APPROXIMATION AND REALIZATION USING GENERALIZED ORTHONORMAL BASES

P.S.C. Heuberger, T.J. de Hoog and P.M.J. Van den Hof

Mechanical Engineering Systems and Control Group
Delft University of Technology
Mekelweg 2, 2628 CD, Delft, the Netherlands
Fax : +31 15 2784717 and email: p.s.c.heuberger@wbmt.tudelft.nl

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Abstract

This paper considers the approximation of linear systems by means of orthonormal basis functions, which are generated by stable all-pass functions. These basis functions induce the so called *Hambo* transform, which transforms scalar systems into square systems of i/o dimension equal to the order of the all-pass function considered. We will consider the construction of the Markov parameters of the system representation in the transform domain and show how these can be used to realize minimal state space representations for the exact and partial knowledge case. Additionally a projection mechanism is presented to allow inverse transformation of any sequence of Markov parameters in the transform domain. This mechanism is illustrated with an example.

1 Introduction

Any inner (stable all-pass) function $G_b(z)$ with McMillan degree n_b and minimal, balanced realization (A, B, C, D) generates a set of vector functions

$$V_k = (zI - A)^{-1}BG_b^{k-1}(z),\ k \in \mathbb{N} \tag{1}$$

whose scalar elements constitute an orthonormal basis of the space of stable, strictly proper, linear time invariant (LTI), transfer functions (which we will denote by H_2 in the following) [1]. The all-pass and balancedness properties can be summarized by:

$$\begin{bmatrix} A^T & C^T \\ B^T & D^T \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_{n_b} & 0 \\ 0 & 1 \end{bmatrix} \tag{2}$$

Systems satisfying this relation are also called orthogonal systems [6].

The functions $V_k(z)$ will be called the system based orthonormal basis functions in the following. This basis construction by means of all-pass functions (or generalized shift-operators) can be seen as a generalization of the well-known Laguerre and Kautz bases. Laguerre functions, for instance, are generated by the all-pass function with realization $\{a, \sqrt{1-a^2}, \sqrt{1-a^2}, -a\}$, with a the Laguerre parameter (or time-scaling factor). Also the standard basis z^{-k}, $k \in \mathbb{N}$ can easily be seen to be a special case of this more general construction, in which case the generating all-pass function is simply z^{-1}.

Orthonormal basis functions play a fundamental role in the analysis and approximation of systems and signals. The recent surge of interest in wavelets serves as an illustration of this. Interest in the system based orthogonal basis functions lies mainly in the field of system approximation and identification [4, 12, 10]. In that case a model of the form $\hat{G}(z) = \sum_{k=1}^{n} L_k^T V_k(z)$ is used, where L_k are the model parameters which are estimates of the expansion coefficients of the system $G(z)$. The application of these basis functions in the area of system identification has several important benefits. Firstly, it gives the possibility of incorporating prior knowledge about system dynamics into the model structure, which will ensure that the number of coefficients in the basis function expansion to be estimated can be kept relatively small while giving a small approximation error (bias). This sparsity of coefficients implies a favorable variance to bias ratio for the estimate. Secondly the model structure is linearly parametrized and with a least squares prediction error identification criterion this yields a convex optimization problem which can be solved analytically.

Important for the analysis of this identification method are the signal and system transforms that are induced by the basis functions, similar to the Laguerre transform [5]. In [10] properties of these so called *Hambo* transforms were fruitfully used to derive variance expressions for the basis function model estimate, much like the expressions for the standard FIR modeling case. It was shown that the variance can be kept relatively small in any desired frequency range by appropriately choosing the basis functions.

The Hambo system transform translates operations on time sequences into operations on the coefficient sequences

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(or Hambo signal transforms) pertaining to the expansions of these sequences in the generalized basis. In this sense it transfers system analysis from the "time" domain to an (abstract) transform domain. For example, the question how to construct a minimal state-space realization of the model $\hat{G}$ on the basis of the $\{L_k\}_{k=1..n}$ can be shown to come down to the construction of the Markov parameters of the transformed system. In [7, 8] it was shown that these so called *generalized* Markov parameters can be computed directly from the coefficients L_k. In this paper we will use the Hambo transform properties to derive similar results. In addition this derivation yields some very efficient methods for computing the generalized Markov parameters.

A difficulty with the Hambo system transform lies in the fact that it maps from H_2 into a proper subspace of $H_2^{n_b \times n_b}$. Applying manipulations such as model reduction to a Hambo transformed system might take us out of this subspace, making it impossible to transform back to H_2, using the inverse transform relations. A characterization of the image space is needed in order to be able to restrict to it, for instance by means of (orthogonal) projection. As we will show the generalized Markov parameters and their dependency on the coefficient sequence opens up a new way to project onto the subspace of Hambo system transforms.

First we will briefly review the Hambo transform and its principal properties.

2 The Hambo transform

Since the functions $V_k(z)$, $k \in \mathbb{N}$ constitute an orthonormal basis of the space H_2 of stable, strictly proper transfer functions we can expand any element $x(z)$ of that space in terms of this basis, as follows:

$$x(z) = \sum_{k=1}^{\infty} L_k^T V_k(z), \tag{3}$$

where equality is in the sense of the 2-norm and $L_k \in \mathbb{R}^{n_b}$. At the same time the pulse responses of the $V_k(z)$ will constitute an orthonormal basis of $\ell_2[1,\infty)$. Equivalently we can therefore write

$$x(t) = \sum_{k=1}^{\infty} L_k^T V_k(t), \tag{4}$$

where $x(t)$ and $V_k(t)$ denote the time-domain versions of $x(z)$ and $V_k(z)$.

We now define the Hambo *signal* transform of $x(z) \in H_2$ as the z-transform of the sequence $\{L_k\} \in \ell_2^{n_b}[1,\infty)$, with z replaced by λ to avoid confusion.

Definition 1 *The Hambo* signal *transform of a signal $x(t) \in \ell_2[1,\infty)$ is given by*

$$\tilde{x}(\lambda) = \sum_{k=1}^{\infty} L_k \lambda^{-k}. \tag{5}$$

Suppose that we have a signal $u(t) \in \ell_2[1,\infty)$ and a stable, finite dimensional, strictly proper transfer function $G(z)$. Let $y(z)$ be generated from $u(z)$ and $G(z)$ according to $y(z) = G(z)u(z)$. It can be shown that there exists a finite dimensional, stable transfer function $\tilde{G}(\lambda)$ of i/o dimension $n_b \times n_b$ that satisfies

$$\tilde{y}(\lambda) = \tilde{G}(\lambda)\tilde{u}(\lambda). \tag{6}$$

Proof is by construction and can be found in [1, 2].

Definition 2 *Given any system $G(z) \in H_2$ that satisfies $y(z) = G(z)u(z)$, then its Hambo* system *transform $\tilde{G}(\lambda)$ is the transfer matrix of i/o dimension $n_b \times n_b$ that satisfies*

$$\tilde{y}(\lambda) = \tilde{G}(\lambda)\tilde{u}(\lambda), \tag{7}$$

with $\tilde{y}(\lambda), \tilde{u}(\lambda)$ the Hambo signal transforms of $y(z)$ and $u(z)$.

The transform operation is linear and satisfies $\widetilde{F(z)G(z)} = \tilde{F}(\lambda)\tilde{G}(\lambda)$. The transformed system $\tilde{G}(\lambda)$ shares some interesting properties with $G(z)$ such as McMillan degree, H_∞ and Hankel norm [1]. An intriguing result is that it can be obtained from $G(z)$ by simple variable substitution:

$$\tilde{G}(\lambda) = G(z)|_{z^{-1}=N(\lambda)} \tag{8}$$

with $N(\lambda)$ the all-pass function given by

$$N(\lambda) = A + B(\lambda I - D)^{-1}C. \tag{9}$$

Another property of the transform that we will need later on is the fact that if the poles of the transfer function $G(z)$ are contained in the set of poles of $G_b(z)$, then its Hambo transform satisfies the 2-parameter relation [1]:

$$\tilde{G}(\lambda) = W_0 + W_1\lambda^{-1}, \tag{10}$$

with W_0 and W_1 matrices of dimension $n_b \times n_b$. As a special case it holds that

$$\tilde{G}_b(\lambda) = \lambda^{-1}. \tag{11}$$

Finally we will mention the relation by which the inverse Hambo system transform can be calculated

$$V_1^T(z)BG(z) = V_1^T(z)\,\tilde{G}(\lambda)\Big|_{\lambda^{-1}=G_b} B \tag{12}$$

We can of course also write $\tilde{G}(\lambda)$ in terms of its Markov parameters:

$$\tilde{G}(\lambda) = \sum_{k=0}^{\infty} M_k \lambda^{-k}. \tag{13}$$

The M_k will be called the *generalized* Markov parameters in the following. We will come back later to the fact that in general the Hambo transform is not strictly proper but only proper (so $M_0 \neq 0$). But first we will show how we can derive the M_k by using the aforementioned properties of the Hambo transform.

3 Generalized Markov parameters

As stated earlier we, can write any stable, strictly proper transfer function $G(z) \in H_2$ as

$$G(z) = \sum_{k=1}^{\infty} L_k^T V_1(z) G_b(z)^{k-1}. \quad (14)$$

We will now split up the vector valued $V_1(z)$ into n_b scalar basis functions which we will denote $\phi_i(z) = e_i^T V_1(z)$, $i = 1...n_b$, with e_i the i-th Euclidean basis vector. Accordingly we split up the vector valued coefficient vectors L_k into their components l_k^i. This gives us the relation

$$G(z) = \sum_{k=1}^{\infty} \sum_{i=1}^{n_b} l_k^i \phi_i(z) G_b(z)^{k-1}. \quad (15)$$

From this relation and the properties of the Hambo system transform it follows that:

$$\tilde{G}(\lambda) = \sum_{k=1}^{\infty} \sum_{i=1}^{n_b} l_k^i \tilde{\phi}_i(\lambda) \lambda^{-(k-1)}, \quad (16)$$

with $\tilde{\phi}_i(\lambda)$ denoting the Hambo *system* transform of $\phi_i(z)$. The $\tilde{\phi}_i(\lambda)$ can be written in terms of a 2-parameter relation because we know the poles of $\phi_i(z)$ are contained in the set of poles of $G_b(z)$. Hence we get:

$$\tilde{G}(\lambda) = \sum_{k=1}^{\infty} \sum_{i=1}^{n_b} l_k^i \left(P_i + Q_i \lambda^{-1}\right) \lambda^{-(k-1)}, \quad (17)$$

which immediately gives us the desired Markov parameters:

$$M_k = \sum_{i=1}^{n_b} l_{k+1}^i P_i + \sum_{i=1}^{n_b} l_k^i Q_i, \; k \in \mathbb{N}, \quad (18)$$

or written in more concise form:

$$M_k = L_{k+1}^T \otimes \mathcal{P} + L_k^T \otimes \mathcal{Q}, \quad (19)$$

with $\mathcal{P} = [\, P_1^T \; \cdots \; P_{n_b}^T \,]^T$ and $\mathcal{Q} = [\, Q_1^T \; \cdots \; Q_{n_b}^T \,]^T$. Notice that from this last expression it follows that, although the underlying (transformed) system is strictly proper (i.e. $L_0 = 0$) , its Hambo transform will in general not be strictly proper but only proper. So strictly speaking the Hambo transform maps from H_2 to the space of stable, proper transfer functions of i/o dimension $n_b \times n_b$.

In a similar manner we can compute the generalized Markov parameters of the one-sample shifted system $zG(z)$ by using the easily shown fact that $zV_1(z) = B + AV_1(z)$. The generalized Markov parameters of this shifted system are given by

$$M_k = (L_{k+1}^T B) I + (L_{k+1}^T A) \otimes \mathcal{P} + (L_k^T A) \otimes \mathcal{Q}. \quad (20)$$

We will need these Markov parameters later on for the implementation of a realization algorithm.

The question remains how the P_i and Q_i can be calculated. To do this we first observe that by means of the Hambo inverse relation (12) it follows that the 2-parameter relation is equivalent to

$$V_1(z)\phi_i(z) = P_i^T V_1(z) + Q_i^T V_1(z) G_b(z). \quad (21)$$

By equating the realizations of the left and right hand side terms of this expression we eventually find that P_i and Q_i can be found as the solutions of the discrete-time Lyapunov equations

$$\begin{aligned} AP_iA^T + Ae_iB^T &= P_i \\ AQ_iA^T + BCe_iB^T + BCP_iA^T &= Q_i. \end{aligned} \quad (22)$$

4 A realization algorithm based on the generalized Markov parameters

In this section we will review the minimal realization algorithm that was derived by Szabó and Bokor (in a different formulation) [7]. It is based on the idea of representing the change from standard basis (z^{-k}) to generalized basis $V_k(z)$ by unitary transformations. Let H denote the infinite Hankel matrix pertaining to the representation of $G(z)$ in the standard basis and $\tilde{H}$ the infinite block Hankel matrix pertaining to the generalized basis. There then exist unitary operators T_{H_2} and $T_{H_2^\perp}$ [1] that satisfy:

$$H = T_{H_2}^T \tilde{H} T_{H_2^\perp}. \quad (23)$$

The i-th block row (of dimension $n_b \times \infty$) of the matrix operator T_{H_2} and $T_{H_2^\perp}$ can be shown to be given by the pulse responses of the systems $(zI - A)^{-1} B G_b^{i-1}(z)$ and $(zI - A^T)^{-1} C^T G_b^{i-1}(z)$ respectively.

Substituting relation (23) into the standard Ho-Kalman algorithm gives us the following slightly adapted version.

Algorithm 1 *Let the infinite block Hankel matrix $\tilde{H}$ pertaining to the system $G(z)$ have a full rank decomposition $\tilde{H} = \tilde{\Gamma}\tilde{\Delta}$, and let $\underline{\tilde{H}}$ be the Hankel matrix related to the shifted system $zG(z)$. Then a minimal realization $\{A_m, B_m, C_m\}$ of $G(z)$ is obtained through*

$$\begin{aligned} A_m &= \tilde{\Gamma}^\dagger \underline{\tilde{H}} \tilde{\Delta}^\dagger, \\ B_m &= \tilde{\Delta} T_{H_2^\perp} e_1, \\ C_m &= e_1^T T_{H_2}^T \tilde{\Gamma}, \end{aligned}$$

where $\dagger$ denotes Moore-Penrose pseudo-inverse.

The above algorithm involves operations of infinite matrices upon the full rank decomposition of $\tilde{H}$. It is therefore based on the assumption that the generalized

[1] The notation here stems from the fact that T_{H_2} represents a change of basis in H_2 while $T_{H_2^\perp}$ represents the corresponding change of basis in $H_2^\perp$, the orthogonal complement of H_2 in L_2.

Markov parameters are known exactly up to infinity. This is obviously not a very realistic assumption in the general case. In practice this will only occur when the basis function coefficients, and thereby the generalized Markov parameters, are zero from a certain point onwards.

Partial realization

The more general case where only partial knowledge of the coefficients is available has been solved as well, see [3] for a detailed description. Here we restrict ourselves to a brief outline of the main ingredients of the partial realization case. The main problem for the case where only a finite number of generalized Markov parameters can be calculated stems from the fact that equation (23) is only valid for the infinite case. And since only a finite part of $\tilde{H}$ is known this relation can not be used directly. The solution to this problem is straightforward. It involves the creation of an infinite $\tilde{H}$, by applying a classical partial realization algorithm (see for instance [9]) in the transform domain.
The same procedure is applied to derive an infinite $\underline{\tilde{H}}$. Note that these two partial realizations in the transform domain immediately yield full rank decompositions of the Hankel matrices $\tilde{H}$ and $\underline{\tilde{H}}$, in the form of controllability and observability matrices:

$$\tilde{H} = \tilde{O} \cdot \tilde{C} \qquad \underline{\tilde{H}} = \underline{\tilde{O}} \cdot \underline{\tilde{C}}, \tag{24}$$

which can be plugged into Algorithm 1. This results in a number of Lyaponov and Sylvester equations, that produce the desired minimal state space realization of the underlying system. Note that where in the classical case only one Ho-Kalman step is required, the generalized case needs two of these steps, caused by the non-trivial relation between $\tilde{H}$ and $\underline{\tilde{H}}$. See section 6 for an example.

Remark 1 *These realization algorithms operate on the generalized Markov parameters, which can be calculated from the series expansion $G(z) = \sum_{k=1}^{\infty} L_k V_k(z)$. An interesting question is whether one could devise such an algorithm directly in terms of the coefficients $\{L_k\}$. Thereto one should investigate the system*

$$\tilde{g}(\lambda) := \sum_{k=1}^{\infty} L_k^T \lambda^{-k} \tag{25}$$

In the classical case (basis $\{z^{-k}\}$) the functions $\tilde{G}(\lambda)$ and $\tilde{g}(\lambda)$ coincide, in the generalized case this is no longer the case. Where $\tilde{G}(\lambda)$ is a matrix function in $H_2^{n_b \times n_b}$, the function $\tilde{g}(\lambda)$ is a vector function in $H_2^{n_b}$. This transition from matrix function to vector function can result in a decrease of the McMillan degree.
Consider for example the system $G(z) = G_b^k$ (degree $k \times n_b$) and assume that $D = 0$. This G has transforms $\tilde{G}(\lambda)) = \lambda^{-k}$ and $\tilde{g}(\lambda)) = C^T \lambda^{-k}$. The latter has McMillan degree k.

5 Projection onto the class of Hambo transforms

Apart from providing us with the possibility to compute minimal realizations based upon the expansion coefficients, the generalized Markov parameters also make it possible to characterize the subspace X of Hambo transformed systems in a convenient manner. Expression (18) can equivalently be written as

$$\begin{bmatrix} \mathrm{vec}(M_0) \\ \mathrm{vec}(M_1) \\ \mathrm{vec}(M_2) \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{P} & 0 & 0 & \cdots \\ \mathbf{Q} & \mathbf{P} & 0 & \cdots \\ 0 & \mathbf{Q} & \mathbf{P} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ \vdots \end{bmatrix}, \tag{26}$$

with $\mathrm{vec}(\cdot)$ representing a column stacking operation and $\mathbf{P}$ and $\mathbf{Q}$ given by:

$$\begin{aligned} \mathbf{P} &= [\mathrm{vec}(P_1) \;\; \mathrm{vec}(P_2) \;\; \cdots \;\; \mathrm{vec}(P_{n_b})] \\ \mathbf{Q} &= [\mathrm{vec}(Q_1) \;\; \mathrm{vec}(Q_2) \;\; \cdots \;\; \mathrm{vec}(Q_{n_b})]. \end{aligned} \tag{27}$$

Equation (26), which can be written in shorthand notation as $\mathbf{M} = \mathbf{ZL}$, shows that the subspace X of Hambo transformed systems can be characterized as the image space of the linear mapping from $\ell_2^{n_b}$, or

$$X = \mathrm{Im}(Z). \tag{28}$$

Any system $P(\lambda) \in H_2^{n_b \times n_b}$ but not in X can be approximated by a $\hat{P}(\lambda) \in X$ in a least squares sense. The approximant is given in terms of its Markov parameters $\hat{\mathbf{M}}$ by

$$\hat{\mathbf{M}} = \mathbf{Z}\hat{\mathbf{L}} = \mathbf{Z}(\mathbf{Z}^T\mathbf{Z})^{-1}\mathbf{Z}^T\mathbf{M}. \tag{29}$$

A drawback of this method is that although it provides a characterization of the subspace X of Hambo transformed systems, there is no limit to the McMillan degree of the resulting approximant. A projection of a finite dimensional system might be infinite dimensional itself.

6 Example

In this section we first demonstrate a possible application of the projection method, in the context of model reduction. Secondly the power of the partial realization method is shown.

The system considered here is a 6-th order LTI system which was obtained through system identification of one of the input/output channels of a nonlinear model of a fluidized catalytic cracking unit [11]. An important feature of this process it that is incorporates a combination of fast and slow dynamics, illustrated by the step response of the system, as shown in figure 2. The poles of the system are:

$$\begin{pmatrix} -0.0036 \pm 0.7837i \\ 0.7446 \pm 0.2005i \\ 0.9941 \\ 0.8758 \end{pmatrix}. \tag{30}$$

We will investigate what the consequences are of applying model reduction through balanced truncation in the Hambo transform domain and in the original domain. Based on the knowledge we have (e.g. from the step response) we choose to incorporate two poles in our basis generating function, 0.5 and 0.9.

The Hambo system transform is calculated by computing the expansion coefficient vectors L_k up to $k = n$ where we take $n > 6$. This can be done without approximation by solving a Sylvester equation. Next we compute from the L_k the associated generalized Markov parameters M_k up to $n-1$. From these Markov parameters we can realize the transformed system $\tilde{G}(\lambda)$ in state space form.

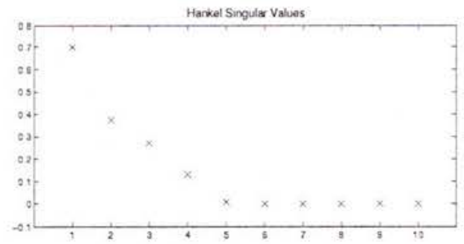

Figure 1: The first 10 Hankel singular values of the projected model.

The next step is to apply balanced truncation to this model. We reduce it to a 4-th order model. This reduced model does not necessarily lie within the image space of the Hambo transform. To compute an approximate model that does lie in that space we apply orthogonal projection onto the image space as described in equation (29). But now we have as a drawback that the projected model might not have McMillan degree equal to 4. If we consider the Hankel singular values of this reduced and projected model, see figure 1, we see that there is a considerable gap between the 4th and the 5th singular value. This suggests that we can try to apply approximate realization (reducing to model order 4) using the partial realization algorithm described in [3], in order to obtain a 4-th order approximation $\hat{G}(z)$. The poles of the resulting model are:

$$\begin{pmatrix} 0.7561 \pm 0.2128i \\ 0.8864 \\ 0.9940 \end{pmatrix}. \tag{31}$$

In figure 2 the resulting approximation is illustrated in terms of its step response to the system. Also a comparison is made to a 4-th order model obtained by applying balanced truncation directly to the model $G(z)$.

In figure 3 the Bode plots of the system, the model $\hat{G}(z)$ and the model obtained through direct balanced truncation are compared. It is clear that, for this example, the model obtained by truncating in the Hambo domain has a better overall fit to the system. Apparently transformation to the Hambo domain and subsequent reduction, implies that a certain frequency weighting takes place, depending on the choice of poles, that is different from the

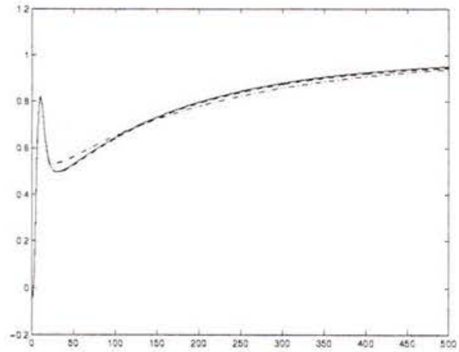

Figure 2: Comparison of the step responses of the system (solid), the 4-th order model $\hat{G}(z)$ (dashed) and a model obtained by direct application of balanced truncation (dash-dotted)

implicit frequency weighting that occurs when balanced truncation is applied. We must mention however that the Hambo domain reduction approach presented here, works particularly well for this example as the singular value gap that occurs is considerable. If we were to reduce to 3rd order, the resulting approximation is considerably worse.

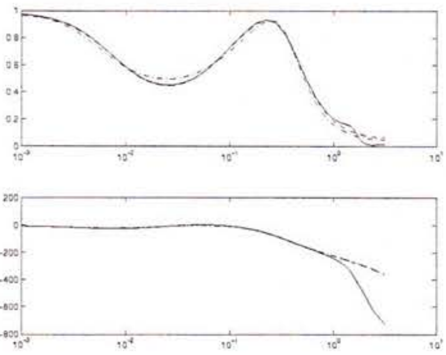

Figure 3: Comparison of the frequency responses of the system (solid), the 4-th order model $\hat{G}(z)$ (dashed) and a model obtained by direct application of balanced truncation (dash-dotted). Upper: Amplitude, Lower: Phase.

realization

The power of the generalized partial realization algorithm is shown in Figures 4 and 5, where for the same system as before the first 8 expansion coefficients L_k where calculated with a (deliberately bad) basis generated by poles in $0.5 \pm 0.5i$. The partial realization algorithm renders the exact system, whereas the full order model

$$\hat{G}(z) = \sum_{k=1}^{8} L_k V_k(z)$$

is lacking information on the low frequency behavior. This illustrates that the sequence of coefficients carries more information than is reflected by the full order approximation.

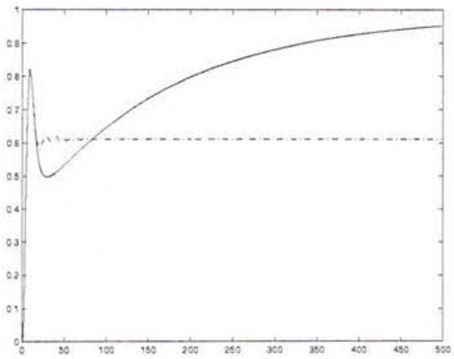

Figure 4: Comparison of the step responses of the system (solid) and the 16-th order model $\hat{G}(z)$ (dash-dotted).

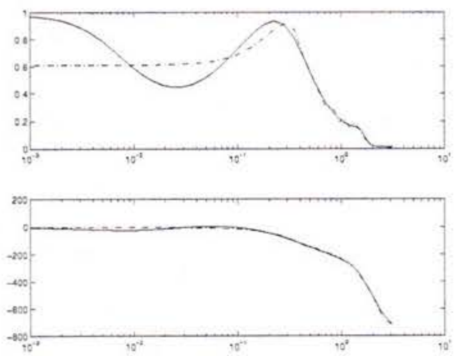

Figure 5: Comparison of the frequency responses of the system (solid) and the 16-th order model $\hat{G}(z)$ (dash-dotted). Upper: Amplitude, Lower: Phase.

7 Conclusions

We have shown a method to efficiently compute the Markov parameters of linear system representations with respect to a generalized orthonormal basis. These generalized Markov parameters can be interpreted as the Markov parameters of the Hambo transformed system and they can be computed directly from the expansion coefficients of the basis function representation. Knowledge of the generalized Markov parameters can be used for the calculation of a minimal state-space realization of the underlying system by means of adapted versions of the Ho-Kalman algorithm, as well for the complete as for the partial knowledge case. The generalized Markov parameters also provide a means to characterize the class of Hambo transformed systems in such a way that it becomes possible to project onto it. This feature can for instance be used in the context of model manipulation and reduction in the transform domain.

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