T R_4 \widetilde{A} + B^T \widetilde{C}. \end{aligned}$$

Given this algorithm, the question will arise why and when it would be of use. This will typically be the case in an approximation or identification setting, using an orthonormal basis function approach. One of these situations is where a finite number of expansion coefficients is estimated, i.e. estimation of the parameters $\{L_k\}$ in a model structure

$$y(t) = \sum_{k=1}^{N} L_k^T V_k(q) u(t) + e(t),$$

and the number of significant coefficients L_k is relatively high, such that a direct state space realization would result in a high order model. One approach in this case would be to apply model reduction techniques to obtain a lower order model. However, such a procedure will not make use of the intrinsic information contained in the expansion coefficients or the directly related Markov parameters of the transformed system.

The merit of Algorithm 3 is that it makes full use of this information and hence will improve the quality of the resulting approximation. It will furthermore give much more insight in the McMillan degree of the underlying system.

It is important to note that the presented partial realization algorithm intrinsically requires two applications of the standard Ho-Kalman mechanism. In the classical case this can be reduced to one application, because there a realization of $\{M_k\}_{k=1}^{N_0}$ immediately results in a realization of $\{\overleftarrow{M}_k\}_{k=1}^{N_0}$. Also in the generalized case these sets with Markov parameters are obviously closely related and it might be possible to further simplify the algorithm.

7. APPROXIMATE REALIZATION

The generalized algorithms can be applied in an approximate fashion as is the case with the standard Ho-Kalman algorithm (Kung, 1978). Most commonly the full rank decomposition of $\widetilde{H}$ will be computed by means of a singular value decomposition. One can then simply truncate the SVD by setting the smaller singular values to zero, and proceed as in the non-approximative case. When algorithm 2 is applied in an approximate sense, in the situation where all expansion coefficients are known, the resulting realization will be exactly the same as the one obtained by application of the Ho-Kalman algorithm in the original domain. This is caused by the unitary transformations in Proposition 5.

The situation becomes different in the case where only a finite number of Markov parameters is given. Note that in that partial realization case we need only to truncate the SVD of $\widetilde{H}$ and not that of $\overleftarrow{\widetilde{H}}$. The consequences of this method when compared to the standard partial realization algorithm still have to be further explored.

8. CONCLUSIONS

The problem has been addressed of constructing a minimal state space realization on the basis of a sequence of expansion coefficients in a generalized orthonormal basis function expansion. The classical Ho-Kalman algorithm, designed for minimal realization on the basis of Markov parameters, has been extended for expansions in a general class of basis functions. Whereas in the classical situation one algorithm essentially solves both the exact (infinite data) and the partial (finite data) problem, in the generalized case different algorithms result. The presented algorithms can also be used for constructing reduced order state space models on the basis of estimated expansion coefficients.

9. REFERENCES

Audley, D.R. and W.J. Rugh (1973). On the H-matrix system representation. *IEEE Trans. Autom. Control, 18*, 235-242.

Heuberger, P.S.C., P.M.J. Van den Hof and O.H. Bosgra (1995). A generalized orthonormal basis for linear dynamical systems. *IEEE Trans. Autom. Control, AC-40*, 451–465.

Heuberger, P.S.C. and P.M.J. Van den Hof (1996). The Hambo transform: a signal and systems transform induced by generalized orthonormal basis functions. In: *Prepr. 13th Triennial IFAC World Congress*, San Francisco, 357-362.

Heuberger, P.S.C., Szabó, Z., de Hoog, T.J., Van den Hof, P.M.J. and J. Bokor (1998). Realization algorithms for expansions in generalized orthonormal basis functions. *Internal report N-225*, Mechanical Engineering Systems and Control Group, Delft University of Technology.

Ho, B.L., and R.E. Kalman (1966). Effective construction of linear state-variable models from input-output functions. *Regelungstechnik, vol 14*, 545-548.

Kautz, W.H. (1954). Transient synthesis in the time domain. *IRE Trans. Circ. Th., CT-1*, 29–39.

Kung, S. (1978). A new identification and model reduction algorithm via singular value decompositions. *Proc 12th Asilomar Conf. Circuits Syst. and Computers*, Pacific Grove, CA, Nov. 6-8, 1978, 705-714.

Ninness, B.M. and F. Gustafsson (1997). A unifying construction of orthonormal bases for system identification. *IEEE Trans. Autom. Control, AC-42*, 515-521.

Szabó, Z. and J. Bokor (1997). Minimal state space realization for transfer functions represented by coefficients using generalized orthonormal basis. In: *Proceedings 36th IEEE Conference on Decision and Control*, San Diego, Ca, 169-174.

Tether, A.J., (1970). Construction of minimal linear state-variable models from finite input-output data. *IEEE Trans. Autom. Control, AC-15*, 427-436.

Van den Hof, P.M.J., P.S.C. Heuberger and J. Bokor (1995). System identification with generalized orthonormal basis functions. *Automatica, 31*, 1821-1834.

Wahlberg, B. (1991). System identification using Laguerre models. *IEEE Trans. Autom. Control, AC-36*, 551–562.

Wahlberg, B. (1994). System identification using Kautz models. *IEEE Trans. Autom. Control, AC-39*, 1276–1282.