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Brief Paper

On the relation between a bias-eliminated least-squares (BELS) and an IV estimator in closed-loop identification☆

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Abstract

A bias-correction method for closed-loop identification, introduced in the literature as the bias-eliminated least-squares (BELS) method (Zheng & Feng, *Automatica* 31 (1995) 1019), is shown to be equivalent to a basic instrumental variable estimator applied to a predictor for the closed-loop system. This predictor is a function of the plant parameters and the known controller. Corresponding to the related method using a least-squares criterion, the method is referred to as the tailor-made IV method for closed-loop identification. The indicated equivalence greatly facilitates the understanding and the analysis of the BELS method. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: System identification; Closed-loop identification; Prediction error methods; Instrumental variables

1. Introduction

Least-squares methods based on the bias-correction principle aim at providing unbiased plant parameter estimates, while using linear-in-the-parameters model structures, see e.g. Sagara and Wada (1977) and Zheng (1998). They retain all merits of the LS method and make it possible to cope with the bias problem in the identification of systems subject to colored disturbances. Recently these kinds of methods have also been developed for identification under closed-loop conditions (Zheng & Feng, 1995; Zheng, 1996). The proposed method, called the bias-eliminated least-squares (BELS) method, is able to estimate unbiased plant parameters in indirect closed-loop system identification. In Söderström, Zheng, and Stoica (1999) it has been shown, based on the work of Stoica, Söderström, and Šimonytė (1995), that the bias-eliminated least-squares estimator proposed in Zheng (1998) for open-loop system identification is identical to a basic instrumental variable estimator. For the closed-loop identification case, the BELS method is analyzed in Zhang, Wen, and Soh (1997), where a relation is claimed with a particular (and rather complex) frequency weighted IV method, applied to the input and output measurement data, gathered under closed-loop conditions. In Zheng and Feng (1995) the BELS estimate is analyzed for the situation where the order of the controller is not smaller than the order of the open-loop plant. In Zheng (1996) the method is generalized to avoid this restriction.

In this paper, it will be shown that, whatever the controller order may be, the closed-loop BELS method is equivalent to the so-called tailor-made instrumental variable method, where the predictor for the closed-loop system is used to generate the prediction error and the external reference signals are used as instrumental variables. This connects to the (least squares) tailor-made identification method for closed-loop identification that was recently introduced in the literature (Van Donkelaar

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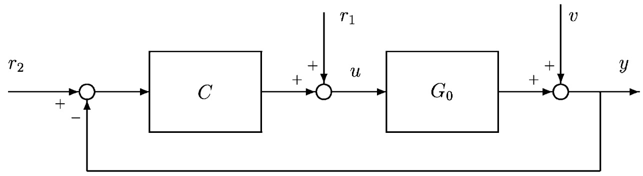

Fig. 1. Closed-loop configuration.

& Van den Hof, 1997; Landau & Karimi, 1997). This equivalence greatly facilitates the understanding and analysis of the BELS method.

2. Preliminaries

Consider a linear SISO closed-loop system shown in Fig. 1. The process is denoted with $G_0(z)$ and the controller with $C(z)$; $u(t)$ is the process input signal, $y(t)$ the process output signal and $v(t)$ describes the disturbances acting on the loop. The external signals $r_1(t)$, $r_2(t)$ are assumed to be uncorrelated with the noise disturbance $v(t)$. For ease of notation we also introduce the signal $r(t) = r_1(t) + C(q)r_2(t)$. With this notation the closed-loop system can be described as

$$\mathscr{S}: \quad y(t) = \frac{G_0}{1 + CG_0} r(t) + \frac{1}{1 + CG_0} v(t). \tag{1}$$

A parametrized process model is considered

$$\mathscr{G}: \quad G(q,\theta) = \frac{B(q^{-1},\theta)}{A(q^{-1},\theta)} = \frac{b_1 q^{-1} + \cdots + b_n q^{-n}}{1 + a_1 q^{-1} + \cdots + a_n q^{-n}} \tag{2}$$

and the process model parameters are stacked columnwise in the parameter vector

$$\theta = [a_1 \quad \cdots \quad a_n \quad b_1 \quad \cdots \quad b_n]^{\mathrm{T}} \in \mathbb{R}^{2n}. \tag{3}$$

The real plant G_0 is considered to satisfy $G_0(q) = B_0(q^{-1})/A_0(q^{-1})$, while in these expressions q^{-1} is the delay operator, and the numerator and denominator degree is n_0. The mth order controller C is assumed to be known and specified by

$$C(q) = \frac{Q(q^{-1})}{P(q^{-1})} = \frac{q_0 + q_1 q^{-1} + \cdots + q_m q^{-m}}{1 + p_1 q^{-1} + \cdots + p_m q^{-m}} \tag{4}$$

with the pair (P, Q) assumed to be coprime. The closed-loop transfer function (1) can be rewritten in polynomial fraction form

$$y(t) = \frac{B_{\mathrm{cl}}^0}{A_{\mathrm{cl}}^0} r(t) + \frac{1}{A_{\mathrm{cl}}^0} \xi(t) \tag{5}$$

with $\xi(t) = A_0 P v(t)$. The polynomials B_{cl}^0 and A_{cl}^0 will generically have orders $n_0 + m$.

For parametrizing the closed-loop transfer function $G_0/(1 + CG_0)$ the following model structure is used:

$$B_{\mathrm{cl}}(q^{-1},\Theta) = \beta_1 q^{-1} + \cdots + \beta_{r_B} q^{-r_B}, \tag{6}$$

$$A_{\mathrm{cl}}(q^{-1},\Theta) = 1 + \alpha_1 q^{-1} + \cdots + \alpha_{n+m} q^{-(n+m)} \tag{7}$$

and the closed-loop parameters are collected in the parameter vector

$$\Theta = [\alpha^{\mathrm{T}} \quad \beta^{\mathrm{T}}]^{\mathrm{T}} = [\alpha_1 \quad \cdots \quad \alpha_{n+m} \quad \beta_1 \quad \cdots \quad \beta_{r_B}]^{\mathrm{T}} \in \mathbb{R}^{n+m+r_B}. \tag{8}$$

For $r_B \geqslant (n_0 + m)$ the closed-loop model structure will be flexible enough to exactly represent the reference to output transfer function in the closed-loop system (5).

The BELS method that is considered in this paper attempts to estimate the process parameters by an indirect closed-loop identification. This means that the closed-loop transfer function (5) is identified, after which process parameters (3) are determined.

The relation between (open-loop) process parameters and closed-loop parameters is determined by the linear equation[2]

$$\Theta = M\theta + \rho, \tag{9}$$

where ρ is a known vector and M is a known full-column rank matrix, given by

$$M = \begin{pmatrix} P_c & Q_c \\ 0 & \bar{P}_c \end{pmatrix} \in \mathbb{R}^{(n+m+r_B) \times 2n}, \tag{10}$$

$$\rho = (p_1 \quad \cdots \quad p_m \quad 0 \quad \cdots \quad 0)^{\mathrm{T}} \in \mathbb{R}^{(n+m+r_B)}. \tag{11}$$

$P_c, Q_c \in \mathbb{R}^{(n+m) \times n}$ are Sylvester matrices expanded by $[1 \quad p_1 \quad \cdots \quad p_m]^{\mathrm{T}}$ and $[q_0 \quad q_1 \quad \cdots \quad q_m]^{\mathrm{T}}$ respectively, e.g.

$$P_c = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ p_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ p_m & \vdots & \ddots & 1 \\ 0 & \ddots & \vdots & p_1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & p_m \end{bmatrix}. \tag{12}$$

$\bar{P}_c \in \mathbb{R}^{r_B \times n}$ is also a Sylvester matrix defined by

$$\bar{P}_c = \begin{bmatrix} P_c \\ 0_{(r_B - n - m) \times n} \end{bmatrix}.$$

[2] Note that in Zheng and Feng (1995) and Zheng (1996) the corresponding equation is written as $\Theta = M\theta - \rho$; the difference in sign is due to the fact that in the mentioned references denominator parameters appear with a negative sign in the parameter vectors.

3. Bias-eliminated least-squares method

The BELS method for closed-loop identification as discussed in Zheng and Feng (1995) and Zheng (1996) is designed to provide an unbiased estimate for the process model $G(q,\theta)$, while pertaining to simple algorithmic schemes as the linear regression type of estimates. Accurate noise modeling (i.e. finding a noise-shaping filter that models the disturbance signal v) is not considered part of the problem. The method comprises the following main steps:

- Estimate an ARX model for the closed-loop system (5) on the basis of data r and y; this estimate is denoted by $\hat{\Theta}_{\rm ls}$.
- This estimate generally will be biased due to the fact that $\xi(t)$ in (5) will not be white noise; however the bias on $B_{\rm cl}(q^{-1},\hat{\Theta}_{\rm ls})/A_{\rm cl}(q^{-1},\hat{\Theta}_{\rm ls})$ can be estimated and subtracted from the closed-loop estimate.
- The corrected closed-loop parameter is converted to an equivalent open-loop process parameter by solving for (9) in a least-squares sense.

In short

$$\text{Closed-loop data} \xrightarrow[\text{ARX}]{} \hat{\Theta}_{\rm ls} \xrightarrow[\text{Computation}]{} \hat{\Theta}_{\rm corr} \xrightarrow[\text{LS}]{} \hat{\theta}_{\rm bels}. \tag{13}$$

3.1. ARX estimate

An ARX estimate for the closed-loop system is obtained through

$$\hat{\Theta}_{\rm ls}(N) = \hat{R}_{\varphi\varphi}(N)^{-1}\hat{R}_{\varphi y}(N),$$

where

$$\hat{R}_{\varphi\varphi}(N) = \frac{1}{N}\sum_{t=1}^{N}\varphi(t)\varphi^{\rm T}(t), \tag{14}$$

$$\hat{R}_{\varphi y}(N) = \frac{1}{N}\sum_{t=1}^{N}\varphi(t)y(t), \tag{15}$$

$$\varphi(t) = [-y(t-1) \quad \cdots \quad -y(t-n-m) \quad r(t-1) \quad \cdots \quad r(t-r_B)]^{\rm T}. \tag{16}$$

3.2. Bias correction

The bias correction principle is based on the following reasoning. If the ARX model structure is rich enough to capture all dynamics of the closed-loop system (i.e. if the system is in the model set), then

$$\hat{\Theta}_{\rm ls}(N) = \Theta_0 + \hat{R}_{\varphi\varphi}^{-1}(N)\hat{R}_{\varphi\xi}(N), \tag{17}$$

where Θ_0 is the coefficient vector of the real closed-loop plant, and $\hat{R}_{\varphi\xi}(N) = (1/N)\sum_{t=1}^{N}\varphi(t)\xi(t)$. Then, under minor regularity conditions on the data, the least-squares estimate $\hat{\Theta}_{\rm ls}(N)$ is known to converge for $N \to \infty$ with probability 1 to

$$\Theta_{\rm ls}^{*} = \Theta_0 + R_{\varphi\varphi}^{-1}R_{\varphi\xi}$$

with $R_{\varphi\varphi} = \bar{E}\varphi(t)\varphi^{\rm T}(t)$ and $R_{\varphi\xi} = \bar{E}\varphi(t)\xi(t)$, where the notation $\bar{E}[.] = \lim_{N\to\infty}(1/N)\sum_{t=0}^{N-1}E[.]$ is adopted from the prediction error framework of Ljung (1987). As the noise disturbance ξ is assumed to be uncorrelated with the reference signal r, the bias in the asymptotic estimate is given by

$$\Delta^{*} := R_{\varphi\varphi}^{-1}R_{\varphi\xi} = R_{\varphi\varphi}^{-1}\begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix}R_{y\xi}$$

with $R_{y\xi} = \bar{E}\{[-y(t-1) \ \cdots \ -y(t-n)]^{\rm T}\cdot\xi(t)\}$. Based on this expression, an estimate for Δ is obtained by considering

$$\hat{\Delta}(N) := \hat{R}_{\varphi\varphi}^{-1}(N)\begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix}\hat{R}_{y\xi}(N).$$

The unknown $\hat{R}_{y\xi}(N)$ in this relation can be obtained by the following reasoning.

As matrix M in (9) has full column rank, there exists a full column rank matrix $H \in \mathbb{R}^{(n+m+r_B)\times(m+r_B-n)}$ that satisfies $H^{\rm T}M = 0$. Multiplying Eq. (17) by $H^{\rm T}$ and using Eq. (9) for Θ_0, it follows that

$$H^{\rm T}\hat{R}_{\varphi\varphi}^{-1}(N)\begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix}\hat{R}_{y\xi}(N) = H^{\rm T}(\hat{\Theta}_{\rm ls}(N) - \rho). \tag{18}$$

This is a set of $m + r_B - n$ equations with $n + m$ unknowns in $\hat{R}_{y\xi}(N)$, requiring $r_B \geqslant 2n$ to have at least as many equations as unknowns. There are two situations to be distinguished.

- $m \geqslant n$ (see Zheng & Feng, 1995). r_B is chosen according to $r_B = n + m$, and Eq. (18) is an overdetermined set of equations that is solved in least-squares sense, leading to

$$\hat{\Delta}(N) = \hat{R}_{\varphi\varphi}^{-1}(N)\begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix}\left[H^{\rm T}\hat{R}_{\varphi\varphi}^{-1}(N)\begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix}\right]^{+} \cdot H^{\rm T}[\hat{\Theta}_{\rm ls}(N) - \rho] \tag{19}$$

  with $(\cdot)^{+}$ denoting the matrix pseudo-inverse.

- $m < n$ (see Zheng, 1996). By choosing $r_B = m + n$ the number of equations in (18) is not sufficient to uniquely determine $\hat{\Delta}$. In Zheng (1996) this is solved by applying a dynamic prefilter to the reference signal such that effectively a system with higher numerator degree is estimated. This is equivalent to simply choosing $r_B = 2n$, thus obtaining the situation that (18) is uniquely solvable for $\hat{R}_{y\xi}(N)$. An estimate $\hat{\Delta}(N)$ can then be constructed according to

$$\hat{\Delta}(N) = \hat{R}^{-1}_{\varphi\varphi}(N)\begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix}\left[H^{\mathrm{T}}\hat{R}^{-1}_{\varphi\varphi}(N)\begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix}\right]^{-1} \cdot H^{\mathrm{T}}[\hat{\Theta}_{\mathrm{ls}}(N) - \rho]. \quad (20)$$

Combining both situations it appears that r_B can be set to $r_B = \max(2n, n+m)$. The bias elimination can now be performed by constructing the corrected closed-loop parameter vector:

$$\hat{\Theta}_{\mathrm{corr}}(N) = \hat{\Theta}_{\mathrm{ls}}(N) - \hat{\Delta}(N). \quad (21)$$

Finally, the plant parameter estimate $\hat{\theta}_{\mathrm{bels}}$ is obtained by solving (9) in a least-squares sense:

$$\hat{\theta}_{\mathrm{bels}}(N) = (M^{\mathrm{T}}M)^{-1}M^{\mathrm{T}}(\hat{\Theta}_{\mathrm{corr}}(N) - \rho). \quad (22)$$

It has been shown in Zheng and Feng (1995) and Zheng (1996) that this resulting parameter estimate is asymptotically unbiased.

4. Tailor-made IV identification

4.1. Main result

In the first step of the BELS method the closed-loop system is estimated with a general ARX (black-box) model structure. However, as we know that the closed-loop system has a particular structure (1) with a known controller C, this structure can also be imposed on the parametrization of the closed loop.

When defining

$$\bar{B}_{\mathrm{cl}}(q^{-1},\theta) = B(q^{-1},\theta)P(q^{-1}), \quad (23)$$

$$\bar{A}_{\mathrm{cl}}(q^{-1},\theta) = A(q^{-1},\theta)P(q^{-1}) + B(q^{-1},\theta)Q(q^{-1}), \quad (24)$$

a parametrization of the closed-loop system has been obtained, in terms of the process parameter θ. In the literature this is known as a tailor-made parametrization, and has been applied before in prediction error identification with least-squares criteria, see e.g. Van Donkelaar and Van den Hof (1997) and Landau and Karimi (1997).

Next the main result is formulated.

Proposition 1. *Consider a data generating system according to* (1), *such that the closed-loop system is asymptotically stable, and consider the BELS estimate* $\hat{\theta}_{\mathrm{bels}}(N)$ *given by* (22), *with* $r_B = \max(2n, n+m)$, *and r persistently exciting of sufficiently high order.*

Define the weighted tailor-made IV estimate $\hat{\theta}_{\mathrm{iv},F}(N)$ *as the solution to the set of* $2n$ *equations*:

$$\frac{1}{N}\sum_{t=1}^{N} \varepsilon(t,\hat{\theta}_{\mathrm{iv},F})\eta(t) = 0, \quad (25)$$

$$\varepsilon(t,\theta) = \bar{A}_{\mathrm{cl}}(q^{-1},\theta)y(t) - \bar{B}_{\mathrm{cl}}(q^{-1},\theta)r(t), \quad (26)$$

$$\eta(t) := F\varphi_r(t), \quad F \in \mathbb{R}^{2n \times r_B}, \quad (27)$$

with

$$\varphi_r(t) = [r(t-1) \quad \cdots \quad r(t-r_B)]^{\mathrm{T}} \in \mathbb{R}^{r_B} \quad (28)$$

with $F = I_{2n}$ *in the situation* $m \leqslant n$, *and* $F = M^{\mathrm{T}}\hat{R}^{\mathrm{T}}_{\varphi_r\varphi}(\hat{R}_{\varphi_r\varphi}\hat{R}^{\mathrm{T}}_{\varphi_r\varphi})^{-1}$ *in the situation* $m > n$.

Then

$$\hat{\theta}_{\mathrm{bels}}(N) = \hat{\theta}_{\mathrm{iv},F}(N).$$

Proof. A full proof is added in the appendix. □

Remarks.

- If the order of the process model exceeds the controller order ($n \geqslant m$), the parameter estimate is a simple tailor-made IV estimate, where the closed-loop prediction error is made orthogonal to delayed versions of the external reference signal. Related estimation algorithms based on a least-squares criterion $\sum_t \varepsilon^2(t,\theta)$ have been considered in Van Donkelaar and Van den Hof (1997) and Landau and Karimi (1997). The indicated equivalence greatly facilitates the understanding and analysis of the BELS estimator. Moreover, it also allows the analysis of the estimator under conditions where the real process is not considered to be present in the model set. Note that in the formulation of the main result it is *not* assumed that G_0 has order n.
- In the situation $m < n$, it is suggested by Zheng (1996) to introduce an auxiliary filter operating on the reference signals in order to increase the number of numerator parameters to identify. Here it is shown, as was also indicated by Zhang et al. (1997), that this dynamic prefilter is superfluous. The problem can be handled by simply choosing $r_B = 2n$, i.e. by deliberately increasing the number of numerator parameters to estimate.
- When $m > n$, the estimator is obtained by using a linear combination of delayed samples of the reference signal, to act as an instrument in the IV estimator.
- In this paper we have dealt with strictly proper plant models, having at least one time delay ($b_0 = 0$). The situation of proper plant models in the BELS estimate is exactly similar to appropriate adaptation of matrix and vector dimensions, provided that in the closed loop there is at least one time delay, i.e. $b_0 q_0 = 0$. The tailor-made IV method does not require the presence of such a loop delay.

4.2. Interpretation of matrix F

In order to interpret the role of matrix F, let us analyze $\varepsilon(t,\hat{\theta}_{\mathrm{bels}})$, the equation error of the BELS estimator. As the connection between IV and BELS estimators has been stated, the equation error can be written as

$$\varepsilon(t,\hat{\theta}_{\mathrm{iv},F}) = y(t) - \varphi^{\mathrm{T}}(t)(M\hat{\theta}_{\mathrm{iv},F} + \rho). \tag{29}$$

This equation can be simplified by analyzing the constituting expressions. At first, it can be noticed that the controller denominator $P(q^{-1})$ can be used as a prefilter for the output $y(t)$. Then, with

$$\bar{y}(t) = P(q^{-1})y(t) \tag{30}$$

it follows that

$$\bar{y}(t) = y(t) - \varphi^{\mathrm{T}}(t)\rho. \tag{31}$$

The second expression $\varphi^{\mathrm{T}}(t)M$ can be rephrased by considering the plant description

$$y(t) = \psi^{\mathrm{T}}(t)\theta_0 + A_0(q^{-1})v(t)$$

with

$$\psi(t) = [-y(t-1) \quad \cdots \quad -y(t-n) \quad u(t-1) \quad \cdots \quad u(t-n)]^{\mathrm{T}} \in \mathbb{R}^{2n}.$$

In a filtered version this reads

$$\bar{y}(t) = \bar{\psi}^{\mathrm{T}}(t)\theta_0 + P(q^{-1})A_0(q^{-1})v(t), \tag{32}$$

where $\bar{\psi}(t) := P(q^{-1})\psi(t)$. Similarly,

$$y(t) = \varphi^{\mathrm{T}}(t)(M\theta_0 + \rho) + P(q^{-1})A_0(q^{-1})v(t)$$

leading to

$$\bar{y}(t) = \varphi^{\mathrm{T}}(t)M\theta_0 + P(q^{-1})A_0(q^{-1})v(t)$$

which combined with (32) shows that

$$\bar{\psi}^{\mathrm{T}}(t) = \varphi^{\mathrm{T}}(t)M.$$

Using this expression in (29) leads to

$$\varepsilon(t,\hat{\theta}_{\mathrm{iv},F}) = \bar{y}(t) - \bar{\psi}^{\mathrm{T}}(t)\hat{\theta}_{\mathrm{iv},F}. \tag{33}$$

For a further interpretation two cases have to be considered, according to the orders of the controller and the model.

Case $m \leqslant n$: If the controller order is smaller than or equal to the model order, it has been stated that F is equal to the identity matrix and the tailor-made IV estimate satisfies

$$\hat{R}_{\varphi_r\varepsilon} = 0. \tag{34}$$

By substituting (33) in (34), this yields

$$\hat{R}_{\varphi_r\bar{y}} - \hat{R}_{\varphi_r\bar{\psi}}\hat{\theta}_{\mathrm{iv},F} = 0. \tag{35}$$

If the signal $r(t)$ is persistently exciting, of sufficient order, the squared matrix $\hat{R}_{\varphi_r\bar{\psi}} \in \mathbb{R}^{2n\times 2n}$ is invertible and thus the IV estimate is given by

$$\hat{\theta}_{\mathrm{iv}} = \hat{R}^{-1}_{\varphi_r\bar{\psi}}\hat{R}_{\varphi_r\bar{y}}. \tag{36}$$

Case $m > n$: In the case where the controller order is greater than the model order, the vector φ_r is made up of $(n+m)$ components (see the proposition). It follows that the matrix $\hat{R}_{\varphi_r\bar{\psi}} \in \mathbb{R}^{(n+m)\times 2n}$ is not invertible. Thus, the matrix F is added in order to make it invertible, i.e. to make $\hat{R}_{\varphi_r\bar{\psi}}$ regular. In this case, F is equal to $M^{\mathrm{T}}\hat{R}^{\mathrm{T}}_{\varphi_r\varphi}(\hat{R}_{\varphi_r\varphi}\hat{R}^{\mathrm{T}}_{\varphi_r\varphi})^{-1}$ and the tailor-made IV estimate satisfies

$$\hat{R}_{F\varphi_r\varepsilon} = 0. \tag{37}$$

By substituting (33) in (37), it follows

$$\hat{R}_{F\varphi_r\bar{y}} - \hat{R}_{F\varphi_r\bar{\psi}}\hat{\theta}_{\mathrm{iv},F} = 0. \tag{38}$$

If $r(t)$ is persistently exciting of sufficient order, the matrix $\hat{R}_{F\varphi_r\bar{\psi}} \in \mathbb{R}^{2n\times 2n}$ is invertible and the IV estimate can be written as

$$\hat{\theta}_{\mathrm{iv}} = \hat{R}^{-1}_{F\varphi_r\bar{\psi}}\hat{R}_{F\varphi_r\bar{y}}. \tag{39}$$

The matrix $\hat{R}_{F\varphi_r\bar{\psi}}$ can be regarded as the product of two matrices F and $\hat{R}_{\varphi_r\bar{\psi}}$. $F \in \mathbb{R}^{2n\times(n+m)}$ and $\hat{R}_{\varphi_r\bar{\psi}} \in \mathbb{R}^{(n+m)\times 2n}$ both have rank $2n$. Thus, the product $F\hat{R}_{\varphi_r\bar{\psi}}$, or equivalently $\hat{R}_{F\varphi_r\bar{\psi}}$, is squared (dimensions $2n \times 2n$) and has rank $2n$.

5. Relation with other work

Recently, it was claimed in Zhang et al. (1997) that the BELS method of Zheng and Feng (1995) and Zheng (1996) when applied to closed-loop data, is equivalent to a particular frequency weighted optimal IV estimator. In this analysis use is made of earlier results from Stoica et al. (1995), in the open-loop context.

However, the estimator analyzed in Zhang et al. (1997) is different from the BELS estimator of Zheng and Feng (1995) and Zheng (1996) as will be shown next.

In Zhang et al. (1997) the following situation is considered (for either $m \geqslant n$ or $m < n$):

$$r_B = n + m, \quad M \text{ is given by Eq. (10).} \tag{40}$$

The matrix, perpendicular to M is chosen as

$$H_1^{\mathrm{T}}M = 0 \quad \text{with } H_1 \in \mathbb{R}^{(2n+2m)\times(n+m)}, \tag{41}$$

where in the situation $m > n$, H_1 has a smaller column dimension than the matrix H used before in this paper. As a result, this leads to an estimator different from the one considered in Zheng and Feng (1995).

In order to apply the reasoning of Stoica et al. (1995) in the closed-loop context, a linear regression has to be found between the closed-loop regression vector $\varphi(t)$ and the open-loop one $\psi(t)$. As a result, the filter $P(q^{-1})$ is used to change the affine map ($\Theta_0 = M\theta_0 + \rho$) into a linear one ($\bar{\psi}^{\mathrm{T}}(t) = M^{\mathrm{T}}\varphi(t)$). Then, the prefiltered open-loop system relation is defined by

$$P(q^{-1})y(t) = P(q^{-1})\psi^{\mathrm{T}}(t)\theta_0 + P(q^{-1})A_0 e(t) \tag{42}$$

or equivalently,

$$\bar{y}(t) = \bar{\psi}^{\mathrm{T}}(t)\theta_0 + \xi(t). \tag{43}$$

The definition of $\bar{y}(t)$ and $\bar{\psi}(t)$ assumes that a delay is operating on the loop, due to a hold. Then, a weighted optimal IV estimator operating on the plant input and output signals is denoted by

$$\hat{\theta}_{\mathrm{iv}}(N) = (\hat{G}^{\mathrm{T}} W_{\mathrm{iv}} \hat{G})^{-1} \hat{G}^{\mathrm{T}} W_{\mathrm{iv}} \hat{p}, \tag{44}$$

where

$$\hat{G} = \frac{1}{N}\sum_{t=1}^{N} z(t)\bar{\psi}^{\mathrm{T}}(t), \tag{45}$$

$$\hat{p} = \frac{1}{N}\sum_{t=1}^{N} z(t)\bar{y}^{\mathrm{T}}(t), \tag{46}$$

$$z(t) = [r(t-1) \quad \cdots \quad r(t-n-m)]^{\mathrm{T}} \in \mathbb{R}^{n+m}, \tag{47}$$

$$W_{\mathrm{iv}} \text{ is an optimal weight} \tag{48}$$

and it is stated in Zhang et al. (1997) that

$$\hat{\theta}_{\mathrm{iv}}(N)|_{W_{\mathrm{iv}} = S^{\mathrm{T}}S^{-1}} = \hat{\theta}_{\mathrm{bels}}(N)$$

with $\hat{S} = [0 \;\; I_{n+m}]\hat{R}_{\varphi\varphi}(N)M(M^{\mathrm{T}}M)^{-1}$, assuming that it is nonsingular.

In the proof of this result the property is used that both weighted IV and BELS estimators can be written as

$$\hat{\theta}_{\mathrm{bels}}(N) = \hat{\theta}_{\mathrm{iv}}(N) = (\hat{K} - \hat{C}\hat{D}^{-1}\hat{F})\hat{p}, \tag{49}$$

where

$$\hat{C} = (M^{\mathrm{T}}M)^{-1}M^{\mathrm{T}}\hat{R}_{\varphi\varphi}^{-1}(N)\begin{pmatrix} I_{n+m} \\ 0 \end{pmatrix} \in \mathbb{R}^{2n\times(n+m)}, \tag{50}$$

$$\hat{K} = (M^{\mathrm{T}}M)^{-1}M^{\mathrm{T}}\hat{R}_{\varphi\varphi}^{-1}(N)\begin{pmatrix} 0 \\ I_{n+m} \end{pmatrix} \in \mathbb{R}^{2n\times(n+m)}, \tag{51}$$

$$\hat{D} = H_1^{\mathrm{T}}\hat{R}_{\varphi\varphi}^{-1}(N)\begin{pmatrix} I_{n+m} \\ 0 \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}, \tag{52}$$

$$\hat{F} = H_1^{\mathrm{T}}\hat{R}_{\varphi\varphi}^{-1}(N)\begin{pmatrix} 0 \\ I_{n+m} \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}. \tag{53}$$

However, Eq. (49) only holds true if matrix $\hat{D}$ is invertible (and thus square). This implies that matrix H_1 has dimensions $(2n+2m)\times(n+m)$. In the method of Zheng and Feng (1995), as considered in this paper, the matrix H has either dimensions $(2n+2m)\times 2m$ (situation $m \geqslant n$) or $(3n+m)\times(n+m)$ (situation $m < n$). As these dimensions do not match with the dimension of H_1, the method developed in Zhang et al. (1997) gives consistent estimates but cannot be associated with the BELS estimator considered here, neither in the $m \geqslant n$ case nor in the $m < n$ case.

6. Conclusions

It has been shown that a BELS estimator for closed-loop identification is equivalent to an instrumental variable estimator, where the predictor considered reflects the closed-loop system, and where external reference signals act as instrumental variables. This requires a tailor-made parametrization of the closed-loop system, as has been used in the literature before in a least-squares setting. The relation between BELS and IV greatly facilitates the understanding and analysis of the former method.

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Appendix

Proof of Proposition 1. The equations that constitute $\hat{\Theta}_{\mathrm{bels}}$ are collected in the following set of equations:

$$\hat{\Theta}_{\mathrm{bels}} = M\hat{\theta}_{\mathrm{bels}} + \rho, \tag{54}$$

$$\hat{\theta}_{\mathrm{bels}} = (M^{\mathrm{T}}M)^{-1}M^{\mathrm{T}}(\hat{\Theta}_{\mathrm{corr}} - \rho), \tag{55}$$

$$\hat{\Theta}_{\mathrm{corr}} = \hat{R}_{\varphi\varphi}^{-1}\hat{R}_{\varphi y} - \hat{\Delta}, \tag{56}$$

$$\hat{\Delta} = \hat{R}_{\varphi\varphi}^{-1}\begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix}\hat{R}_{y\xi}, \tag{57}$$

$$H^{\mathrm{T}}\hat{R}_{\varphi\varphi}^{-1}\begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix}\hat{R}_{y\xi} = H^{\mathrm{T}}(\hat{\Theta}_{\mathrm{ls}} - \rho), \tag{58}$$

$$\hat{\Theta}_{\mathrm{ls}} = \hat{R}_{\varphi\varphi}^{-1}\hat{R}_{\varphi y}, \tag{59}$$

where (58) can only be solved exactly if $r_B = 2n$. If $r_B > 2n$ (for example, if $m \geqslant n$) this equation has to be solved for $\hat{R}_{y\xi}$ in a least-square sense. To this end we denote

$$\mu := H^{\mathrm{T}}\hat{R}_{\varphi\varphi}^{-1}\begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix}\hat{R}_{y\xi} - H^{\mathrm{T}}(\hat{R}_{\varphi\varphi}^{-1}\hat{R}_{\varphi y} - \rho)$$

which by use of (57)–(59) can be shown to satisfy

$$\mu = H^{\mathrm{T}}(\hat{\Delta} - \hat{R}_{\varphi\varphi}^{-1}\hat{R}_{\varphi y} + \rho).$$

Minimization of $\mu^{\mathrm{T}}\mu$ over $\hat{R}_{y\xi}$ is equivalent to minimization of $\mu^{\mathrm{T}}\mu$ over $\hat{\Delta}$ provided that the structural constraint (57) on $\hat{\Delta}$ is added. This structural constraint can be formulated as $[0 \;\; I_{r_B}]\hat{R}_{\varphi\varphi}\hat{\Delta} = 0$, or equivalently

$$\hat{R}_{\varphi_r\varphi}\hat{\Delta} = 0. \tag{60}$$

Minimizing $\mu^T\mu$ under constraint (60) is achieved by solving the corresponding Lagrangian equation leading to

$$\begin{pmatrix} HH^T & \hat{R}^T_{\varphi_r\varphi} \\ \hat{R}_{\varphi_r\varphi} & 0 \end{pmatrix}\begin{pmatrix} \hat{\Delta} \\ \lambda \end{pmatrix} = \begin{pmatrix} HH^T(\hat{R}^{-1}_{\varphi\varphi}\hat{R}_{\varphi y} - \rho) \\ 0 \end{pmatrix}, \tag{61}$$

where $\lambda \in \mathbb{R}^{r_B}$ is the Lagrange multiplier.

The left matrix is square with dimensions $(2r_B + m + n) \times (2r_B + m + n)$. However in its current form it cannot be simply inverted. Therefore, Eq. (61) is directly combined with the remaining Eqs. (54)–(56) of the general solution for $\hat{\Theta}_{\text{bels}}$. Premultiplication of (54) with the nonsingular matrix $[M \; H]^T$ then leads to

$$\begin{pmatrix} M^T & M^T & 0 \\ H^T & 0 & 0 \\ 0 & HH^T & \hat{R}^T_{\varphi_r\varphi} \\ 0 & \hat{R}_{\varphi_r\varphi} & 0 \end{pmatrix}\begin{pmatrix} \hat{\Theta}_{\text{bels}} \\ \hat{\Delta} \\ \lambda \end{pmatrix} = \begin{pmatrix} M^T\hat{R}^{-1}_{\varphi\varphi}\hat{R}_{\varphi y} \\ H^T\rho \\ HH^T(\hat{R}^{-1}_{\varphi\varphi}\hat{R}_{\varphi y} - \rho) \\ 0 \end{pmatrix}. \tag{62}$$

Premultiplying the third block row with the nonsingular matrix $[M \;\; H]^T$, and using $H^TM = 0$ and $H^TH = I$, the set of equations becomes

$$\begin{pmatrix} M^T & M^T & 0 \\ H^T & 0 & 0 \\ 0 & H^T & H^T\hat{R}^T_{\varphi_r\varphi} \\ 0 & 0 & M^T\hat{R}^T_{\varphi_r\varphi} \\ 0 & \hat{R}_{\varphi_r\varphi} & 0 \end{pmatrix}\begin{pmatrix} \hat{\Theta}_{\text{bels}} \\ \hat{\Delta} \\ \lambda \end{pmatrix} = \begin{pmatrix} M^T\hat{R}^{-1}_{\varphi\varphi}\hat{R}_{\varphi y} \\ H^T\rho \\ H^T(\hat{R}^{-1}_{\varphi\varphi}\hat{R}_{\varphi y} - \rho) \\ 0 \\ 0 \end{pmatrix}. \tag{63}$$

Combining the first and third block row shows that

$$\begin{pmatrix} M^T \\ 0 \end{pmatrix}\hat{\Theta}_{\text{bels}} + \begin{pmatrix} M^T \\ H^T \end{pmatrix}\hat{\Delta} + \begin{pmatrix} 0 \\ H^T\hat{R}^T_{\varphi_r\varphi} \end{pmatrix}\lambda = \begin{pmatrix} M^T \\ H^T \end{pmatrix}\hat{R}^{-1}_{\varphi\varphi}\hat{R}_{\varphi y} - \begin{pmatrix} 0 \\ H^T \end{pmatrix}\rho, \tag{64}$$

which can be used to solve for $\hat{\Delta}$. Using the fact that $[M \; H]^T$ is invertible, and exploiting the relations $H^T\rho = H^T\hat{\Theta}_{\text{bels}}$ and $\hat{R}_{\varphi_r\varphi} = [0 \;\; I_{r_B}]\hat{R}_{\varphi\varphi}$, Eq. (64) is used to remove the $\hat{\Delta}$-dependent term in the last block row of (63), leading to

$$\hat{R}_{\varphi_r\varphi}\hat{\Theta}_{\text{bels}} + \hat{R}_{\varphi_r\varphi}\begin{pmatrix} M^T \\ H^T \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ H^T \end{pmatrix}\hat{R}^T_{\varphi_r\varphi}\lambda = \hat{R}_{\varphi_r y}. \tag{65}$$

Substituting $M^T\hat{R}^T_{\varphi_r\varphi}\lambda = 0$, (65) simplifies to

$$\hat{R}_{\varphi_r\varphi}\hat{\Theta}_{\text{bels}} + \hat{R}_{\varphi_r\varphi}\hat{R}^T_{\varphi_r\varphi}\lambda = \hat{R}_{\varphi_r y}. \tag{66}$$

As a result the set of equations (63) is equivalent to

$$\begin{pmatrix} \begin{pmatrix} M^T \\ 0 \end{pmatrix} & \begin{pmatrix} M^T \\ H^T \end{pmatrix} & \begin{pmatrix} 0 \\ H^T\hat{R}^T_{\varphi_r\varphi} \end{pmatrix} \\ H^T & 0 & 0 \\ \hat{R}_{\varphi_r\varphi} & 0 & \hat{R}_{\varphi_r\varphi}\hat{R}^T_{\varphi_r\varphi} \\ 0 & 0 & M^T\hat{R}^T_{\varphi_r\varphi} \end{pmatrix}\begin{pmatrix} \hat{\Theta}_{\text{bels}} \\ \hat{\Delta} \\ \lambda \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} M^T \\ H^T \end{pmatrix}\hat{R}^{-1}_{\varphi\varphi}\hat{R}_{\varphi y} - \begin{pmatrix} 0 \\ H^T \end{pmatrix}\rho \\ H^T\rho \\ \hat{R}_{\varphi_r y} \\ 0 \end{pmatrix}. \tag{67}$$

The $(n + m + 2r_B)$ unknowns ($\hat{\Theta}_{\text{bels}} \in \mathbb{R}^{n+m+r_B}$ and $\lambda \in \mathbb{R}^{r_B}$) are uniquely determined by the three last block rows ($n + m + 2r_B$ equations) of the previous system. This comes down to

$$H^T\hat{\Theta}_{\text{bels}} = H^T\rho, \tag{68}$$

$$\hat{R}_{\varphi_r\varphi}\hat{\Theta}_{\text{bels}} + \hat{R}_{\varphi_r\varphi}\hat{R}^T_{\varphi_r\varphi}\lambda = \hat{R}_{\varphi_r y}, \tag{69}$$

$$M^T\hat{R}^T_{\varphi_r\varphi}\lambda = 0. \tag{70}$$

Eq. (69) is used to determine λ

$$\lambda = (\hat{R}_{\varphi_r\varphi}\hat{R}^T_{\varphi_r\varphi})^{-1}(\hat{R}_{\varphi_r y} - \hat{R}_{\varphi_r\varphi}\hat{\Theta}_{\text{bels}}).$$

Then, substituting λ by its expression in Eq. (70) yields

$$M^T\hat{R}^T_{\varphi_r\varphi}(\hat{R}_{\varphi_r\varphi}\hat{R}^T_{\varphi_r\varphi})^{-1}(\hat{R}_{\varphi_r y} - \hat{R}_{\varphi_r\varphi}\hat{\Theta}_{\text{bels}}) = 0.$$

By substituting $\hat{\Theta}_{\text{bels}} = M\hat{\theta}_{\text{bels}} + \rho$ and

$$y(t) = \varphi^T[M\hat{\theta}_{\text{bels}} + \rho] + \varepsilon(t,\hat{\theta}_{\text{bels}})$$

it follows that

$$M^T\hat{R}^T_{\varphi_r\varphi}(\hat{R}_{\varphi_r\varphi}\hat{R}^T_{\varphi_r\varphi})^{-1}\hat{R}_{\varphi_r\varepsilon(t,\hat{\theta}_{\text{bels}})} = 0$$

or, equivalently

$$\hat{R}_{F\varphi_r\varepsilon(t,\hat{\theta}_{\text{bels}})} = 0.$$

This proves the result for $m \geqslant n$, corresponding to $r_B = n + m$.

In the situation $m \leqslant n$, corresponding to $r_B = 2n$, the matrix F is square and invertible under persistency of excitation conditions on r. As a result the set of equations can equivalently be characterized by $F = I_{r_B}$.

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