Proceedings of the American Control Conference
Arlington, VA June 25-27, 2001

Controller tuning freedom under plant identification uncertainty: double Youla beats gap in robust stability

Sippe G. Douma [a] Paul M.J. Van den Hof [a][1] Okko H. Bosgra [b]

[a] Signals, Systems and Control Group, Department of Applied Physics,
Delft University of Technology, Lorentzweg 1, 2628 CJ Delft, The Netherlands
[b] Mechanical Engineering Systems and Control Group,
Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands.
s.g.douma, p.m.j.vandenhof@tnw.tudelft.nl, o.h.bosgra@wbmt.tudelft.nl

Abstract

In iterative schemes of identification and control one of the particular and important choices to make is the choice for a model uncertainty structure, capturing the uncertainty concerning the estimated plant model. This is typically done in some norm-bounded form, in order to guarantee robust stability and/or robust performance when redesigning the controller. Structures that are used in the recent literature encompass e.g. gap metric uncertainty, coprime factor uncertainty, and the Vinnicombe gap metric uncertainty. In this paper we study the effect of these choices when our aim is to maximize the (re)tuning freedom for a present controller under conditions of robust stability. Particular attention will be given to the representation of plant uncertainty and controller tuning freedom in terms of Youla parameters. This so-called double Youla parametrization provides a norm-bounded set of robustly stabilizing controllers that is larger than corresponding sets that are achieved by using any of the other uncertainty structures.

1 Problem Set-up

We consider linear time-invariant finite dimensional systems and controllers in $\mathbb{R}H_\infty$, in a feedback configuration depicted in figure 1, denoted by $H(G_0,C)$, where G_0 is the plant to be (modelled and) controlled, and C a present and known controller to be redesigned. The closed-loop dynamics of $H(G_0,C)$ are described by the transfer matrix

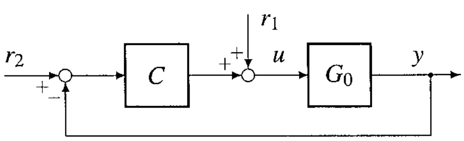

Figure 1: Feedback interconnection $H(G_0,C)$.

$$T(G_0,C) = \begin{bmatrix} G_0 \\ I \end{bmatrix} (I+CG_0)^{-1}[C \;\; I],$$

which maps the vector of variables $col(r_1,r_2)$ into $col(y,u)$. The closed-loop system is stable if and only if $T(G_0,C) \in \mathbb{R}H_\infty$.

The problem field that we consider can be formulated as follows:

> Consider an (unknown) plant G_0 controlled by a known controller C, redesign the controller so as to achieve a better control performance for the controlled plant G_0.

There are several different aspects that can be distinguished in this problem, as e.g.

- One can construct an identified (uncertainty) model of the plant G_0 on the basis of experimental data, e.g. composed of a nominal model and some norm-bounded model or parameter uncertainty. See e.g. [13, 10, 6, 1].
- The redesigning of the controller can be performed on the basis of a single model (nominal design possibly extended with robustness verifications), a (norm-bounded) uncertainty model (robust design), or on no model at all (as e.g. in iterative feedback tuning [11]).

If in the controller redesign a (norm-bounded) uncertainty model is taken into account, then the worst-case performance of the newly designed control system can be optimized. This approach is e.g. followed in [6] where the control design step is a robust control design optimizing the worst-case performance cost. If the uncertainty set that is identified contains the underlying real plant, guaranteed performance bounds will hold for the controlled real plant also. In this approach the control design utilizes all (uncertain) information on the plant that is available. The resulting control design algorithm becomes relatively complex (μ-synthesis in the work of [6]).

[1]This work is part of the research program of the 'Stichting voor Fundamenteel Onderzoek der Materie (FOM)', which is financially supported by the 'Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO)'.

When in the controller redesign only a nominal model is taken into account for the design itself, and an uncertainty model for the plant is used a posteriori to verify the robustness of this design, there is a need for robustness tests concerning stability (and possibly performance).
In this contribution we focus on the latter situation, assuming that the controller C has to be redesigned (retuned) into C_{new} by some (not specified) design method, and that the aim is to construct a (norm-bounded) area around C that characterizes the tuning freedom for C_{new} under conditions of robust stability, i.e. under the condition that C_{new} stabilizes all models in the identified uncertainty set.
The answer to this question, i.e. the size of the norm-bounded set of controllers, is typically dependent on the uncertainty structure that is chosen to represent the plant identification uncertainty. In this paper different structures will be analysed and compared. In particular a gap metric uncertainty structure will be applied and will be shown to lead to results that are more conservative than the results that are obtained when employing a so-called double Youla representation of plant uncertainty and controller retuning freedom.

2 Preliminaries

A coprime factor framework will be used to represent plants and controllers, employing both right coprime and left coprime factorizations:

$$\begin{aligned} G(s) &= N(s)D^{-1}(s) = \tilde{D}^{-1}(s)\tilde{N}(s) \\ C(s) &= N_c(s)D_c^{-1}(s) = \tilde{D}_c^{-1}(s)\tilde{N}_c(s) \end{aligned} \tag{1}$$

where (N,D) and (N_c,D_c) are right coprime factorizations *(rcf)* and $(\tilde{N},\tilde{D})$ and $(\tilde{N}_c,\tilde{D}_c)$ are left coprime factorizations *(lcf)* over $\mathbb{R}H_\infty$. [17] The coprime factorizations are normalized *(nrcf)*, *(nlcf)* if they additionally satisfy

$$\bar{N}^*\bar{N}+\bar{D}^*\bar{D}=I \qquad \bar{\tilde{N}}\bar{\tilde{N}}^*+\bar{\tilde{D}}\bar{\tilde{D}}^*=I$$

where $(\cdot)^*$ denotes complex conjugate transpose.
Let G and C have coprime factorizations as in (1) and let $\Lambda,\tilde{\Lambda} \in \mathbb{R}H_\infty$ be defined as

$$\Lambda=\bar{\tilde{N}}_c\bar{N}+\bar{\tilde{D}}_c\bar{D} \qquad \tilde{\Lambda}=\bar{\tilde{N}}\bar{N}_c+\bar{\tilde{D}}\bar{D}_c, \tag{2}$$

then $H(G,C)$ is stable iff $\Lambda^{-1} \in \mathbb{R}H_\infty$ which is equivalent to the condition $\tilde{\Lambda}^{-1} \in \mathbb{R}H_\infty$ ([17]).

3 Robust stability results for double-Youla representations

Uncertainty on a plant G_0 can be described in very many different ways. In a norm-bounded formulation, there are options for additive, multiplicative, coprime-factor, gap-metric uncertainties, all having their particular robust stability tests. See e.g. [5] for an overview in a rather uniform (coprime factor) framework.
When considering robust performance tests on norm-bounded uncertainty sets, it has been motivated in [6] that for general classes of performance measures, norm-bounded uncertainty in a dual Youla parametrization framework has particular advantages. In this parametrization, a norm-bounded plant uncertainty set is considered of the form:

$$\mathcal{P}(G_x)=\{G \mid G_\Delta=\frac{N_x+D_c\Delta_R}{D_x-N_c\Delta_R}, \|\Delta_R\| \leq \gamma_G\}$$

with the present controller C given by the *rcf* $C=N_cD_c^{-1}$ and a nominal model G_x (stabilized by C) given by $G_x=N_xD_x^{-1}$. In terms of stability, the dual-Youla parametrization has the basic property that an element in $\mathcal{P}$ is stabilized by C if and only if the corresponding Δ_R is stable.
Similar to characterizing plant uncertainty, a retuning or adaptation of the controller can be represented as a Youla-type "perturbation" on the present controller C. This results in the so-called double Youla parametrization, indicated in Figure 2, where

$$C_{new}:=C_\Delta=(N_c+D_x\Delta_C)(D_c-N_x\Delta_C)^{-1}.$$

The following stability results apply to this situation ([16, 14]).

Proposition 1 *Let G_x and C have normalized coprime factorizations as described above, and let $H(G_x,C)$ be stable. Denote*

$$\begin{aligned} G_\Delta &= (\bar{N}_x+\bar{D}_c\bar{\Delta}_R)(\bar{D}_x-\bar{N}_c\bar{\Delta}_R)^{-1} &(3)\\ C_\Delta &= (\bar{N}_c+\bar{D}_x\bar{\Delta}_C)(\bar{D}_c-\bar{N}_x\bar{\Delta}_C)^{-1}. &(4) \end{aligned}$$

Then for $\bar{\Delta}_R,\bar{\Delta}_C \in \mathbb{RH}_\infty$
(a) $H(G_\Delta,C_\Delta)$ is stable if and only if $H(\bar{\Delta}_R,\bar{\Delta}_C)$ is stable;
(b) $H(G_\Delta,C_\Delta)$ is stable if there exist some unimodular $Q,Q_c \in \mathbb{RH}_\infty$ such that

$$\left\|Q^{-1}\bar{\Delta}_C Q_c\right\|_\infty \cdot \left\|Q_c^{-1}\bar{\Delta}_R Q\right\|_\infty < 1 \tag{5}$$

The unimodular matrices Q and Q_c reflect the freedom in choosing the coprime factorizations of G_x and C.
Based on the sufficient condition (b) for stability the following result can be formulated.

Proposition 2 *Given a nominal model G_x and a nominal controller C, with nrcf's as described before, such that $H(G_x,C) \in \mathbb{R}H_\infty$. Define a set of plants $\mathcal{G}_Y(\gamma_G)$ and a set of controllers $\mathcal{C}_Y(\gamma_C)$ as*

$$\mathcal{G}_Y(\gamma_G):=\{G_\Delta=(\bar{N}_x+\bar{D}_c\bar{\Delta}_R)(\bar{D}_x-\bar{N}_c\bar{\Delta}_R)^{-1} \mid \|\bar{\Delta}_R\|_\infty<\gamma_G\}$$
$$\mathcal{C}_Y(\gamma_C):=\{C_\Delta=(\bar{N}_c+\bar{D}_x\bar{\Delta}_C)(\bar{D}_c-\bar{N}_x\bar{\Delta}_C)^{-1} \mid \|\bar{\Delta}_C\|_\infty<\gamma_C\}$$

Then all plants in $\mathcal{G}_Y(\gamma_G)$ are stabilized by all controllers contained in the set $\mathcal{C}_Y(\gamma_C)$ if and only if

$$\gamma_G \cdot \gamma_C < 1.$$

This Proposition serves as a means to specify the allowable area for retuning the controller C so as to guarantee robust stability for all models in the plant uncertainty set. Since the result is based on a small gain criterion, the resulting set of controllers equals the set of all controllers stabilizing the set $\mathcal{G}_Y(\gamma_G)$. Note that the freedom in choosing the coprime factorization of G_x and C has not been exploited yet.

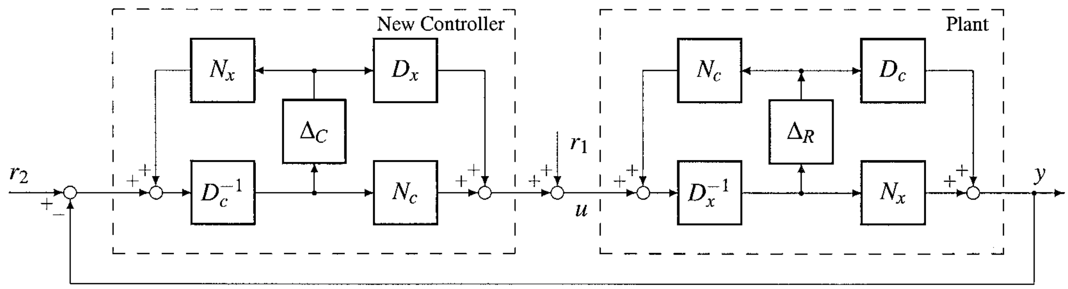

Figure 2: Double Youla parametrization.

4 Gap metric results

When considering the gap metric as a measure for bounding plant uncertainty a similar analysis can be given as presented in the previous section. The gap metric distance between two systems G_x, G_Δ is defined by

$$\delta(G_x, G_\Delta) = \max\{\vec{\delta}(G_x, G_\Delta), \vec{\delta}(G_\Delta, G_x)\}$$

where the *directed gap* is:

$$\vec{\delta}(G_x, G_\Delta) = \inf_{Q \in H_\infty} \| \begin{bmatrix} \bar{N}_x \\ \bar{D}_x \end{bmatrix} - \begin{bmatrix} \bar{N}_\Delta \\ \bar{D}_\Delta \end{bmatrix} \bar{Q} \|_\infty \qquad (6)$$

where $(\bar{N}_x, \bar{D}_x)$ and $(\bar{N}_\Delta, \bar{D}_\Delta)$are *nrcf's* G_x and G_Δ.
The stability result that is applicable to our problem set up is the following, see e.g. [9, 18].

Proposition 3 *Let $H(G_x, C)$ be stable. Then $H(G_\Delta, C_\Delta)$ is stable if*

$$\delta(G_x, G_\Delta) + \delta(C, C_\Delta) < \|T(G_x, C)\|_\infty^{-1}. \qquad (7)$$

This sufficient condition for stability leads to the following formulation in terms of stabilizing sets of controllers.

Proposition 4 *Given a nominal model G_x and a nominal controller C such that $H(G_x, C) \in \mathbb{R}H_\infty$. The set $\mathcal{G}_\delta(\delta_G)$ defined as*

$$\mathcal{G}_\delta(\delta_G) := \{G_\Delta | \quad \delta(G_x, G_\Delta) \le \delta_G\}$$

is stabilized by all controllers contained in the set $\mathcal{C}_\delta(\delta_C)$ defined as

$$\mathcal{C}_\delta(\delta_C) := \{C_\Delta | \quad \delta(C_x, C_\Delta) < \delta_C\}$$

if and only if $\delta_C \le \|T(G_x, C)\|_\infty^{-1} - \delta_G$.

In this proposition the maximum size of retuning range (or the allowed "perturbation" from the present controller) is specified that is allowed under guarantee of robust stability.

5 Comparison of the two uncertainty structures

Theorem 1 *Given a nominal model G_x and a nominal controller C such that $H(G_x, C) \in \mathbb{R}H_\infty$. Consider any plant $G_\Delta \in \mathbb{R}H_\infty$, and determine the minimal values of $\bar{\gamma}_G$ and $\bar{\delta}_G$ such that $G_\Delta \in \mathcal{G}_Y(\bar{\gamma}_G)$ and $G_\Delta \in \mathcal{G}_\delta(\bar{\delta}_G)$. Then the sets of controllers that result from Propositions 2 and 4, i.e.*

$$\begin{aligned} \mathcal{C}_Y &= \mathcal{C}_Y(\bar{\gamma}_G^{-1}) \\ \mathcal{C}_\delta &= \mathcal{C}_\delta(\bar{\delta}_C), \quad \bar{\delta}_C = \|T(G, C)\|_\infty^{-1} - \bar{\delta}_G \end{aligned}$$

relate to each other according to

$$\mathcal{C}_\delta \subseteq \mathcal{C}_Y.$$

Proof. A sketch of proof is added in the appendix.

The result of this theorem implies that when describing plant uncertainty in either a gap metric bound or a norm bound in a dual-Youla representation, the latter format allows for a larger set of controllers that guarantee robust stability. In other words: the related robust stability test for the Youla-structured uncertainty is less conservative than the test for the gap metric.
The formulation of the theorem allows that the two sets of controllers actually are equal. The fact that the two sets are not equal is shown in the next section by a (counter) example which is taken from [14].
One of the principal differences in the two uncertainty structures is that a gap-metric distance between two plants is controller independent. A Youla formulation of the "distance" between two plants is taken under the presence of (and therefore dependent on) a particular controller. In the latter situation the closed-loop properties of the two plants can therefore more particularly be taken into account.

6 Example

An example is considered in which robust stability is guaranteed by the condition of Proposition 2, but not by the gap-metric condition of Proposition 4. The systems of concern have the following transfer functions:

$$\begin{aligned} G_x &= \frac{-s+1}{4s^3+0.4s^2+4s} \\ C &= \frac{17s^2-2.3s+10}{s^2+3.3s+11} \\ G_\Delta &= \frac{0.2s^7+3s^6+5.4s^5+7.8s^4-22s^3+5.2s^2-21s+3.2}{10s^7+31s^6+150s^5+123s^4+218s^3+87s^2+69s+7.1} \\ C_\Delta &= \frac{30s^7+87s^6+131s^5+148s^4+130s^3+63s^2+41s+9.3}{s^7+8.3s^6+38s^5+83s^4+107s^3+97s^2+62s+13} \end{aligned}$$

The Bode diagrams of these systems have been depicted in Fig. 3. The Figures 3.a and c display that G_x (solid) and G_Δ (dashed) are strikingly different. The difference $G_x - G_\Delta$ (dotted) is quite large: its frequency response magnitude is at least 40% of $|G_x(i\omega)|$ over all frequencies, and it is even larger than 60% at those frequencies where $|G_x(i\omega)C(i\omega)| \approx 1$. The controller variation seems to be moderate, but $|C(i\omega) - C_\Delta(i\omega)|$ is larger than 15% of $|C(i\omega)|$ over all frequencies, and it is up to 70% at the frequencies where $|G_xC| \approx 1$.
G_Δ and C_Δ are modelled as perturbations of the normalized coprime factors of G_x, C. The corresponding plant and controller perturbations Δ_R and Δ_C are shown in Fig. 4. The $\mathbb{H}_\infty$-norms of these perturbations are $\|\Delta_R\|_\infty = 0.968$ and $\|\Delta_C\|_\infty = 0.764$. The product of these norms is 0.734, so that even larger plant and controller perturbations are allowed in view of the robust stability test of Proposition 2.
For the robust stability test based on the gap-metric condition of Proposition 4 we have the following numbers:

$$\begin{aligned} \delta(G_x, G_\Delta) &= 0.917 \\ \delta(C, C_\Delta) &= 0.286 \\ \|T(G_x, C)\|_\infty^{-1} &= 5.73 \cdot 10^{-2}. \end{aligned}$$

Clearly $\delta(G_x, G_\Delta) + \delta(C, C_\Delta)$ is much larger than $\|T(G_x, C)\|_\infty^{-1}$. Hence from (7) it cannot be concluded that $H(G_\Delta, C_\Delta)$ is robustly stable. Moreover, as $\delta(G_x, G_\Delta) > \|T(G_x, C)\|_\infty^{-1}$ and $\delta(C, C_\Delta) > \|T(G_x, C)\|_\infty^{-1}$, the gap-metric condition fails even to guarantee stability of $H(G_\Delta, C)$ or of $H(G_x, C_\Delta)$. Finally, the small value of $\|T(G_x, C)\|_\infty^{-1}$ indicates that $H(G_x, C)$ has poor robustness properties in gap metric sense, while $H(G_x, C)$ is robustly stable against rather large perturbations as shown in Fig. 3.

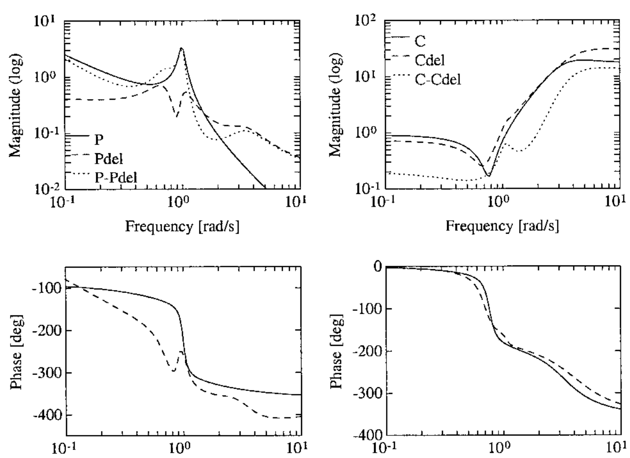

Figure 3: Bode diagrams of the nominal and perturbed plants and controllers. a: Magnitudes of G_x (solid), G_Δ (dashed) and $G_x - G_\Delta$ (dotted); b: Magnitudes of C (solid), C_Δ (dashed) and $C - C_\Delta$ (dotted); c: Phases, see a; d: Phases, see b.

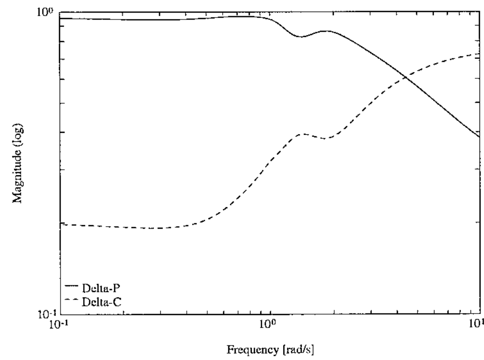

Figure 4: Plant perturbation Δ_R (solid) and controller perturbation Δ_C (dashed).

7 Extension to $\Lambda - gap$ and Vinnicombe gap

The analysis as presented in this paper so far can readily be extended to other uncertainty structures as well, as e.g. the Λ-gap and the Vinnicombe gap.
The Λ-gap $\vec{\delta}_\Lambda(G_x, G_\Delta)$ between two plants G_x and G_Δ is defined as ([3, 4, 5])

$$\vec{\delta}_\Lambda(G_x, G_\Delta) = \inf_{Q \in \mathbb{R}H_\infty} \left\| \begin{pmatrix} \bar{N}_x \\ \bar{D}_x \end{pmatrix} \Lambda^{-1} - \begin{pmatrix} \bar{N}_\Delta \\ \bar{D}_\Delta \end{pmatrix} Q \right\|_\infty, \quad (8)$$

with $(\bar{N}_x, \bar{D}_x)$ and $(\bar{N}_\Delta, \bar{D}_\Delta)$ *nrcf's* of G_x and G_Δ, and Λ as defined in (2).
The Vinnicombe or ν-gap metric is defined as ([18]):

$$\delta_\nu(G_x, G_\Delta) = \begin{cases} \bullet \ \left\| \begin{bmatrix} -\bar{\tilde{D}}_x & \bar{\tilde{N}}_x \end{bmatrix} \begin{bmatrix} \bar{N}_\Delta \\ \bar{D}_\Delta \end{bmatrix} \right\|_\infty, \\ \text{if } \det\left(\begin{bmatrix} \bar{\tilde{N}}_x^* & \bar{\tilde{D}}_x^* \end{bmatrix} \begin{bmatrix} \bar{N}_\Delta \\ \bar{D}_\Delta \end{bmatrix} \right) \neq 0 \quad \forall\omega \\ \text{and } W\left(\det\left(\begin{bmatrix} \bar{\tilde{N}}_x^* & \bar{\tilde{D}}_x^* \end{bmatrix} \begin{bmatrix} \bar{N}_\Delta \\ \bar{D}_\Delta \end{bmatrix} \right)\right) = 0 \\ \bullet \ 1, \qquad \text{otherwise,} \end{cases} \quad (9)$$

where $W(g)$ denotes the winding number about the origin of $g(s)$ as s follows the standard Nyquist D-contour.
The robust stability results -known from the literature- that now can be exploited for our purpose of specifying a norm-bounded area around C under robust stability guarantees read as follows ([4, 18]).

Proposition 5 *Let $H(G_x, C)$ be stable. Then $H(G_\Delta, C_\Delta)$ is stable if*
(a) $\vec{\delta}_\Lambda(G, G_\Delta) + \vec{\delta}_\Lambda(C, C_\Delta) < 1$ (Λ-gap condition) or
(b) $\|T(G,C)\|_\infty \delta_\nu(G, G_\Delta) + \|T(G,C)\|_\infty \delta_\nu(C, C_\Delta) < 1$ (ν-gap condition)[1].

These sufficient conditions for stability lead to the following formulation in terms of stabilizing sets of controllers.

[1] The considered condition is not the least conservative ν-gap condition available in the literature. [18] Here the specific aim is to explicitly parametrize the set of controllers around a nominal controller C.

Proposition 6 *Given a nominal model G_x and a nominal controller C such that $H(G_x,C) \in \mathbb{R}H_\infty$. The set $\mathcal{G}_\Lambda(\delta_{\Lambda,G})$ defined as*

$$\mathcal{G}_\Lambda(\delta_{\Lambda,G}) := \{G_\Delta | \quad \delta_\Lambda(G_x,G_\Delta) \leq \delta_{\Lambda,G}\}$$

is stabilized by all controllers contained in the set $\mathcal{C}_\Lambda(\delta_C)$ defined as

$$\mathcal{C}_\Lambda(\delta_{\Lambda,C}) := \{C_\Delta | \quad \delta_\Lambda(C_x,C_\Delta) < \delta_{\Lambda,C}\}$$

if and only if $\delta_{\Lambda,C} \leq 1 - \delta_{\Lambda,G}$.

Proposition 7 *Given a nominal model G_x and a nominal controller C such that $H(G_x,C) \in \mathbb{R}H_\infty$. The set $\mathcal{G}_\nu(\delta_{\nu,G})$ defined as*

$$\mathcal{G}_\nu(\delta_{\nu,G}) := \{G_\Delta | \quad \delta_\nu(G_x,G_\Delta) \leq \delta_{\nu,G}\}$$

is stabilized by all controllers contained in the set $\mathcal{C}_\nu(\delta_C)$ defined as

$$\mathcal{G}_\nu(\delta_{\nu,C}) := \{C_\Delta | \quad \delta_\nu(C_x,C_\Delta) < \delta_{\nu,C}\}$$

if and only if $\delta_{\nu,C} \leq \|T(G_x,C)\|_\infty^{-1} - \delta_{\nu,G}$.

Based on these robust stability results one can now consider the same problem as is considered in the formulation of Theorem 1, leading to maximum sized controller sets:

Theorem 2 *Given a nominal model G_x and a nominal controller C such that $H(G_x,C) \in \mathbb{R}H_\infty$. Consider any plant $G_\Delta \in \mathbb{R}H_\infty$, and determine the minimal values of $\bar{\gamma}_G$, $\bar{\delta}_\Lambda$ and $\bar{\delta}_\nu$ such that $G_\Delta \in \mathcal{G}_Y(\bar{\gamma}_G)$, $G_\Delta \in \mathcal{G}_\delta(\bar{\delta}_{\Lambda,G})$ and $G_\Delta \in \mathcal{G}_\delta(\bar{\delta}_{\nu,G})$. Then the sets of controllers that result from Propositions 2, 4 and 7, i.e.*

$$\begin{aligned}
\mathcal{C}_Y &= \mathcal{C}_Y(\bar{\gamma}_G^{-1}) \\
\mathcal{C}_\Lambda &= \mathcal{C}_\Lambda(\bar{\delta}_{\Lambda,C}), \quad \bar{\delta}_{\Lambda,C} = 1 - \bar{\delta}_{\Lambda,G} \\
\mathcal{C}_\nu &= \mathcal{C}_\nu(\bar{\delta}_{\nu,C}), \quad \bar{\delta}_{\nu,C} = \|T(G,C)\|_\infty^{-1} - \bar{\delta}_{\nu,G}
\end{aligned}$$

relate to each other according to

$$\begin{aligned}
\mathcal{C}_\delta \subseteq \mathcal{C}_\Lambda \subseteq \mathcal{C}_Y \\
\mathcal{C}_\delta \subseteq \mathcal{C}_\nu \subseteq \mathcal{C}_Y \quad .
\end{aligned}$$

For a formal proof of this, see the extended paper [8].

8 Concluding remarks

We have used the double Youla parametrization for purpose of specifying the maximum allowable tuning range for a new controller to deviate from the present controller while retaining robust stability. It is demonstrated that the result obtained when using this uncertainty structure is less conservative than when using the gap metric. An example has been provided to support these results. The results imply that model uncertainty characterized in terms of a dual Youla-parametrization not only is advantageous from a performance point of view, but also for the situation where attention is restricted to robust stability aspects. Related results are provided for stability conditions in terms of the Λ-gap and ν-gap.

Acknowledgement

The authors would like to thank X. Bombois for the fruitful discussions which contributed to the paper.

References

[1] Bombois, X., M. Gevers and G. Scorletti (1999). Controller validation for a validatied model set. *Proc. European Control Conference*, Karlsruhe, Germany, paper 869.

[2] Bombois, X., M. Gevers and G. Scorletti (1999). Controller validation based on an identified model. *Proc. 38th IEEE Conf. Decision and Control*, pp. 2816-2821.

[3] Bongers, P.M.M. (1991). On a new robust stability margin. *Recent Advances in Mathematical Theory of Systems, Control, Networks and Signal Processing*, Proc. of the Int. Symposium MTNS–91, pp. 377–382.

[4] Bongers, P.M.M. (1994). *Modeling and Identification of Flexible Wind Turbines and a Factorizational Approach to Robust Control.* PhD thesis, Delft Univ. of Techn., Mech. Eng. Systems and Control Group.

[5] de Callafon, R.A., P.M.J. Van den Hof and P.M.M. Bongers (1996). A unified approach to stability robustness for uncertainty descriptions based on fractional model representations. *IEEE Trans. Autom. Control, AC-41*, 723–727.

[6] de Callafon, R.A. and P.M.J. Van den Hof (1997). Suboptimal feedback control by a scheme of iterative identification and control design. *Math. Mod. of Syst., 3*, 77-101.

[7] de Callafon, R.A. and P.M.J. Van den Hof (2001). Multivariable feedback relevant system identification of a wafer stepper system. To appear in: *IEEE Trans. Contr. Syst. Tech., 9, no2.*

[8] Douma, S.G., P.M.J. Van den Hof and O.H. Bosgra (2000). *Robust Stability Results under Simultaneous Plant and Controller Perturbations.* Internal report, September 2000. http://www.tn.tudelft.nl/mmr/downloads/pubdouma.

[9] Georgiou, T.T. and M.C. Smith (1990). Optimal robustness in the gap metric. *IEEE Trans. Autom. Control, AC-35*, 673–686.

[10] Hakvoort, R.G. and P.M.J. Van den Hof (1997). Identification of probabilistic system uncertainty regions by explicit evaluation of bias and variance errors. *IEEE Trans. Autom. Control, AC-42*, 1516-1528.

[11] Hjalmarsson, H., S. Gunnarsson and M. Gevers (1994). A convergent iterative restricted complexity control design scheme. *Proc. 33rd Conf. Decision and Control*, Lake Buena Vista, FL, pp. 1735-1740.

[12] Horn, R.A. and C.R. Johnson (1991). *Topics in Matrix Analysis.* Cambridge University Press, England.

[13] Ninness, B.M. and G.C. Goodwin (1995). Estimation of model quality. *Automatica, 31*, 1771-1797.

[14] Schrama, R.J.P., P.M.M. Bongers and O.H. Bosgra (1992). Robust stability under simultaneous perturbations of

linear plant and controller. *Proc. 31st IEEE Conf. Decision and Control*, Tucson, AZ, pp. 2137-2139.

[15] Sefton, J.A. and R.J. Ober (1993). On the gap metric and coprime factor perturbations. *Automatica*, 29, 723-734

[16] Tay, T.T., J.B. Moore and R. Horowitz (1989). Indirect adaptive techniques for fixed controller performance enhancement. *Int. J. Control, 50*, 1941–1959.

[17] Vidyasagar, M. (1985). *Control Systems Synthesis: A Factorization Approach.* M.I.T. Press, Cambridge, USA.

[18] Vinnicombe, G. (1993). Frequency domain uncertainty and the graph topology. *IEEE Trans. Autom. Control, AC-38*, 1371–1383.

9 Appendix: Sketch of Proof

The proof of theorem 1 consists of showing that there exist sets $\mathcal{G}_Y(\bar{\gamma}_G)$ and $\mathcal{C}_Y(\bar{\gamma}_C)$, as defined in proposition 2, which embed, respectively, the sets $\mathcal{G}_\delta(\bar{\delta}_G)$ and $\mathcal{C}_\delta(\bar{\delta}_C)$ of theorem 1 and which satisfy the stability condition of proposition 2, i.e $\bar{\gamma}_G \cdot \bar{\gamma}_C < 1$.

Lemma 1 *Given a nominal plant $G_x = \bar{N}_x\bar{D}_x^{-1}$ and controller $C = \bar{N}_c\bar{D}_c^{-1}$ such that $H(G_x, C) \in \mathbb{R}H_\infty$. Then every plant G_Δ stabilized by C and every controller C_Δ stabilized by G_x can be expressed in a (dual) Youla factorization ([17]), i.e.*
$G_\Delta = \left(\bar{N}_x + \bar{D}_c\bar{\Delta}_R\right)\left(\bar{D}_x - \bar{N}_c\bar{\Delta}_R\right)^{-1}$ *and*
$C_\Delta = \left(\bar{N}_c + \bar{D}_x\bar{\Delta}_C\right)\left(\bar{D}_c - \bar{N}_x\bar{\Delta}_C\right)^{-1}$*, and*

$$\begin{aligned}
\|\bar{\Delta}_R\|_\infty &\leq \|\tilde{\Lambda}^{-1}\|_\infty \delta(G,G_\Delta)\left(1 - \|\Lambda^{-1}\|_\infty \delta(G,G_\Delta)\right)^{-1} \\
\|\bar{\Delta}_C\|_\infty &\leq \|\Lambda^{-1}\|_\infty \delta(C,C_\Delta)\left(1 - \left(\|\tilde{\Lambda}^{-1}\|_\infty \delta(C,C_\Delta)\right)\right)^{-1}.
\end{aligned}$$

Lemma 1 follows from exploiting the freedom in coprime factorizations with respect to a unimodular premultiplication. Each G_Δ can be written in a Youla factorization and in terms of a coprime factorization related to the directed gap $\vec{\delta}(G_\Delta, G)$ of (6),

$$\begin{bmatrix} \bar{N}_\Delta \\ \bar{D}_\Delta \end{bmatrix} Q_G = \begin{bmatrix} \bar{N}_x + \bar{D}_c\bar{\Delta}_R \\ \bar{D}_x - \bar{N}_c\bar{\Delta}_R \end{bmatrix} Q. \tag{10}$$

Here Q_G denotes the minimizing argument of definition (6) of the directed gap. Q_G is unique and unimodular (i.e. $Q_G, Q_G^{-1} \in \mathbb{R}H_\infty$) implying the left hand side of (10) to represent a coprime factorization indeed [15]. The unimodular matrix Q accomplishes the equality and is uniquely defined due to the normalizations of the coprime factors. From (10) we can write:

$$\begin{aligned}
\begin{bmatrix} \bar{N}_\Delta \\ \bar{D}_\Delta \end{bmatrix} Q_G Q^{-1} &= \begin{bmatrix} \bar{N}_x \\ \bar{D}_x \end{bmatrix} + \begin{bmatrix} \bar{D}_c \\ -\bar{N}_c \end{bmatrix} \bar{\Delta}_R \\
\begin{bmatrix} -\bar{D}_c \\ \bar{N}_c \end{bmatrix} \bar{\Delta}_R &= \begin{bmatrix} \bar{N}_x \\ \bar{D}_x \end{bmatrix} - \begin{bmatrix} \bar{N}_x \\ \bar{D}_x \end{bmatrix} Q^{-1} \\
&\quad + \left\{ \begin{bmatrix} \bar{N}_x \\ \bar{D}_x \end{bmatrix} - \begin{bmatrix} \bar{N}_\Delta \\ \bar{D}_\Delta \end{bmatrix} Q_G \right\} Q^{-1}.
\end{aligned} \tag{11}$$

This expression is simplified by multiplication from the left with $\tilde{\Lambda}^{-1}\begin{bmatrix} -\tilde{\bar{D}}_x & \tilde{\bar{N}}_x \end{bmatrix}$, using the fact that $\tilde{\bar{D}}_x\bar{N}_x = \tilde{\bar{N}}_x\bar{D}_x$, leading to

$$\bar{\Delta}_R = \tilde{\Lambda}^{-1}\begin{bmatrix} -\tilde{\bar{D}}_x & \tilde{\bar{N}}_x \end{bmatrix}\left\{ \begin{bmatrix} \bar{N}_x \\ \bar{D}_x \end{bmatrix} - \begin{bmatrix} \bar{N}_\Delta \\ \bar{D}_\Delta \end{bmatrix} Q_G \right\} Q^{-1}, \tag{12}$$

where $\tilde{\Lambda}$ is as defined in (2).
Multiplication of (11) from the left with $\Lambda^{-1}\begin{bmatrix} \tilde{\bar{N}}_c & \tilde{\bar{D}}_c \end{bmatrix}$ yields

$$Q = I - \Lambda^{-1}\begin{bmatrix} \tilde{\bar{N}}_c & \tilde{\bar{D}}_c \end{bmatrix}\left\{ \begin{bmatrix} \bar{N}_x \\ \bar{D}_x \end{bmatrix} - \begin{bmatrix} \bar{N}_\Delta \\ \bar{D}_\Delta \end{bmatrix} Q_G \right\}. \tag{13}$$

Applying the singular value relation [12]

$$\underline{\sigma}(A - B) \geq \underline{\sigma}(A) - \bar{\sigma}(B)$$

to (13) and using the fact that $\bar{\sigma}(Q^{-1}(\omega)) = \frac{1}{\underline{\sigma}(Q(\omega))}$ for all ω, results in

$$\begin{aligned}
\bar{\sigma}\left(Q^{-1}(\omega)\right) &\leq \left(1 - \|\Lambda^{-1}\|_\infty \vec{\delta}(G,G_\Delta)\right)^{-1} \qquad \forall\omega \\
\|Q^{-1}\|_\infty &\leq \left(1 - \|\Lambda^{-1}\|_\infty \vec{\delta}(G,G_\Delta)\right)^{-1}.
\end{aligned} \tag{14}$$

The bounds of lemma 1 then follow easily from (12) and (14) upon noting that $\begin{bmatrix} -\tilde{\bar{D}}_x & \tilde{\bar{N}}_x \end{bmatrix}$ is co-inner. The appearance of the gap itself in lemma 1 follows from a similar derivation for the other directed gap $\vec{\delta}(G, G_\Delta)$ and the fact that the gap is the maximum of both.
With lemma 1 the following proposition is easily seen to hold.

Proposition 8 *The sets $\mathcal{G}_Y(\bar{\gamma}_G)$ and $\mathcal{G}_Y(\bar{\gamma}_C)$ as defined in (2), with*

$$\begin{aligned}
\bar{\gamma}_G &= \bar{\delta}_G \|\tilde{\Lambda}^{-1}\|_\infty \left(1 - \bar{\delta}_G \|\Lambda^{-1}\|_\infty\right)^{-1} \\
\bar{\gamma}_C &= \bar{\delta}_C \|\tilde{\Lambda}^{-1}\|_\infty \left(1 - \bar{\delta}_C \|\Lambda^{-1}\|_\infty\right)^{-1}
\end{aligned}$$

embed the sets $\mathcal{G}_\delta(\bar{\delta}_G)$ and $\mathcal{C}_\delta(\bar{\delta}_C)$, respectively. Moreover, they satisfy proposition 2, i.e. $\bar{\gamma}_G \cdot \bar{\gamma}_C < 1$.

The last inequality follows from the fact that every $G_\Delta \in \mathcal{G}_\delta(\bar{\delta}_G)$ and $C_\Delta \in \mathcal{C}_\delta(\bar{\delta}_C)$ satisfies the gap condition (7), i.e $\|T(G,C)\|_\infty \bar{\delta}_C < 1 - \|T(G,C)\|_\infty \bar{\delta}_G$. Using the fact that $\|T(G,C)\|_\infty = \|\tilde{\Lambda}^{-1}\|_\infty = \|\Lambda^{-1}\|_\infty$ we have,

$$\bar{\gamma}_C < \left(1 - \bar{\delta}_G \|\tilde{\Lambda}^{-1}\|_\infty\right)\left(\bar{\delta}_G \|\tilde{\Lambda}^{-1}\|_\infty\right)^{-1},$$

from which proposition 8 follows readily.