IV METHODS FOR CLOSED-LOOP SYSTEM IDENTIFICATION

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Abstract: In this paper, several instrumental variable (IV) and instrumental variable-related methods for closed-loop system identification are considered and set in an extended IV framework. Extended IV methods require the appropriate choice of particular design variables, as the number and type of instrumental signals, data prefiltering and the choice of an appropriate norm of the extended IV-criterion. The optimal IV estimator achieves minimum variance, but requires the exact knowledge of the noise model. For the closed-loop situation several IV methods, such as tailor-made IV, IV4 and BELS are put in an extended IV framework and characterized by different choices of design variables. Their variance properties are considered and illustrated with a simulation example.

Keywords: Closed-loop system identification; linear estimators; instrumental variables.

1. INTRODUCTION

For many industrial production processes, safety and production restrictions are often strong reasons for not allowing identification experiments in open-loop. In such situations, experimental data can only be obtained under so-called closed-loop conditions. The main difficulty in closed-loop identification is due to the correlation between the disturbances and the control signal, induced by the loop. Several classical alternatives are available to cope with this problem, broadly classified into three main approaches: direct, indirect and joint input/output (Söderström and Stoica 1989, Ljung 1999). Some particular versions of these methods have been developed more recently in the area of control-relevant identification as e.g. the two-stage, the coprime factor, the dual-Youla methods. An overview of these recent developments can be found in Van den Hof (1998) and Forssell and Ljung (1999).

When looking at methods that can consistently identify plant models of systems operating in closed-loop while relying on simple linear (regression) algorithms, instrumental variable (IV) techniques seem to be rather attractive, but at the same time also not very often applied. On the other hand, when dealing with highly complex processes that are high dimensional in terms of inputs and outputs, it can be attractive to rely on methods that do not require non-convex optimization algorithms.

For closed-loop identification a basic IV estimator has been proposed (Söderström *et al.* 1987), and more recently a so-called tailor-made IV algorithm (Gilson and Van den Hof 2001), where the closed-loop plant is parametrized using (open-loop) plant parameters. The class of algorithms denoted by BELS (for Bias-Eliminated Least-Squares), e.g. Zheng (1996), is also directed towards the use of linear regression algorithms only. It has recently been shown that these algorithms are also particular forms of IV estimation schemes (Gilson and Van den Hof 2001). Then, when comparing the several available IV algorithms,

the principal question to address should be: how to achieve the smallest variance of the estimate. For extended IV methods an optimality result has been developed in the open-loop case, showing consequences for the optimal choice of weights, filters, and instruments. This result can be extended to the closed-loop case.
In this paper the several IV and IV-related methods are set in an extended IV framework, and the consequences for the several design variables (related to optimal variance) are considered. Since for optimal variance, the noise model has to be known exactly, several bootstrap methods are proposed for approximating this required information from measurement data. The comparison between the different proposed methods is illustrated in a simulation example, showing that the optimal estimator can be accurately approximated by an appropriate choice of the design parameters.

2. PRELIMINARIES

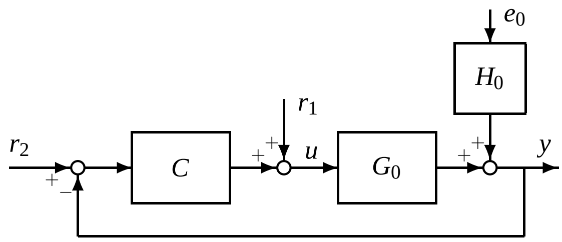


Fig. 1. Closed-loop configuration.

Consider a linear SISO closed-loop system shown in figure 1. The process is denoted by G_0 and the controller by C; $u(t)$ describes the process input signal, $y(t)$ the process output signal and $\{e_0(t)\}$ is a sequence of independent identically distributed random variables with variance λ_0. The external signals $r_1(t)$, $r_2(t)$ are assumed to be uncorrelated with $e_0(t)$. For ease of notation we also introduce the signal $r(t) = r_1(t) + C(q)r_2(t)$. With this notation, the data generating system becomes

$$\mathscr{S} : \begin{cases} y(t) = G_0(q)u(t) + H_0(q)e_0(t) \\ u(t) = r(t) - C(q)y(t) \end{cases} \tag{1}$$

The real plant G_0 is considered to satisfy $G_0(q) = B_0(q^{-1})/A_0(q^{-1})$, while in these expressions q^{-1} is the delay operator, and the numerator and denominator polynomials have degree n_0. The m-th order controller C is assumed to be known and specified by

$$C(q) = \frac{Q(q^{-1})}{P(q^{-1})} \tag{2}$$

with the pair of polynomials (P,Q) assumed to be coprime.

The closed-loop transfer function can be written as

$$y(t) = \frac{G_0}{1+CG_0} r(t) + \frac{H_0}{1+CG_0} e_0(t) \tag{3}$$

or in polynomial fraction form

$$y(t) = \frac{B_{cl}^0}{A_{cl}^0} r(t) + \frac{1}{A_{cl}^0} \xi(t) \tag{4}$$

with $\xi(t) = A_0 P H_0 e_0(t)$. The polynomials B_{cl}^0 and A_{cl}^0 will generically have orders $n_0 + m$. A parametrized process model is considered

$$\mathscr{G} : G(q,\theta) = \frac{B(q^{-1},\theta)}{A(q^{-1},\theta)} = \frac{b_1 q^{-1} + \cdots + b_n q^{-n}}{1 + a_1 q^{-1} + \cdots + a_n q^{-n}},$$

and the process model parameters are stacked columnwise in the parameter vector

$$\theta = \begin{bmatrix} a_1 & \cdots & a_n & b_1 & \cdots & b_n \end{bmatrix}^T \in \mathbb{R}^{2n}. \tag{5}$$

Furthermore, let us denote by $\varphi(t)$ and by $\psi(t)$ the closed-loop and open-loop regressors respectively, defined as

$$\begin{aligned} \varphi^T(t) &= [-y(t-1) \cdots -y(t-n-m) \; r(t-1) \cdots r(t-r_B)] \\ \psi^T(t) &= [-y(t-1) \cdots -y(t-n) \; u(t-1) \cdots u(t-n)] \\ \varphi_r^T(t) &= [r(t-1) \; \cdots \; r(t-r_B)], \end{aligned}$$

and r_B a user-specified integer. Additionally we use the following notation

$$\bar{\psi}(t) = P(q^{-1})\psi(t) \tag{6}$$
$$\bar{y}(t) = P(q^{-1})y(t). \tag{7}$$

3. IV METHODS IN AN EXTENDED IV FRAMEWORK

3.1 *Tailor-made IV identification (M1)*

The tailor-made IV method (referred to M1 in the following) as discussed in (Gilson and Van den Hof 2001) is designed to provide an unbiased estimate for the process model $G(q,\theta)$, while pertaining to simple linear regression type of estimates. Accurate noise modelling (i.e. estimating H_0) is not considered part of the problem.
Consider the (closed-loop) prediction error

$$\begin{aligned} \varepsilon(t,\theta) &= \bar{A}_{cl}(q^{-1},\theta)y(t) - \bar{B}_{cl}(q^{-1},\theta)r(t) \qquad (8) \\ \bar{B}_{cl}(q^{-1},\theta) &= B(q^{-1},\theta)P(q^{-1}) \\ \bar{A}_{cl}(q^{-1},\theta) &= A(q^{-1},\theta)P(q^{-1}) + B(q^{-1},\theta)Q(q^{-1}), \end{aligned}$$

parametrized in the plant parameter θ (tailor-made parametrization), and the prediction error alternatively written as

$$\varepsilon(t,\theta) = \bar{y}(t) - \bar{\psi}^T(t)\theta. \tag{9}$$

Then the tailor-made IV estimate of θ is determined as the solution to the set of equations

$$\frac{1}{N}\sum_{t=1}^{N} \varepsilon(t,\hat{\theta}_{tiv,F})\eta(t) = 0, \tag{10}$$

where $\eta(t) = F\varphi_r(t)$, and $F \in \mathbb{R}^{2n \times r_B}$ a user-chosen matrix with rank $2n$.
The choice $r_B = 2n$, $F = I_{2n}$ leads to a simple basic IV estimator, taking as instruments $2n$ delayed samples of

the reference signal, that is supposed to bepersisntely exciting of sufficiently high order. For $r_B > 2n$, the matrix F constructs $2n$ instruments out of r_B delayed reference samples, by taking particular linear combinations.

3.2 *BELS method*

The so-called bias-eliminated least-squares method (BELS) as proposed by (Zheng and Feng 1995, Zheng 1996) has been shown to be a particular form of tailor-made IV estimator (Gilson and Van den Hof 2001). It has two different formats, dependent on the relation between n (model order) and m (controller order). For $m \leq n$ the BELS estimator is equivalent to the tailor-made estimate with $r_B = 2n$ and $F = I_{2n}$. For $m > n$ it is obtained by choosing $r_B = n+m$ and

$$F = M^T \hat{R}^T_{\varphi_r\varphi}(N)(\hat{R}_{\varphi_r\varphi}(N)\hat{R}^T_{\varphi_r\varphi}(N))^{-1}. \qquad (11)$$

with $\hat{R}_{\varphi_r\varphi}(N) = \frac{1}{N}\sum_{t=1}^{N}\varphi_r(t)\varphi^T(t)$, and $M \in \mathbb{R}^{(n+m+r_B)\times 2n}$ a full-column rank matrix dependent on controller dynamics. For a full description of the relation between BELS and tailor-made IV, see Gilson and Van den Hof (2001).

3.3 *Tailor-made and extended IV identification*

In order to analyse the variance properties of the estimators presented above, they are positioned in the framework of extended IV estimators. An extended IV estimate of θ_0 is obtained by generalizing the so-called basic IV estimates of θ by prefiltering the data and by using an augmented instrument $z(t) \in \mathbb{R}^{n_z}$ ($n_z \geq 2n$) so that an over-determined set of equations is obtained

$$\hat{\theta}_{eiv}(N) = \arg\min_{\theta} \left\| \left[\frac{1}{N}\sum_{t=1}^{N} z(t)L(q^{-1})\psi^T(t)\right]\theta - \left[\frac{1}{N}\sum_{t=1}^{N} z(t)L(q^{-1})y(t)\right]\right\|_Q^2,$$

where $L(q^{-1})$ is a stable prefilter and $\|x\|_Q^2 = x^T Q x$, with Q a positive definite weighting matrix.

Proposition 1. The tailor-made IV estimates presented in sections 3.1 and 3.2 with the particular choice of F given in (11) satisfies

$$\hat{\theta}_{iv}(N) = \arg\min_{\theta} \left\|\hat{R}_{\varphi_r\bar{\psi}}(N)\theta - \hat{R}_{\varphi_r\bar{y}}(N)\right\|_Q^2 \qquad (12)$$

with $\hat{R}_{\varphi_r\bar{\psi}}(N) = \frac{1}{N}\sum_{t=1}^{N}\varphi_r(t)\bar{\psi}^T(t)$, $\hat{R}_{\varphi_r\bar{y}}(N) = \frac{1}{N}\sum_{t=1}^{N}\varphi_r(t)\bar{y}(t)$, and

$$Q = \left(\hat{R}_{\varphi_r\varphi}\hat{R}^T_{\varphi_r\varphi}\right)^{-1} \in \mathbb{R}^{(n+m)\times(n+m)}.$$

It is equivalent to an extended IV estimator where

- the instruments are chosen such that $z(t) = \varphi_r(t)$ and $n_z = r_B = n+m$,
- the prefilter $L(q^{-1})$ is taken equal to the controller denominator $P(q^{-1})$,
- the regressor $\psi(t)$ depends on delayed input $u(t)$ and output $y(t)$ values.

Proof. A full proof is added in the appendix.

The notation (N) is omitted for ease of notation. The covariance matrix of this extended IV estimate is computed in the scalar case by using equation (8.30) in Söderström and Stoica (1989). Under the assumption $G_0 \in \mathcal{G}$

$$P_{eiv} = \lambda_0 \left(R^T_{\varphi_r\bar{\psi}} Q R_{\varphi_r\bar{\psi}}\right)^{-1} R^T_{\varphi_r\bar{\psi}} Q R_{z_T z_T} Q R_{\varphi_r\bar{\psi}} \left(R^T_{\varphi_r\bar{\psi}} Q R_{\varphi_r\bar{\psi}}\right)^{-1} \qquad (13)$$

where [1]

$$R_{\varphi_r\bar{\psi}} = \bar{\mathbb{E}}\varphi_r(t)\bar{\psi}^T(t) = \bar{\mathbb{E}}\varphi_r(t)P(q^{-1})\psi^T(t) \qquad (14)$$

$$R_{z_T z_T} = \bar{\mathbb{E}} z_T(t) z_T^T(t) \qquad (15)$$

$$z_T(t) = \sum_{i=0}^{\infty} t_i \varphi_r(t-i) \qquad (16)$$

and $\{t_i\}$ is determined by the monic filter

$$T(q^{-1}) = P(q^{-1})A_0(q^{-1})H_0(q^{-1}) = \sum_{i=0}^{\infty} t_i q^{-i}. \qquad (17)$$

Remark. In the situation $r_B = 2n$ and $F = I_{2n}$, the tailor-made IV estimate is an extended IV estimate with $n_z = r_B = 2n$, $Q = I$ and $L(q^{-1}) = P(q^{-1})$. Thus, according to equation (13) and under the assumption $G_0 \in \mathcal{G}$, the expression for the covariance matrix of this estimate simplifies to

$$P_{tiv} = \lambda_0 R^{-1}_{\varphi_r\bar{\psi}} R_{z_T z_T} R^{-T}_{\varphi_r\bar{\psi}}. \qquad (18)$$

4. OPTIMAL CLOSED-LOOP IV

The choice of the instruments $z(t)$, of n_z, of the weighting matrix Q and of the prefilter $L(q^{-1})$ may have a considerable effect on the covariance matrix P_{eiv}. The lower bound of P_{eiv} for any unbiased identification method is given by the Cramer-Rao bound, which for the open-lop situation is specified in e.g. Ljung (1999) and Söderström and Stoica (1983). For the closed-loop case a lower bound of P_{eiv} has been provided in Forssell and Chou (1998), but restricted to the case of an ARMAX type of model. However, the more general case can also be analyzed in the closed-loop framework. Indeed, as explained in (Ljung 1999), the Cramer-Rao bound gives a lower bound for any unbiased estimation problem. Therefore, it applies also to the closed-loop IV technique and the lower bound of (13) is given by (normality is assumed here: $\sqrt{N}(\hat{\theta} - \theta^*) \in AsN(0, P_{eiv})$)

$$P^{opt}_{eiv} = \lambda_0 \left[\bar{\mathbb{E}}\breve{\psi}(t)\breve{\psi}^T(t)\right]^{-1}, \qquad (19)$$

where

$$\breve{\psi}^T(t) = \left[\frac{d}{d\theta}\hat{y}(t|\theta)\right]^T\Bigg|_{\theta=\theta_0} = [A_0(q^{-1})H_0(q^{-1})]^{-1}\tilde{\psi}^T(t)$$

[1] The notation $\bar{\mathbb{E}}[.] = \lim_{N\to\infty}\frac{1}{N}\sum_{t=0}^{N-1}\mathbb{E}[.]$ is adopted from the prediction error framework of Ljung (1999)

is the gradient and $\tilde{\psi}(t)$ denotes the noise-free part of $\psi(t)$. P_{eiv}^{opt} is achieved by taking e.g.

$$z(t) = \lambda_0^{-1}\left[\left(A_0(q^{-1})H_0(q^{-1})\right)^{-1}\tilde{\psi}^T(t)\right]^T \quad (20)$$

$$n_z = 2n,\ Q = I \quad (21)$$

$$L(q^{-1}) = [A_0(q^{-1})H_0(q^{-1})]^{-1}. \quad (22)$$

Then, the optimal IV estimator can only be obtained if the true noise model $A_0(q^{-1})H_0(q^{-1})$ is exactly known and therefore optimal accuracy cannot be achieved in practice.

5. APPROXIMATE IMPLEMENTATIONS OF THE OPTIMAL CLOSED-LOOP IV

In order to give some clues to the closed-loop identification method users, it would be interesting to compare the tailor-made IV method with the optimal IV one. However, as the latter cannot be achieved in practice, approximate implementaions of the optimal IV method will be considered. For this purpose one will need to take care that

- a noise model is available in order to construct the prefilter $L(q^{-1})$ and the instruments $z(t)$,
- a first model of $G_0(q)$ is needed to compute the noise free part of the regressor $\tilde{\psi}(t)$.

The choice of the instruments and prefilter in the IV method affects the asymptotic variance, while consistency properties are generically secured. This suggests that minor deviations from the optimal value (which is not available in practice) will only cause second-order effects in the resulting accuracy. Therefore it could be sufficient to use consistent, but not necessarily efficient estimates of G_0 and H_0 when constructing the instruments and the prefilter (Ljung 1999).
The following sections present two practical solutions to approximate the optimal closed-loop IV estimator. Only linear regressions are used for retaining the simplicity of the IV method.

5.1 *Extensions of the IV4 method (M2, M3)*

Several bootstrap IV methods have been proposed in the open-loop situation, in an attempt to approximate the optimal IV method, see e.g. (Young 1976, Söderström and Stoica 1983, Ljung 1999). A first solution consists thus in extending one of these algorithms to the closed-loop situation; here the IV4 method (Ljung 1999) will be considered. The major difference between open-loop and closed-loop cases is that in the latter, also the input is correlated with the noise. Therefore, the instruments have to be uncorrelated with the noise part of $u(t)$ but correlated with the noise-free part of $u(t)$.

Method M2.

Step 1. Write the model structure as a linear regression

$$\hat{y}(t,\theta) = \psi(t)^T\theta. \quad (23)$$

Estimate θ by a least-squares method and get $\hat{\theta}_1$ along with the corresponding transfer function $\hat{G}_1(q)$, of order n.

Step2. Generate the instruments $z_1(t)$ as

$$\tilde{y}_1(t) = \frac{C(q)\hat{G}_1(q)}{1+C(q)\hat{G}_1(q)}r(t) \quad (24)$$

$$\tilde{u}_1(t) = \frac{1}{1+C(q)\hat{G}_1(q)}r(t) \quad (25)$$

$$z_1(t) = [-\tilde{y}_1(t-1)\cdots-\tilde{y}_1(t-n)\tilde{u}_1(t-1)\cdots\tilde{u}_1(t-n)]^T$$

$z_1(t)$ can be seen as an estimation of the noise-free part of the regressor $\psi(t)$. Determine the IV estimate of θ in (23) as

$$\hat{\theta}_2 = \hat{R}_{z_1\psi}^{-1}\hat{R}_{z_1y} \quad (26)$$

The corresponding estimated transfer function is given by $\hat{G}_2(q) = \frac{\hat{B}_2(q^{-1})}{\hat{A}_2(q^{-1})}$, of order n.

Step 3. Let $\hat{w}(t) = \hat{A}_2(q^{-1})y(t) - \hat{B}_2(q^{-1})u(t)$ and postulate an AR model of order $2n$ for $\hat{w}(t)$: $L(q^{-1})\hat{w}(t) = e(t)$. Estimate $L(q^{-1})$ using a least-squares method and denote the result by $\hat{L}(q^{-1})$.

Step 4. Generate the instruments $z_2(t)$ as

$$\tilde{y}_2(t) = \frac{C(q)\hat{G}_2(q)}{1+C(q)\hat{G}_2(q)}r(t), \tilde{u}_2(t) = \frac{1}{1+C(q)\hat{G}_2(q)}r(t)$$

$$z_2(t) = [-\tilde{y}_2(t-1)\cdots-\tilde{y}_2(t-n)\tilde{u}_2(t-1)\cdots\tilde{u}_2(t-n)]^T$$

Using these instruments $z_2(t)$ and the prefilter $\hat{L}(q^{-1})$, determine the IV estimate of θ in (23) as

$$\hat{\theta}_{M2} = \hat{R}_{z_2\psi_T}^{-1}\hat{R}_{z_2y_T}, \quad (27)$$

where $\psi_T(t) = \hat{L}(q)\psi(t)$ and $y_T(t) = \hat{L}(q)y(t)$.

The asymptotic covariance matrix of the final estimates is the Cramer-Rao bound, provided the true noise model is an autoregression of order $2n$.

Method M3. This method can be improved by using a more sophisticated noise modeling procedure, e.g. by replacing the third step of the M2 algorithm by the `armasel` procedure developed in Broersen (2002), including an appropriate order selection step. This procedure consists in estimating several autoregressive models of different orders and in applying a nonasymptotic order selection criterion based on estimates of prediction error expectation.

5.2 *Another closed-loop "optimal" IV method (M4)*

Noise and process models have to be known in order to construct the instruments and the prefilter. Since, the second order statistical property is not of crucial importance, a simple solution consists in estimating these models by using a high-order least-squares estimator. The result will be obviously biased but a bias in the first step does not lead to a bias in the final model.

Method M4.

Step 1. Write the model structure as a linear regression (equation 23), and estimate θ by a high-order least-squares method.

Then, get $\hat{\theta}_1$ along with the process and noise models $\hat{G}_1(q) = \frac{\hat{B}_1(q^{-1})}{\hat{A}_1(q^{-1})}$, $\hat{H}_1(q) = \frac{1}{\hat{A}_1(q^{-1})}$ respectively.

Step 2. Compute the prefilter $\hat{L}(q^{-1}) = \hat{A}_1(q^{-1})\hat{H}_1(q) = 1$ in the case of an ARX model. Compute the noise-free part of the regressor

$$\tilde{\psi}(t) = [-\tilde{y}_1(t-1) \cdots -\tilde{y}_1(t-n)\tilde{u}_1(t-1) \cdots \tilde{u}_1(t-n)]^T$$

with $\tilde{y}_1(t)$ and $\tilde{u}_1(t)$ computed as in equations (24)-(25). Generate the instruments as

$$z(t) = \{[\hat{A}_1(q^{-1})\hat{H}_1(q^{-1})]^{-1}\tilde{\psi}^T(t)\}^T \qquad (28)$$

Step 3. Using the instrument $z(t)$ and the prefilter $\hat{L}(q^{-1})$, determine the IV estimate in (23) as

$$\hat{\theta}_{M4} = \hat{R}_{z\psi}^{-1}\hat{R}_{zy}. \qquad (29)$$

6. EXAMPLE

The following numerical example is used to compare the performance of the proposed approaches. The process to be identified is described by equation (1), where

$$G_0(q) = \frac{0.5q^{-1}}{1-0.8q^{-1}}, \;\; n = 1 \qquad (30)$$

$$C(q) = \frac{0.0012+0.0002q^{-1}-0.001q^{-2}}{0.5-0.9656q^{-1}+0.4656q^{-2}}, m=2 \quad (31)$$

$$H_0(q) = \frac{1-1.56q^{-1}+1.045q^{-2}-0.3338q^{-3}}{1-2.35q^{-1}+2.09q^{-2}-0.6675q^{-3}} \quad (32)$$

$r(t)$ is a deterministic sequence (realization of a random binary signal) and $e_0(t)$ is a white noise uncorrelated with $r(t)$. The process parameters are estimated by means of the M1 to M4 methods. Moreover, the results from the basic closed-loop IV method developed by Söderström *et al.* (1987) are also analyzed. This method referenced as M5, consists in using the delayed version of the reference signal as instruments; the estimate is thus given by

$$\hat{\theta}_{cliv} = \left[\sum_{t=1}^{N}\zeta(t)\psi^T(t)\right]^{-1}\left[\sum_{t=1}^{N}\zeta(t)y(t)\right] \qquad (33)$$

$$\zeta(t) = \left[r(t) \;\; r(t-1) \;\; \cdots \;\; r(t-2n)\right]^T \qquad (34)$$

For illustration purposes, all of these methods are compared to a benchmark which consists in applying the true noise and process models for generating the prefilter and the instruments.

The process parameters are estimated on the basis of closed-loop data sequences of length $N = 1000$. Monte Carlo simulations of 100 experiments have been performed for a signal to noise ratio

$$SNR = 10\log\left(\frac{P_{y_d}}{P_e}\right) = 15 \;\; \text{dB}, \qquad (35)$$

where P_x denotes the power of the signals x, and y_d is the noise-free output signal.

In figure 2, the Bode diagrams of the 100 models identified by the six methods are represented. Furthermore, the following function is computed and represented in figure 3 for each algorithm

$$g(\omega) = \frac{1}{MC}\sum_{k=1}^{MC}|G_0(e^{i\omega}) - \hat{G}_k(e^{i\omega})| \qquad (36)$$

where MC denotes the number of Monte Carlo experimentations and $\hat{G}_k(e^{i\omega})$ the transfer function estimated during the k^{th} Monte Carlo experimentation. Figures 2 and 3 show that M3 gives the best results (no bias, lower standard-deviation), really close to those of the benchmark. The two approximate versions of the optimal IV algorithm (M3, M4) and the closed-loop IV method (M5) give better results than the tailor-made IV one (M1) in that case. Moreover, the method based on the least-square high-order model (M4) seems to be more appropriate than the extension of the IV4 method to this closed-loop case (M2).

Furthermore, the 2-norm of the difference between the real and estimated transfer functions is also computed for each method

$$Norm = \frac{1}{MC}\sum_{k=1}^{MC}\int|G_0(e^{i\omega}) - \hat{G}_k(e^{i\omega})|^2 d\omega \quad (37)$$

The results are given in table 1 and confirm the previous graphic results: the bootstrap IV methods considered in the paper give better results than the tailor-made IV or the BELS techniques.

method	bench.	M1	M2	M3	M4	M5
Norm	1.921	4.766	2.893	2.223	2.591	3.685

Table 1. *Norm*

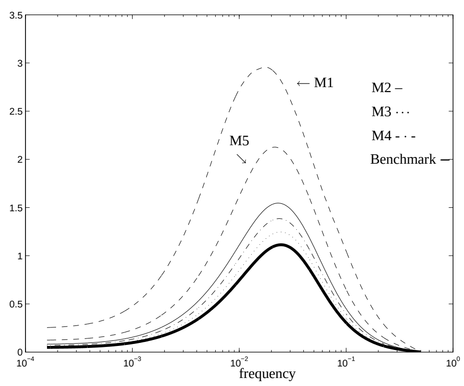
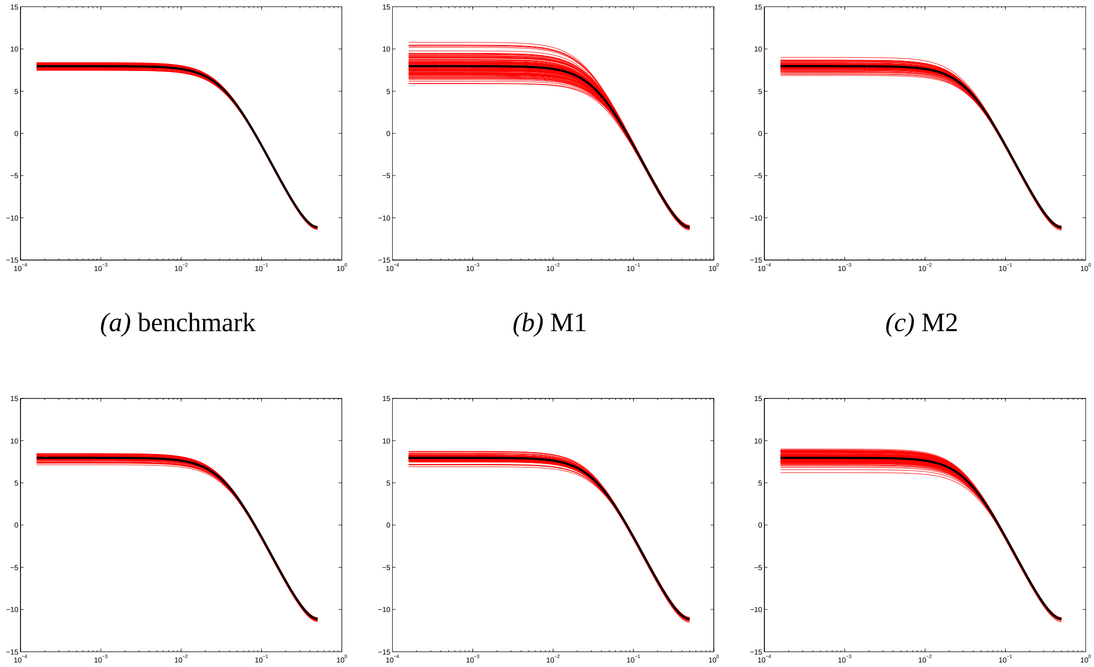


Fig. 3. $g(\omega)$

7. CONCLUSION

Several IV and IV-related estimators for closed-loop system identification have been studied and set in an extended-IV framework. Several methods have been developed to determine the design parameters which allow to approximate the optimal closed-loop IV estimator. In conclusion, the recently suggested Tailor-made IV methods and BELS methods lead to unbiased plant estimates in closed loop. However for arriving at estimates with attractive variance properties it is preferably to apply bootstrap IV methods as considered in this paper.



(a) benchmark *(b)* M1 *(c)* M2

(d) M3 *(e)* M4 *(f)* M5

Fig. 2. Bode amplitude plots of the process (black) and of the estimates (grey)

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APPENDIX

Proof of Proposition 1. Using (9), the solution to (10) can be written as

$$\hat{\theta}_{tiv}(N) = \left[\sum_{t=1}^{N} \eta(t)\bar{\psi}^T(t)\right]^{-1} \left[\sum_{t=1}^{N} \eta(t)\bar{y}(t)\right].$$

By using equation (6), $\eta(t) = F\varphi_r(t)$, and then by replacing F by its expression (11), $\hat{\theta}_{tiv}$ can also be written as

$$\begin{aligned}\hat{\theta}_{tiv} = {} & [\hat{R}^T_{\varphi_r\bar{\psi}}(\hat{R}_{\varphi_r\varphi}\hat{R}^T_{\varphi_r\varphi})^{-1}\hat{R}_{\varphi_r\bar{\psi}}]^{-1} \\ & \cdot \hat{R}^T_{\varphi_r\bar{\psi}}(\hat{R}_{\varphi_r\varphi}\hat{R}^T_{\varphi_r\varphi})^{-1}\hat{R}_{\varphi_r\bar{y}}\end{aligned} \tag{38}$$

The structure of this expression is

$$\hat{\theta}_{tiv} = (A^T Q A)^{-1} A^T Q B \tag{39}$$

with

$$A = \hat{R}_{\varphi_r\bar{\psi}},\ Q = (\hat{R}_{\varphi_r\varphi}\hat{R}^T_{\varphi_r\varphi})^{-1},\ B = \hat{R}_{\varphi_r\bar{y}} \tag{40}$$

As a result, $\hat{\theta}_{tiv}$ is the solution to the extended IV problem

$$\hat{\theta}_{tiv} = \arg\min_{\theta} \left\| \hat{R}_{\varphi_r\bar{\psi}}\theta - \hat{R}_{\varphi_r\bar{y}} \right\|_Q^2 \tag{41}$$

with weighting matrix Q given by equation (40).