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On the choice of uncertainty structure in identification for robust control

S.G. Douma P.M.J. Van den Hof

Signals, Systems and Control Group, Department of Applied Physics,
Delft University of Technology, Lorentzweg 1, 2628 CJ Delft, The Netherlands
p.m.j.vandenhof@tnw.tudelft.nl

Abstract

Various techniques of system identification exist providing for a nominal model and uncertainty bound. An important question is what the implications are for the particular choice of the structure in which the uncertainty is described when dealing with robust stability/performance analysis of a given controller and when dealing with robust synthesis. An amplitude-bounded (circular) uncertainty set can equivalently be described in terms of an additive, Youla parameter and ν-gap uncertainty. Closed-loop performance functions based on these sets are again bounded by circles in the frequency domain, allowing for exact worst-case performance calculation and for the evaluation of the consequences of uncertainty for robust design.

1 Introduction

In identification for robust control an identified model has to be accompanied by a bound on its uncertainty, while the representation of this uncertainty should allow for robustness analysis and robust controller synthesis. A large number of such uncertainty descriptions is available from robust control theory, as e.g. a (H_∞)-norm-bounded additive or multiplicative uncertainty on the plant model, a norm-bounded uncertainty on a closed-loop plant representation (e.g. its dual Youla parameter), uncertainties bounded in the gap or ν-gap metric, and real parametric uncertainties, see e.g. [14, 15, 11].
In the past the necessity to deliver model uncertainty bounds has generated a new class of identification techniques directed towards the construction of worst-case (H_∞) error bounds [2]. However in many situations a worst-case bounded approach has been shown to lead to unnecessarily conservative results. On the other hand, a range of identification techniques exists providing for uncertainty structures identified from the data, where also account is given of the stochastic nature of disturbances in the data. The resulting probabilistic or combined probabilistic/worst-case approaches to the problem deliver a variety of uncertainty sets as e.g. parametrically structured (ellipsoidal) uncertainty, norm-bounded additive, non-parametric (boxed, ellipsoidal) additive in the frequency domain [9, 6, 7, 3].
Amongst such a variety of possible uncertainty structures a relevant question is what the implications are of a particular choice of structure for the identification for robust control problem. An ultimate question to be answered would be what is, for a given purpose (robust stability/performance analyis or synthesis), the best model uncertainty structure in which to identify the model set (nominal model and uncertainty bound). And consequently, what would be the best experiment allowing for minimisation of the uncertainty.
In this paper the former problem will be taken at hand. While extensive literature exists dealing with characteristics of each uncertainty structure, answering the posed question requires a thorough comparison and a bridging of the gap between identification and robust control that goes beyond the present state of the art. A first attempt with limited scope only directed towards robust stability issues was made in [4]. This paper is intended to highlight aspects in which the various uncertainty structures differ in their consequences for robust analysis and design and in their potentials to be determined on the basis of realistic experimental data.
In the next section the uncertainty structures and the performance measure is specified. The third section explores the link between the uncertainty structures and their behaviour under a linear fractional transformation. In the last three sections the uncertainty (structure) is analyzed with respect to, respectively, robust stability and robust performance (analysis/synthesis).

2 Framework

We consider single-input-single-output linear time-invariant finite dimensional systems $G(s)$ and controllers $C(s)$. Coprime factorizations of plants and controllers are defined as $G(s) = N(s)D^{-1}(s)$ and $C(s) = N_c(s)D_c^{-1}(s)$ where $N(s), D(s), N_c, D_c \in \mathbb{R}H_\infty$ satisfy the usual conditions [13]. The factorizations are normalized, denoted by $\overline{(\cdot)}$, if they additionally satisfy $\bar{N}(s)^*\bar{N}(s) + \bar{D}(s)^*\bar{D}(s) = 1$, where $(\cdot)^*$ denotes complex conjugate transpose. This paper considers three model sets based on a specific uncertainty structure:

Additive uncertainty

$$\mathcal{G}_a(G_x, W_a) \quad := \quad \{G_\Delta(s) \mid G_\Delta(s) = G_x(s) + \Delta_a(s) , \; |\Delta_a(i\omega)| \leq |W_a(i\omega)| \quad \forall\omega \in \mathbb{R}\}, \quad (1)$$

with $G_x(s)$ a nominal model and $W_a(s)$ a weighting function.

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Youla-uncertainty

$$\begin{aligned}&\mathcal{G}_Y(G_x,C,Q,Q_c,W_Y) := \\ &\Big\{ G_\Delta(s) \mid G_\Delta(s) = \frac{\bar{N}_x(s)+\bar{D}_c(s)\Delta_G(s)}{D_x(s)-N_c(s)\Delta_G(s)}\,, \\ &\left|Q_c^{-1}(i\omega)\Delta_G(i\omega)Q(i\omega)\right| \le |W_Y(i\omega)| \quad \forall \omega\in\mathbb{R}\Big\}\,.\end{aligned} \tag{2}$$

with $G_x(s) = \bar{N}_x(s)\bar{D}_x^{-1}(s)$ a nominal model, $C(s) = \bar{N}_c(s)\bar{D}_c^{-1}(s)$ a present controller and $Q(s)$,$Q_c(s)$ stable and stably invertible weighting functions reflecting the freedom in choosing the coprime factorizations of $G_x(s)$ and $C(s)$ [13]. An additional weighting can be provided by $W_Y(s)$. The Youla parameter $\Delta_G(s)$ is uniquely determined by $\Delta_G(s) = \bar{D}_c^{-1}(s)(1+G_\Delta(s)C(s))^{-1}\,(G_\Delta(s)-G_x(s))\bar{D}_x(s)$.

ν-*gap uncertainty* [14]

$$\begin{aligned}&\mathcal{G}_\nu(G_x,W_\nu) := \\ &\{G_\Delta(s) \mid \kappa(G_\Delta(i\omega),G_x(i\omega)) \le |W_\nu(i\omega)| \quad \forall\omega\in\mathbb{R}\}\,,\end{aligned} \tag{3}$$

with $\kappa(G_\Delta(\omega),G_x(\omega))$ denoting the chordal distance defined, for a plant $G_\Delta(s) = \bar{N}_\Delta(s)\bar{D}_\Delta^{-1}(s)$ with respect to the nominal model $G_x(s) = \bar{N}_x(s)\bar{D}_x^{-1}(s)$, by

$$\begin{aligned}\kappa(G_\Delta(i\omega),G_x(i\omega)) &:= |\bar{N}_x(i\omega)\bar{D}_\Delta(i\omega) - \bar{D}_x(i\omega)\bar{N}_\Delta(i\omega)| \\ &= \frac{|G_x(i\omega)-G_\Delta(i\omega)|}{\sqrt{\left(1+|G_\Delta(i\omega)|^2\right)\left(1+|G_x(i\omega)|^2\right)}}.\end{aligned}$$

Note that at this point the usual stability conditions are not yet imposed on either the $G_\Delta(s), G_x(s), \Delta(s), P(s)$ or $W(s)$. The focus here lies on the frequency domain conditions. In section 4 the stability conditions will be discussed.

We consider a performance measure formulated in the frequency domain:

$$J(G_\Delta,C,V,W) = \bar{\sigma}\left(V(i\omega)T(G_\Delta(i\omega),C(i\omega))W(i\omega)\right), \tag{4}$$

with $\bar{\sigma}$ the maximum singular value and

$$T(G_\Delta,C) := \begin{bmatrix} G_\Delta \\ 1 \end{bmatrix}(1+G_\Delta C)^{-1}\begin{bmatrix} C & 1 \end{bmatrix}. \tag{5}$$

The weighting matrices V and W are diagonal. These diagonal weighting functions allow for a large range of performance specifications (e.g. bounds on the (complementary) sensitivity function).

Remark 1 *From here onwards arguments are omitted to include both an evaluation in terms of transfer functions (e.g.* $G(s)$*) with frequency responses over the whole frequency axis, and an evaluation over a frequency grid with, e.g.,* $G(i\omega_k)\in\mathbb{C}$.

In all three of the above uncertainty sets a bounding condition is imposed on the frequency response of their members $G_\Delta(s)$. At each frequency $G_\Delta(i\omega)$ is constrained to within a circle in the complex plane. That this is the case for $\mathcal{G}_\nu(G_x,W_\nu)$ as well will be made clear in the next section. Further, note that the mapping of the open-loop transfer functions G_Δ to the closed-loop performance functions in (5) can be described with a linear fractional transformation (LFT), $F(P,C) := P_{22}+P_{21}C(1+P_{22}C)^{-1}P_{12}$. The behaviour of uncertainty sets under a linear fractional transformation is the topic of the next section.

3 Effect of linear fractional transformations

3.1 Mapping of circles

As is well known, a linear fractional transformation (Möbius transformation) maps circles into circles. The explicit formulation of the mapping allows for great insight when comparing uncertainty structures and their influence on performance.

Proposition 1 *A set of frequency responses* $F(P,\Delta)$ *described by the (SISO) LFT*

$$F(P,\Delta) = P_{22}+P_{21}\Delta(1+P_{11}\Delta)^{-1}P_{12}\,, \quad \textit{with } \left|W^{-1}\Delta\right| \le 1$$

and a one-dimensional uncertainty block Δ *can at each frequency be described in an additive structure as*

$$F(P,\Delta) = F_{centre}+\Delta_a \quad , \quad \left|W_a^{-1}\Delta_a\right| \le 1,$$

with

$$\begin{aligned}F_{centre} &= P_{22}+\frac{-P_{21}P_{12}P_{11}^*|W|^2}{1-|P_{11}W|^2} \\ W_a &= \frac{|P_{21}P_{12}|}{\left(1-|P_{11}W|^2\right)}|W|,\end{aligned}$$

provided that $|P_{11}W| < 1$. *Whenever* $|P_{11}W| > 1$, *the frequency responses of the set* $F(P,\Delta)$ *lie in the area outside the circle* $F_{centre}+\Delta_a$, $\left|W_a^{-1}\Delta_a\right| \le 1$.

The proposition is formulated in terms of frequency responses as this will be our main interest. Subtitution of $|W|^2$ by $W(s)W^*(s)$ would allow for a formulation in terms of (rational) systems. The same applies to the following sections. The proposition shows that any circular region in the frequency domain is again mapped into a circle. However, the original centre (P_{22}) is not the new centre unless the linear fractional transformation happens to be affine ($P_{11}=0$).

3.2 Performance functions

When using proposition 1 to study the effect of an uncertainty (structure) on the closed-loop performance of a controller, the entry P contains both the controller C and G_x. For example the set of sensitivity functions S_Δ induced by an additive uncertainty set $G_\Delta = G_x+\Delta_a$, $\left|W_a^{-1}\Delta_a\right| \le 1$ and a controller C is given, with $P_{22}=0$, $P_{12}P_{21} = \frac{1}{1+G_xC}$ and $P_{11} = \frac{C}{1+G_xC}$, by

$$\begin{aligned}S_\Delta &= \frac{1}{1+(G_x+\Delta_a)C}\,, \ \left|W_a^{-1}\Delta_a\right| \le 1 \\ &= \frac{(1+CG_x)^{-1}}{1-\left|(1+CG_x)^{-1}CW_a\right|^2}+\Delta_S,\end{aligned}$$

$$|\Delta_S| \quad \leq \quad |W_S| = \frac{\left|(1+CG_x)^{-2}C\right|}{1-\left|(1+CG_x)^{-1}CW_a\right|^2}\,|W_a|\,.$$

The circular representation allows for an analytical expression of minimal and maximal (worst-case) performance when evaluated in the frequency domain. E.g.

$$\max_{\Delta_a,\,|W_a^{-1}\Delta_a|\leq 1} |S_\Delta| = |S_{centre}| + |W_S|\,.$$

The example shows that the robust stability condition (cf. section 4) appears naturally in the denominators and how the nominal sentivity $(1+CG_x)^{-1}$ is not the centre of the set of sensitivity functions. This property becomes critically important when considering non-circular boundaries and/or probability density functions over the uncertainty set.

3.3 Probability density and non-circular bounds

From an identification point of view the probabilistic uncertainty regions usually follow from a complex probability density function [3][7][6][9]. The uncertainty region per frequency is bounded, with respect to a certain probability, either only in terms of the amplitude (circular) or the real and imaginary part separately (ellipsoidal or boxed). A (SISO) LFT, being a conformal mapping, will map closed contours into closed contours and will leave angles locally intact. However, the mapping will in general not preserve shape, as depicted in Figure 1. Worst-case performance analysis and robust stability evaluation on such non-circular sets will require special, adapted procedures.

As indicated in [8] the transformation will change the structure of the probability distribution. For example, when a closed-loop identified object is used to obtain the open-loop plant by recalculation with the present controller the statistical properties change drastically. An unbiased estimate of the closed-loop object does not imply an unbiased estimate of the recalculated open-loop plant. An important exception here is formed by all affine transformations ($P_{11}=0$) such as closed-loop functions of a Youla uncertainty set $\mathcal{G}_Y(G_x,C,Q,Q_c,W_Y)$ based on the present controller C (cf. section 5).

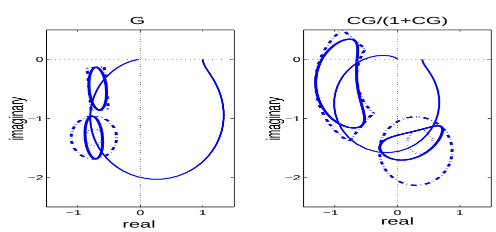

Figure 1: *Transformation of circular, ellipsoidal and boxed uncertainty bounds from plant model G (left) to closed-loop transfer $CG/(1+CG)$ (right).*

3.4 Circular uncertainty structures

All the uncertainty structures of section 2 can equivalently be described by an additive structure in terms of their frequency domain properties. While this follows directly from Proposition 1 for e.g. the Youla parameter uncertainty structure and (inverse) multiplicative structure, the ν-gap structure requires a separate proposition to show this fact. First the use of Proposition 1 for the Youla parameter uncertainty structure is made explicit.

Corollary 1 *The set of frequency responses of all plants $G_\Delta \in \mathcal{G}_Y(G_x,C,Q,Q_c,W_Y)$ (see (2)) is equivalently described as an additive uncertainty set $\mathcal{G}_a(G_{centre},W_a)$ (see (1)) with*

$$G_{centre} = G_x\left(\frac{|D_x|^2}{|D_x|^2-|N_cW_Y|^2}\right) - C^{-1}\left(\frac{-|N_cW_Y|^2}{|D_x|^2-|N_cW_Y|^2}\right)$$

$$W_a = \frac{|\Lambda|}{|D_x|^2-|N_cW_Y|^2}\,|W_Y|\,,$$

where $\Lambda = N_xN_c + D_cD_x$ and $N_x=\bar{N}_xQ$, $D_x=\bar{D}_xQ$, $N_c=\bar{N}_cQ_c$, $D_c=\bar{D}_cQ_c$.

Note that the centre of the Youla uncertainty set is given by a convex combination of the nominal model G_x and the negative inverse of the controller C used in the Youla parametrization. The transformation of the ν-gap uncertainty set to an additive structure is indicated in the following proposition.

Proposition 2 *The set of frequency responses of all plants $G_\Delta \in \mathcal{G}_\nu(G_x,W_\nu)$ (as defined in (3)) is equivalently described as an additive uncertainty set $\mathcal{G}_a(G_{centre},W_a)$ (see (1)) with*

$$G_{centre} = \frac{G_x}{1-\left(1+|G_x|^2\right)|W_\nu|^2}$$

$$W_a = \frac{\sqrt{\left(1-|W_\nu|^2\right)\left(|G_x|^2+1\right)}\,|W_\nu|}{1-\left(1+|G_x|^2\right)|W_\nu|^2}.$$

The fact that both the Youla uncertainty set and the ν-gap uncertainty set allow for an additive description shows that the two sets can be transformed into one another. The explicit formulations above of the uncertainty sets in terms of an additive structure allows for a thorough comparison, as is further explored in the coming section.

3.5 Interpretation

Consider an uncertainty bound in the frequency domain of any shape following from an identification experiment, as e.g. an ellipsoidal region following from [3][7]. It is clear that the smallest (unique) circle embedding the uncertainty can equivalently be described in all structures of section 2 (with different nominal models and weighting functions). Alternatively, this also pleads for an identification criterium minimizing over both nominal model and uncertainty, rather than first identifying a nominal model (with any method) and subsequently bounding the model uncertainty.

In general, however, embedding is sought while maintaining a particular nominal model, in which case all structures will provide different embedding regions. Propositions 1, 2 and Corollary 1 show in which directions the centres move and the circles expand. For example, it is immediate that when embedding an additive set $\mathcal{G}_a(G_x, W_a)$ with a ν-gap set $\mathcal{G}_\nu(G_x, W_\nu)$, the size W_ν (chordal distance) is determined by the element $G_x\left(|G_x - W_a| / |G_x|\right)$. That is, the direction away from the nominal model G_x towards the origin is most costly, in terms of increase of the uncertainty region (see figure 2). However, the increase of the uncertainty region should be judged against the effect on the attainable performance (cf. section 5).

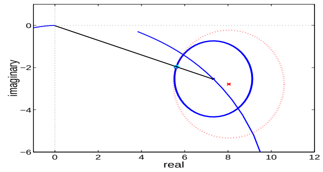

Figure 2: *Embedding an additive uncertainty (solid) with a ν-gap uncertainty (dashed-dot). The point in line with the nominal model* G_x ('x') *will cause the largest chordal distance.*

4 Robust stability

4.1 Robust stability and the frequency domain

Checking for internal stability for all elements $G_\Delta(s) \in \mathcal{G}$ is feasible when certain conditions are imposed on $G_\Delta(s)$. For example, all controllers $C(s)$, satisfying the condition $\left|C(i\omega)\left(1 + C(i\omega) G_x(i\omega)\right)^{-1}\right| < \left|W_a^{-1}(i\omega)\right|$, stabilize the additive set $\mathcal{G}_a(G_x, W_a)$ if and only if $G_x(s)$ is stabilized by $C(s)$ and the number of unstable poles of all $G_\Delta(s)$ is equal to the number of unstable poles of $G_x(s)$. In such conditions for robust stability three parts can be discerned:

i. the condition that $C(i\omega) \neq -G_\Delta^{-1}(i\omega)$ for all $G_\Delta(i\omega) \in \mathcal{G}$ and for all $\omega \in \mathbb{R}$.

ii. stability of C with a nominal model $G_x \in \mathcal{G}$.

iii. conditions on all $G_\Delta \in \mathcal{G}$ with respect to the nominal model G_x.

The first condition on the frequency responses seems most characteristic for different uncertainty structures: $\left|C(i\omega)\left(1 + C(i\omega) G_x(i\omega)\right)^{-1}\right| < \left|W_a^{-1}(i\omega)\right|$ for additive, $\left|Q^{-1}(i\omega)\Delta_C(i\omega)Q_c(i\omega)\right| \leq \left|W_Y^{-1}(i\omega)\right|$ for the Youla parameter uncertainty and $\bar{\sigma}\left(T(C(i\omega), G_x(\omega)\right) < \left|W_\nu^{-1}(i\omega)\right|$ for the ν-gap uncertainty. However, they are simply ensuring the condition that $C(i\omega) \neq -G_\Delta^{-1}(i\omega)$ for all $G_\Delta(i\omega) \in \mathcal{G}$. And from Proposition 1 it was clear that all three uncertainty sets can be transformed into one another with respect to the frequency responses of the members. Naturally the 'nominal' model and weighting function will change. For example, part *i.* of the robust stability condition for the ν-gap set $\bar{\sigma}\left(T(C(i\omega), G_x(i\omega)\right) < \left|W_\nu^{-1}(i\omega)\right|$, is equivalently described by the (additive) condition $\left|C\left(1 + CG_{centre}\right)^{-1}\right| < \left|W_a^{-1}\right|$, with G_{centre} and W_a from Proposition 2. The uncertainty sets do differ in terms of parts *ii*. and *iii*.. That is, they differ in terms of a winding number condition or a condition on unstable poles and zeros.

4.2 Identification and robust stability

Identification techniques as [3][6][7], which take bias effects into consideration, characterize the plant identification uncertainty in terms of bounds on the frequency response. However, we have seen that for robust stability the distinguishing factor between the uncertainty sets is a winding-number/pole/zero condition on the members of the set. It is important to note that this information does not come directly from the identification procedure, but has to be provided based on an assumption. A realistic assumption would be that the identified object is stable. That is, an open-loop identification could lead to an additive uncertainty with the condition that all elements $G_\Delta \in \mathcal{G}_a(G_x, W_a)$ are stable. Or, a closed-loop identification could lead to a Youla parameter uncertainty where a stability condition on Δ_G can automatically be satisfied. The Youla parameter can directly be identified from closed-loop data as described in [12].

5 Robust performance analysis

A controller is said to perform robustly for a set of plants if a certain performance level is reached for all plants in the uncertainty set. A robust performance analysis comes down to a worst-case performance evaluation over the uncertainty set. To this end, μ-analysis or LMI based procedures are available for general (LFT based) uncertainty structures [15]. For SISO systems with a one dimensional uncertainty block, bounded in amplitude, both methods provide for an unconservative answer. However, in the following we will derive analytical expressions since this will allow for much more insight. In case non-circular uncertainty regions are considered, in some cases an adapted μ-analysis could be employed, as for ellipsiodal regions [1], but in general the procedure would become complex.

5.1 Analytical worst-case calculation

Section 3.2 has shown how Proposition 1 allows for an analytical expression of the worst-case performance for all SISO closed-loop performance functions. Moreover Proposition 1 reveals the fact that every increase in uncertainty (W) at a particular frequency results in a decrease of the worst-case performance cost at that frequency (due to W_a). A new experiment could be designed to decrease the worst-case performance at a particular frequency by inducing that amount of power there that could reduce the uncertainty with a factor explicitly given by Proposition 1. Note, however that the new experiment would also yield a new nominal controller.

The Youla parameter uncertainty structure plays a special rôle when considering robust performance for the auxiliary controller C used in the parametrization. For example,

$$\begin{aligned}\frac{G_\Delta C}{1 + CG_\Delta} &= \frac{N_c\left(N_x + D_c\Delta_G\right)}{D_c\left(D_x - N_c\Delta_G\right) + N_c\left(N_x + D_c\Delta_G\right)} \\ &= \frac{G_x C}{1 + CG_x} + \frac{N_c D_c}{D_c D_x + N_c N_x}\Delta_G.\end{aligned}$$

For the auxiliary controller the closed-loop functions of the set $\mathcal{G}_Y(G_x, C, Q, Q_c, W_Y)$ are affine in the uncertainty Δ_G. This implies that the nominal performance will be the centre of the set of performance associated with $\mathcal{G}_Y(G_x, C, Q, Q_c, W_Y)$. Moreover, probability density functions and non-circular uncertainty structures (e.g. boxed/ellipsoidal/irregular) will maintain their shape under the mapping to the performance functions.

In case the weights V and W in (4) are chosen such that the performance function is not SISO, Proposition 1 cannot be used. For this case the following Lemma is presented.

Lemma 1 *Consider the following weighted $T(G_\Delta, C)$ matrix for SISO systems G_Δ and C:*

$$\begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix} \begin{pmatrix} G_\Delta \\ 1 \end{pmatrix} (1+CG_\Delta)^{-1} \begin{pmatrix} C & 1 \end{pmatrix} \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix}.$$

The maximum singular value $\bar{\sigma}(VT(G_\Delta, C)W)$ is given by
$$\bar{\sigma}(VT(G_\Delta, C)W) = \left(\left(S_\Delta - \frac{|V_1|^2}{(|V_1|^2+|V_2|^2|C|^2)} \right) \left(S_\Delta - \frac{|V_1|^2}{(|V_1|^2+|V_2|^2|C|^2)} \right)^* + \frac{|V_1|^2|V_2|^2|C|^2}{(|V_1|^2+|V_2|^2|C|^2)^2} \right) \frac{(|V_1|^2+|V_2|^2|C|^2)(|W_2|^2+|W_1|^2|C|^2)}{|C|^2},$$
where $S_\Delta = (1+CG_\Delta)^{-1}$. A similar result is derived in [14] in connection with loop-shaping.

With Lemma 1 the worst-case performance over a set $\mathcal{G}$ can at each frequency be made explicit using Proposition 1. Any (circular) uncertainty set $\mathcal{G}$ can be transformed with a controller C into an associated set of sensitivity functions $\mathcal{S}_\mathcal{G}(G_x, C, W_a) := \{S_\Delta \mid S_\Delta = S_{centre} + \Delta_a \ , \left|W_a^{-1}\Delta_a\right| < 1\}$. Lemma 1 then shows that the worst-case performance $\bar{\sigma}(VT(G_\Delta, C)W)$ is achieved for that particular $S_{WC} \in \mathcal{S}_\mathcal{G}(G_x, C, W_a)$ most removed from the 'centre' $|V_1|^2 / \left(|V_1|^2 + |V_2|^2 |C|^2\right)$.

5.2 Performance cost and uncertainty sets

Lemma 1 can also be read as to give a description of all sensitivity functions achieving $\bar{\sigma}(VT(G_\Delta, C)W) < 1$. Using the fact that $G_\Delta = \frac{1-S_\Delta}{CS_\Delta}$, all plants achieving $\bar{\sigma}(VT(G_\Delta, C)W) < 1$ with C must have their frequency responses in the region described by the following corollary,

Corollary 2 *All plants G_Δ achieving $\bar{\sigma}(VT(G_\Delta, C)W) < 1$ are characterized by*

$$G_\Delta = \frac{D_c |V_2|^2 |C|^2 + D_c \Delta}{N_c |V_1|^2 - N_c \Delta}, \text{ with } |\Delta| \le W_Y \text{ and}$$
$$W_Y = |C| \sqrt{\left(\frac{\left(|V_1|^2 + |V_2|^2 |C|^2\right)}{\left(|W_2|^2 + |W_1|^2 |C|^2\right)} - |V_1|^2 |V_2|^2 \right)}$$

Due to the LFT structure, this set can also be described in an additive structure with Proposition 1 ,

$$G_\Delta = G_{centre} + \Delta_a \ , \ \left|W_a^{-1}\Delta_a\right| \le 1$$
$$G_{centre} = C^{-1} \frac{|W_Y|^2 + |V_2|^2 |C|^2 |V_1|^2}{|V_1|^4 - |W_Y|^2}$$
$$W_a = \left|C^{-1}\right| \frac{\left| |V_2|^2 |C|^2 + |V_1|^2 \right|}{|V_1|^4 - |W_Y|^2} W_Y.$$

Note again that the set equals the exterior of the circle in case $|V_1|^4 < |W_Y|^2$.

What can Corollary 2 show us with regards to robust performance analysis? It provides us with a clear indication of the 'cost' of an uncertainty region. The worst-case performance over an uncertainty set $\mathcal{G}$ is determined by the measure in which the set of Corollary 2 has to expand in order to contain all the members G_Δ of the set $\mathcal{G}$. As extreme example, a certain nominal performance with G_x is achieved by many a plant G_Δ according to Corollary 2. The set of all these plants (with G_x on the boundary) could be taken as an uncertainty around G_x for free, i.e. without changing the worst-case performance.
The centre of the set of plants performing with the controller C lies in the direction of C^{-1}. This indicates a direction in which the performance cost is most sensitive to an increase in uncertainty. The Youla parametrization was shown to expand in the direction of a convex combination of the nominal model G_x and the negative inverse of the auxiliary controller C_{aux} (cf. Corr. 1). In case the auxiliary controller C_{aux} is close to the controller C to be evaluated, the Youla parameter uncertainty is 'pulled' in the right direction, i.e. in the direction being least sensitive for an increase in the worst-case performance. However, at this point this is still a matter of research. It is further interesting to research the connection between this corollary and the results of [5] based on computational techniques which do include robust stability.

6 Robust performance synthesis

The final goal in identification for control is *designing* a controller achieving a certain worst-case performance over a set of plants. For SISO systems with a one-dimensional uncertainty block, this problem can be solved quite well with standard μ-synthesis or LMI [15]. However, such computational methods do not lend themselves for understanding the influence of uncertainty (structures) on the attainable performance of such methods. Fortunately, for the ν-gap uncertainty structure an analytical expression exists.

Proposition 3 *[14]. Given a ν-gap set $\mathcal{G}_\nu(G_x, W_\nu)$ as defined in (3). Consider the so-called loop-shaped performance measure $\bar{\sigma}\left(T(WG_\Delta, W^{-1}C)\right)$ [10]. Then*

$$\max_{G_\Delta \in G_\nu(G_x, W_\nu)} \bar{\sigma}\left(T(WG_\Delta, W^{-1}C)\right)$$
$$= \sin\left(arc\sin\left(\bar{\sigma}\left(T(WG_x, W^{-1}C)\right)\right)^{-1} - arc\sin(W_\nu)\right)^{-1}.$$

This proposition allows for a robust performance analysis, but more importantly for a robust performance synthesis as well. It reveals that the worst-case performance is minimized by minimizing the nominal performance. And for nominal performance optimization many techniques are available. [5, 10, 15] The considerations in the previous sections show that this result can be generalized to other uncertainty structures. To this end, consider the following lemma.

Lemma 2 *An additive uncertainty set $\mathcal{G}_a(G_a, W_a)$ is alternatively described in terms of a ν-gap uncertainty $\mathcal{G}_\nu(G_\nu, W_\nu)$ with*

$$G_\nu = aG_a; \quad W_\nu = \left(|W_a|^2 \frac{a^2}{(1-a)^2} + 1 \right)^{-\frac{1}{2}},$$

with the factor a the positive solution smaller than 1 to the third order equation $|G_a|^2 a^3 + \left(|W_a|^2 + 1 - 2|G_a|^2\right) a^2 + \left(|G_a|^2 - 2 - |W_a|^2\right) a + 1 = 0.$

On combining Lemma 2 and Proposition 1 all (circular) uncertainty structures can be transformed into a ν-gap uncertainty description. This in turn allows for the application of Proposition 3. For the actual performance synthesis the existing techniques will have to be adapted as the robust stability conditions will change, due to the change in nominal model.
Loop-shaping limits the possible performance specifications. For more general weighting functions V and W, we have to resort to the dual form of Corollary 2. That is, for a particular plant G_Δ the set of all controllers achieving a certain performance can explicitly be given. The intersection of all these controllers when evaluated over a set of plants $\mathcal{G}$ is the set of all robustly performing controllers for that set. Research so far indicates that this set of controllers can again be described in simple geometric terms (either a circle (as for Prop 3) or ellipse). Figure 3 illustrates this point. Here, one could consider not to embed the plant uncertainty set (e.g. coming from multiple nonparametric identification experiments), in terms of fitting a nominal plant and constructing an uncertainty structure. Instead, the uncertainty could be transformed directly with Corollary 2 into a set of frequency responses of robustly performing controllers. Then a controller could be 'identified' by fitting a model to the frequency response data.

7 Conclusions

A first attempt is made to identify and discuss implications of the choice of an uncertainty structure to the robust control problem. An amplitude-bounded (circular) uncertainty set following from system identification can equivalently be described in terms of an additive, Youla parameter and ν-gap uncertainty. Closed-loop performance functions based on these sets are again bounded by circles in the frequency domain, allowing for exact worst-case performance calculation and for the evaluation of the consequences of uncertainty for robust design. Uncertainty sets with noncircular bounds and their underlying probability density functions do not retain their properties when transformed to closed-loop functions.

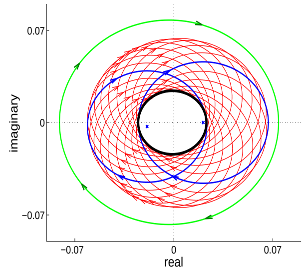

Figure 3: *Depiction at one frequency of the set of frequency responses of robustly performing controllers. For each point of an additive set $G_a(G_x, W_a)$ the (circular) set of controllers achieving a certain performance is given with Corollary 2. The intersection of these sets (thick solid circle) is the set of robustly performing controllers. The thin outer circle depicts the region of all controllers satisfying the robust stability condition $\left|\frac{C}{1+CG_x}\right| < |W_a|$.*

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