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Minimal partial realization from generalized orthonormal basis function expansions ☆

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Abstract

A solution is presented for the problem of realizing a discrete-time LTI state-space model of minimal McMillan degree such that its first N expansion coefficients in terms of generalized orthonormal basis match a given sequence. The basis considered, also known as the Hambo basis, can be viewed as a generalization of the more familiar Laguerre and two-parameter Kautz constructions, allowing general dynamic information to be incorporated in the basis. For the solution of the problem use is made of the properties of the Hambo operator transform theory that underlies the basis function expansion. As corollary results compact expressions are found by which the Hambo transform and its inverse can be computed efficiently. The resulting realization algorithms can be applied in an approximative sense, for instance, for computing a low-order model from a large basis function expansion that is obtained in an identification experiment.

Keywords: Realization theory; Partial expansions; Algorithms; State-space realization; Transforms; All-pass filters; Interpolation; System identification; Model approximation

1. Introduction

The idea of decomposing a system in terms of basis functions is widely applied in system theory and related problems such as system approximation and identification. It is, for instance, common to represent a stable discrete-time system $G(z)$ in the form of its Laurent expansion as

$$G(z)=\sum_{k=1}^{\infty} g_k z^{-k}, \tag{1}$$

where the functions $\{z^{-k}\}$ form an orthonormal basis for the space of functions (1) with $\sum_{k=1}^{\infty} |g_k|^2 < \infty$, denoted as H_2 in the following. The associated expansion coefficients g_k, also known as the Markov parameters, play an important role in systems theory, realization theory, system approximation and identification. The representation of a system in terms of a finite set of Markov parameters is known as finite impulse response (FIR) modeling. In spite of its apparent simplicity the FIR model is widely applied, e.g. in filter synthesis problems in signal processing (Roberts & Mullis, 1987), adaptive filtering problems (Haykin, 1996), and in the context of finite horizon optimal control problems (Richalet, Rault, Testud, & Papon, 1978; Furuta & Wongsaisuwan, 1995).

Generalized orthonormal basis constructions have been proposed that offer the flexibility to tune them so as to

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perform better than the FIR models in particular situations. These are the expansions of the general form

$$G(z)=\sum_{k=1}^{\infty} c_k f_k(z), \tag{2}$$

in which the functions $f_k(z)$ represent general orthonormal basis functions while $c_k \in \mathbb{R}$ are the corresponding expansion coefficients. Examples of such basis function expansions are the well-known Laguerre and two-parameter Kautz basis constructions (Lee, 1960; Kautz, 1954, Wahlberg, 1991, 1994). Further generalizations were proposed in Heuberger (1991), Heuberger, Van den Hof, and Bosgra (1995) and Ninness and Gustafsson (1997). Their general form is given by

$$f_k(z)=\frac{\sqrt{1-|\xi_k|^2}}{z-\xi_k}\prod_{i=1}^{k-1}\frac{1-\xi_i^* z}{z-\xi_i}, \tag{3}$$

where $\{\xi_i\}_{i=1,\dots,k}$ is a collection of poles to be chosen by the user. The origin of these constructions lies in the theory on rational orthonormal bases as developed by Takenaka and Malmquist in the 1920s (Walsh, 1956). The functions constitute a complete orthonormal set in H_2 provided that $\sum_{k=1}^{\infty}(1-|\xi_k|)=\infty$. Typically, the rate of convergence of the series expansion (2) is higher when the pre-chosen poles ξ_i are closer to the poles of the underlying system. The application of these functions in the areas of model reduction and model approximation is analyzed in Mäkilä (1990), Wahlberg and Mäkilä (1996) and Schipp and Bokor (1997). This paper will consider the basis construction that was proposed in Heuberger et al. (1995), sometimes denoted as the Hambo basis, which in terms of (3) is equivalent to a finite pole selection $\{\xi_i\},\ i=1,\dots,n_b$ which is repeated periodically, i.e. $\xi_{k+n_b}=\xi_k, \forall k$.

The problem considered is as follows: given a partial expansion $\{\tilde{c}_k\}_{k=1,\dots,N}$, find a minimal state-space realization (A,B,C,D) of a system $G(z)=D+C(zI-A)^{-1}B$ of smallest order such that $G(z)=\sum_{k=1}^{\infty} c_k f_k(z)$ and $c_k=\tilde{c}_k,\ k=1,\dots,N$. This problem can be viewed as a generalization of the classical minimal partial realization problem that was solved in Ho and Kalman (1966) and Tether (1970). In the case of Laguerre function expansions, the problem has been solved in Nurges (1987). For the Hambo basis case, the realization problem was first considered in Szabó and Bokor (1997) and Szabó, Heuberger, Bokor, and Van den Hof (2000), where the problem was solved for the full-information case ($N\to\infty$). However, in order to provide an algorithm, that is able to deal with finite N, a different approach has to be followed.

In this paper, it will be shown that a solution for this latter case can be constructed by exploiting the so-called Hambo transform theory. This transform theory has also been used to analyze the statistical properties of identification algorithms, estimating series expansion coefficients (see Van den Hof, Heuberger, & Bokor, 1995). As a result this paper has two main contributions:

- the development of an algorithm that solves the minimal partial realization problem for generalized basis functions, and
- the further development and analysis of the underlying Hambo transform theory, by deriving explicit (state-space) expressions for the transform and its inverse.

The presented results will be limited to scalar transfer functions. The generalization to multivariable systems presents no great difficulties.

Besides being interesting from a system theoretic point of view, the analysis in this paper finds its motivation in the application of these bases in system identification (Van den Hof et al., 1995; Szabó, Bokor, & Schipp, 1999). Owing to the linear parameterization (c_k appears linearly in $G(z)$) attractive computational properties result, e.g. in least-squares algorithms, which enables the handling of large-scale problems. However, it is common to estimate a high-order basis function model that is subsequently reduced to a low-order state-space model by means of model reduction. It seems appropriate to replace this model reduction step by an (approximate) realization procedure. When using model reduction the extension to infinity of the estimated expansion is simply set to zero, while approximate realization tries to infer the unknown extension from the dynamics that reveals itself in the partial expansion.

The outline of the paper is as follows. First, some preliminaries about the Hambo basis and Hambo transform theory are recalled in Section 2. In Section 3, some properties of the Hambo operator transform are established that are instrumental for the solution of the realization problem. In Section 4, the Hankel operator framework is presented in which the realization problem is solved for the case where one has knowledge of the full expansion. In Sections 5 and 6, this approach is combined with results from Hambo basis theory to derive the main results. The connection with a related interpolation problem is analyzed in Section 7, while in Section 8 application of the results in the context of system identification is discussed, which is illustrated with an example in Section 9. The proofs of all results are collected in the appendix.

Notation

A^{T}, A^*	transpose, respectively, complex conjugate transpose, of matrix A
$L_2^{p\times m}$	Hilbert space of complex matrix functions of dimension $p\times m$ that are square integrable on the unit circle. The superscript $p\times m$ will be suppressed if $p=m=1$

$H_2^{p\times m}$ Hardy space of complex matrix functions of dimension $p\times m$ which are analytic in the exterior of the unit disc such that

$$\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Trace}(f(re^{i\omega})^* f(re^{i\omega}))\,d\omega<\infty$$

and $f(\infty)=0$.[2] The superscript $p\times m$ will be suppressed if $p=m=1$

RH_2 subspace of *rational* (strictly proper) transfer functions of H_2

$\langle X,Y\rangle_M$ matrix "inner" product between $X\in L_2^{p\times 1}$ and $Y\in L_2^{m\times 1}$ defined as $\langle X,Y\rangle_M=\frac{1}{2\pi}\int_0^{2\pi}X(e^{i\omega})Y(e^{i\omega})^*\,d\omega$. For $p=m=1$ the subscript M will be suppressed

$H_2^{\perp}$ the orthogonal complement of H_2 in L_2[3]

$H_{2,0}^{p\times m}$ the same as $H_2^{p\times m}$, without the restriction that the functions must be zero at infinity

$RH_{2,0}^{p\times m}$ subspace of *rational* (proper) transfer functions of $H_{2,0}^{p\times m}$

$\mathbf{P}_X$ orthogonal projection onto the subspace X

$\mathbb{N}_0$ $\mathbb{N}$ extended with zero

$\ell_2^n(J)$ the space of square summable vector sequences, of vector dimension n, where J denotes the index set of the sequence. The superscript n will be omitted if $n=1$

$\mathbf{e}_i$ the ith canonical Euclidean basis (column) vector.

$\mathrm{diag}\{x_i\}_{i=1,\dots,n}$ diagonal matrix with diagonal elements $x_i,\dots,x_n$.

At various instances in this paper use is made of Sylvester equations of the form

$$AXB+C=X.$$

Existence and uniqueness of the solution matrix X will be everywhere guaranteed by the fact that—wherever used—the matrices A and B both have eigenvalues with modulus strictly smaller than 1.

In this paper, we assume that all transfer functions and state-space realizations have real-valued coefficients, i.e. poles of transfer functions appear in complex conjugate pole pairs.

2. Preliminaries

2.1. Orthogonal basis functions—Hambo basis

One way to construct the Hambo basis functions (Heuberger et al., 1995) is by considering a finite set of poles $\{\xi_i\}_{i=1,\dots,n_b}$ that are stable, i.e. $|\xi_i|<1$, generating an all-pass transfer function

$$G_b(z)=\prod_{i=1}^{n_b}\frac{(1-\xi_i^* z)}{(z-\xi_i)}$$

having a minimal balanced realization (A_b,B_b,C_b,D_b) that satisfies

$$\begin{bmatrix}A_b & B_b\\ C_b & D_b\end{bmatrix}^T\begin{bmatrix}A_b & B_b\\ C_b & D_b\end{bmatrix}=I. \tag{4}$$

Due to these properties the input-to-states transfer functions of G_b:

$$\phi_i(z):=\mathbf{e}_i^T(zI-A_b)^{-1}B_b,\quad i=1,\dots,n_b,$$

form an orthonormal set. An orthogonal basis for H_2 is created by introducing

$$\phi_{i,k}(z)=\phi_i(z)G_b(z)^{k-1},\quad k=1,\dots,\infty.$$

In order to facilitate analysis, we also denote: $V_k=[\phi_{1,k}\ \phi_{2,k}\ \dots\ \phi_{n_b,k}]^T$, leading to $V_1(z)=(zI-A_b)^{-1}B_b$ and the basis function vectors

$$V_k(z)=V_1(z)G_b(z)^{k-1}. \tag{5}$$

Since the functions $\{\phi_{i,k}\}_{i=1,\dots,n_b;k=1,\dots,\infty}$ form an orthonormal basis, any element G of H_2 can be written as

$$G(z)=\sum_{k=1}^{\infty}\sum_{i=1}^{n_b} l_{i,k}\phi_{i,k}(z)\quad\text{with } l_{i,k}=\langle G,\phi_{i,k}\rangle \tag{6}$$

with $l_{i,k}$ being the expansion coefficients. Equality here should be interpreted in the two-norm sense. Throughout this paper, it will be assumed that all state-space realizations and expansion coefficients $l_{i,k}$ are real-valued. Similar to (6) we can write

$$G(z)=\sum_{k=1}^{\infty}L_k^T V_k(z) \tag{7}$$

with $L_k^T:=[l_{1,k}\ l_{2,k}\ \dots\ l_{n_b,k}]=\langle G,V_k\rangle_M$. The all-pass function $G_b(z)$ also generates a basis for the $H_2^{\perp}$ space (and thereby for the entire L_2 space) by repeatedly multiplying $V_1(z)$ with $G_b(z)^{-1}=G_b(1/z)$.

Lemma 1. *Defining* $U_0(1/z):=V_1(z)G_b(1/z)$, *it holds that*

$$U_0(1/z)=\sum_{l=0}^{\infty}A_b^{T\,l}C_b^T z^l=(1/z)((1/z)I-A_b^T)^{-1}C_b^T. \tag{8}$$

Clearly, $U_0(1/z)$ is an element of $H_2^{n_b\perp}$ and $G_b(1/z)$ is an element of $H_2^{\perp}$. This implies that all the functions $U_k(1/z)$ defined as

$$U_k(1/z)=U_0(1/z)G_b(1/z)^k \tag{9}$$

[2] Here, H_2 is identified with the subspace of L_2 with vanishing non-positive Fourier-coefficients. More precisely, for $f\in H_2$, $f(z)=f_1z^{-1}+f_2z^{-2}+\cdots$, and $\sum_{k=1}^{\infty}|f_k|^2<\infty$.

[3] For $f\in H_2^{\perp}$, $f(z)=\sum_{k=0}^{\infty}f_kz^k$, and $\sum_{k=0}^{\infty}|f_k|^2<\infty$.

for $k \in \mathbb{N}_0$ lie in $H_2^{n_b \perp}$. In fact the functions $\mathbf{e}_i^T U_k$, with $1 \leqslant i \leqslant n_b$, and $k \in \mathbb{N}_0$ constitute an orthonormal basis of $H_2^{\perp}$, as is shown, e.g. in Szabó et al. (1999).

2.2. Signal and operator transforms

By the isomorphic property of the z-transform there exist equivalent time-domain representations of $\phi_{i,k}$ that form an orthonormal basis of the signal space $\ell_2(\mathbb{N})$. They are written as $\phi_{i,k}(t)$, or $V_k(t)$ for the vectors, where the index $t \in \mathbb{N}$ denotes time.

The Hambo signal transform (Heuberger & Van den Hof, 1996) of a signal x in $\ell_2(\mathbb{N})$ is then defined by

$$\mathscr{X}(\lambda) = \sum_{k=1}^{\infty} \mathscr{X}(k)\lambda^{-k} \quad \text{with } \mathscr{X}(k) = \langle V_k, x \rangle_M$$

and λ a complex indeterminate.

This signal transform gives rise to a transform operation on a dynamical system, as formulated next.

Proposition 2. *Suppose that $u \in \ell_2(\mathbb{N})$, $G \in RH_{2,0}$ and let $y(z) = G(z)u(z)$. Let $\mathscr{Y}$ and $\mathscr{U}$ denote the Hambo signal transform of y, respectively u. Then,*

$$\mathscr{Y}(m) = \sum_{j=1}^{m} M_{m-j}\mathscr{U}(j) \tag{10}$$

with the Markov parameters M_k given by

$$\begin{aligned} M_k &= \langle V_1(z), V_1(z)G_b(1/z)^k G(z) \rangle_M \\ &= \langle V_1(z)G_b(z)^k, V_1(z)G(z) \rangle_M. \end{aligned} \tag{11}$$

The resulting dynamical system $\tilde{G} \in RH_{2,0}^{n_b \times n_b}$ determined by

$$\tilde{G}(\lambda) = \sum_{k=0}^{\infty} M_k \lambda^{-k}, \tag{12}$$

is referred to as the Hambo operator transform of G.

It follows from this proposition that the *Hambo operator transform* of the scalar system G is a causal, linear time-invariant $n_b \times n_b$ system. Several properties of this transform have been derived (Heuberger & Van den Hof, 1996):

- $\tilde{G}(\lambda)$ is obtained by a simple variable substitution applied to the transfer function $G(z)$. It holds that
$$\tilde{G}(\lambda) = \sum_{k=0}^{\infty} g_k N(\lambda)^k \tag{13}$$
with g_k the Markov parameters of G as in (1) and $N(\lambda)$ a transfer matrix that has the state-space realization (D_b, C_b, B_b, A_b).
- The transform of G_b is a simple shift, i.e.
$$\tilde{G}_b(\lambda) = \lambda^{-1} I_{n_b}. \tag{14}$$
- $\tilde{G}(\lambda)$ and $G(z)$ have the same McMillan degree.

In Section 5, it will be shown how the analysis involved in solving the realization problem produces a means for directly computing a minimal state-space realization of $\tilde{G}(\lambda)$ on the basis of a minimal state-space realization of $G(z)$ and vice versa.

3. Transformation of expansion coefficients

As stated in the previous section, one can expand any $G(z) \in RH_2$ in terms of the Hambo basis function vectors as in (7). We will now recall from Szabó et al. (2000) the connection that exists between the coefficient vector sequence $\{L_k\}$ and the sequence of Markov parameters $\{M_k\}$ of the Hambo operator transform $\tilde{G}(\lambda)$. The relation will prove to be essential for the solution of the generalized realization problems.

Proposition 3. *Let $G \in H_2$ have a generalized expansion as in* (6). *Then,*

(a) *the Markov parameters M_k of the Hambo operator transform $\tilde{G}(\lambda)$ satisfy*

$$M_k = \begin{cases} \sum_{i=1}^{n_b} l_{i,k+1} P_i^T + l_{i,k} Q_i^T, & k \geqslant 1, \\ \sum_{i=1}^{n_b} l_{i,1} P_i^T, & k = 0. \end{cases} \tag{15}$$

(b) $\widetilde{zG(z)}(\lambda) = \sum_{k=1}^{\infty} M_k^{\leftarrow} \lambda^{-k}$ *with*

$$\begin{aligned} M_k^{\leftarrow} = L_{k+1}^T B_b \cdot I &+ \sum_{i=1}^{n_b} \{L_{k+1}^T A_b\}_i P_i^T \\ &+ \{L_k^T A_b\}_i Q_i^T, \quad k \geqslant 1, \end{aligned} \tag{16}$$

where $\{\cdot\}_i$ denotes the ith element of the corresponding vector.

The matrices P_i and Q_i are obtained as unique solutions to the following Sylvester equations:

$$A_b P_i A_b^T + B_b \mathbf{e}_i^T A_b^T = P_i, \tag{17}$$

$$A_b^T Q_i A_b + C_b^T \mathbf{e}_i^T = Q_i. \tag{18}$$

A proof of this proposition can be found in Szabó et al. (2000), except for Eq. (18), which is proven in de Hoog, (2001).

The expression for the Hambo transform of the shifted system $zG(z)$ will turn out to be useful when constructing the realization algorithm in the sequel. The main implication of part (a) of Proposition 3 is that the Markov parameters of $\tilde{G}(\lambda)$ can be derived directly from the expansion coefficients. More precisely, M_k solely depends on the coefficient vectors L_k and L_{k+1}. In the next section, this fact is used to solve the realization problem.

4. Realization on the basis of the infinite expansion

The solution to the classical minimal realization problem, due to Ho and Kalman (1966), is based on the representation of a system $G(z)=\sum_{k=1}^{\infty} g_k z^{-k}$ in the Hankel operator form, reflecting the mapping from past input signals $u \in \ell_2(-\infty,0]$ to future output signals $y \in \ell_2[1,\infty)$. This operator is represented by an infinite Hankel matrix $\mathbf{H}$ that operates on the infinite vectors $\mathbf{u}$ and $\mathbf{y}$, as in

$$\mathbf{y}=\begin{bmatrix} y(1) \\ y(2) \\ y(3) \\ \vdots \end{bmatrix} = \begin{bmatrix} g_1 & g_2 & g_3 & \cdots \\ g_2 & g_3 & g_4 & \\ g_3 & g_4 & g_5 & \\ \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} u(0) \\ u(-1) \\ u(-2) \\ \vdots \end{bmatrix} = \mathbf{H}\mathbf{u}. \tag{19}$$

The Ho–Kalman realization algorithm employs the property that any full-rank decomposition of $\mathbf{H}$ corresponds to a minimal realization (A,B,C):

$$\mathbf{H}=\Gamma\Delta=\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} [B \quad AB \quad A^2B \quad \cdots]. \tag{20}$$

Hence, the B and C matrices of a minimal realization are obtained by extracting the first column of Δ and the first row of Γ, respectively, while the A matrix is obtained by solving the equation

$$\mathbf{H}^{\leftarrow}=\Gamma A \Delta, \tag{21}$$

where $\mathbf{H}^{\leftarrow}$ is the Hankel matrix that is obtained by removing the first column of $\mathbf{H}$ or equivalently by shifting the columns of $\mathbf{H}$ one place to the left. Note that the matrix $\mathbf{H}^{\leftarrow}$ can be viewed as the Hankel matrix associated with the system $zG(z)$. This algorithm yields an exact realization provided that an underlying finite dimensional system exists.

In our situation, the problem is to find this system not on the basis of $\{g_k\}$, but by starting with $\{L_k\}$. To this end, it is expedient to formulate the Hankel operator of system (19) in terms of a matrix representation that considers the signals to be decomposed in terms of the generalized basis functions chosen. We then define the vectors $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{u}}$ containing the expansion coefficient sequences according to

$$\begin{aligned} \tilde{\mathbf{y}} &= [\mathscr{Y}(1)^{\mathrm{T}} \quad \mathscr{Y}(2)^{\mathrm{T}} \quad \cdots]^{\mathrm{T}} \quad \text{and} \\ \tilde{\mathbf{u}} &= [\mathscr{U}(0)^{\mathrm{T}} \quad \mathscr{U}(-1)^{\mathrm{T}} \quad \cdots]^{\mathrm{T}} \end{aligned} \tag{22}$$

Since the coefficients satisfy $\mathscr{Y}(k)=\langle V_k, y\rangle_M$ and $\mathscr{U}(-k)=\langle U_k, u\rangle_M$ one can express the vectors $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{u}}$ as

$$\tilde{\mathbf{y}}=\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \end{bmatrix}\mathbf{y}=\mathbf{T}_1\mathbf{y} \quad \text{and} \quad \tilde{\mathbf{u}}=\begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \end{bmatrix}\mathbf{u}=\mathbf{T}_2\mathbf{u}, \tag{23}$$

where $\mathbf{v}_k$ and $\mathbf{u}_k$ are given by

$$\begin{aligned} \mathbf{v}_k &= [V_k(1) \quad V_k(2) \quad \cdots] \quad \text{and} \\ \mathbf{u}_k &= [U_k(0) \quad U_k(-1) \quad \cdots]. \end{aligned} \tag{24}$$

The matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ consist of inverse z-transforms of the orthonormal basis functions $V_k(z)$ and $U_k(z)$. Hence, they are unitary (orthogonal) matrices: $\mathbf{T}_1^{\mathrm{T}}\mathbf{T}_1=\mathbf{T}_2^{\mathrm{T}}\mathbf{T}_2=I$. From Eqs. (19) and (23), it then follows that one can write

$$\tilde{\mathbf{y}}=\mathbf{T}_1\mathbf{H}\mathbf{T}_2^{\mathrm{T}}\tilde{\mathbf{u}}=\tilde{\mathbf{H}}\tilde{\mathbf{u}}. \tag{25}$$

The matrix $\tilde{\mathbf{H}}$ is the Hankel operator representation associated with expansions of signals in terms of the Hambo basis functions.

Proposition 4. *With $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{u}}$ as defined in Eq.* (22) *it holds that $\tilde{\mathbf{y}}=\tilde{\mathbf{H}}\tilde{\mathbf{u}}$ with $\tilde{\mathbf{H}}$ given by*

$$\tilde{\mathbf{H}}=\begin{bmatrix} M_1 & M_2 & M_3 & \cdots \\ M_2 & M_3 & M_4 & \\ M_3 & M_4 & M_5 & \\ \vdots & & & \ddots \end{bmatrix}, \tag{26}$$

where M_k are the Markov parameters of the Hambo operator transform of $G(z)$ as defined by Eq. (11).

It follows that the matrix $\tilde{\mathbf{H}}$ coincides with the block Hankel matrix that is associated with system $\tilde{G}(\lambda)$, the Hambo operator transform of $G(z)$. This matrix can hence be constructed from the expansion coefficients L_k using the result of Proposition 3.

The construction of a minimal realization (A,B,C) according to (20) and (21) requires a full-rank decomposition of $\mathbf{H}$ and the availability of $\mathbf{H}^{\leftarrow}$.

A full-rank decomposition of $\mathbf{H}$ is obtained by any full-rank decomposition of $\tilde{\mathbf{H}}=\tilde{\Gamma}\tilde{\Delta}$, because since (25) it follows that $\mathbf{H}=\mathbf{T}_1^{\mathrm{T}}\tilde{\Gamma}\cdot\tilde{\Delta}\mathbf{T}_2$ is a full-rank decomposition.

The shifted Hankel matrix $\mathbf{H}^{\leftarrow}$ is obtained by observing that it is the Hankel matrix related to the shifted system $zG(z)$, satisfying $\mathbf{H}^{\leftarrow}=\mathbf{T}_1^{\mathrm{T}}\tilde{\mathbf{H}}^{\leftarrow}\mathbf{T}_2$, where $\tilde{\mathbf{H}}^{\leftarrow}$ is the Hankel matrix related to $\widetilde{zG(z)}$. The Markov parameters of this latter system are specified by Proposition 3(b), and so $\tilde{\mathbf{H}}^{\leftarrow}$ can be constructed.

We can now formulate a realization algorithm for computation of a minimal realization of system G from the coefficient vectors L_k. This algorithm is essentially the same as the minimal realization algorithm in Szabó et al. (2000).

Algorithm 1. *Let L_k for $k=1,\ldots,\infty$ be the expansion coefficient vectors of a system $G\in RH_2$. A minimal state-space realization (A,B,C) of G is obtained as follows*:

(1) *Compute the Markov parameters M_k and $M_k^{\leftarrow}$ from the expansion coefficient L_k vectors according to formulas* (15) *and* (16), *and build the corresponding block Hankel matrices $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{H}}^{\leftarrow}$ (as in Eq.* (26)),
(2) *calculate a full-rank decomposition $\tilde{\mathbf{H}}=\tilde{\Gamma}\tilde{\Delta}$,*
(3) *obtain B and C as the first column of $\tilde{\Delta}\mathbf{T}_2$ and the first row of $\mathbf{T}_1^{\mathrm{T}}\tilde{\Gamma}$, respectively, and A according to $A=\tilde{\Gamma}^{\dagger}\tilde{\mathbf{H}}^{\leftarrow}\tilde{\Delta}^{\dagger}$, with $(\cdot)^{\dagger}$ denoting the pseudo-inverse.*

While Algorithm 1 gives insight into the method of solving the realization problem, it has limited practical value as it requires knowledge of the expansion coefficients of G up to infinity. The situation of a given finite expansion is considered next.

5. Minimal realization on the basis of a finite expansion

For the classical basis, it is well known that when a finite sequence $\{g_k\}_{k=1,\ldots,N}$ is given, the Ho–Kalman algorithm can be applied to a finite submatrix $\mathbf{H}_{N_1,N_2}$ of the full matrix $\mathbf{H}$ (with $N_1+N_2=N$), leading to an exact realization of the underlying system if N is sufficiently large. The Ho–Kalman algorithm in this case is equal to the one sketched at the beginning of Section 4 with the modification that now the Hankel matrices involved are of finite dimension, containing only the parameters that one actually knows.

In this section, we treat the (intermediate) problem that a finite number of expansion coefficients $\{L_k\}_{k=1,\ldots,N}$ is given of a system $G\in RH_2$ with known McMillan degree n. The results obtained here will be instrumental in solving the partial realization problem in the next section, where the McMillan degree of the underlying system is not assumed to be known.

When given a finite number of expansion coefficients $\{L_k\}_{k=1,\ldots,N}$ of a system $G\in RH_2$, this information can be translated to a finite number of Markov parameters $\{M_k\}_{k=0,\ldots,N-1}$ of $\tilde{G}$ according to (15). If N is sufficiently large, allowing the construction of a finite matrix $\tilde{\mathbf{H}}_{N_1,N_2}$ with $N_1+N_2=N-1$ that has the same rank as $\tilde{\mathbf{H}}$, a standard Ho–Kalman algorithm can be applied to arrive at a minimal realization $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ of $\tilde{G}$.

If we would be able to apply an inverse Hambo transform to this realization $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ this would solve our problem; however, such a direct inverse relation maintaining state-space dimension is not yet available. Actually the realization algorithm that is developed here is going to provide us with such an inverse relation.

For constructing a realization of G we need to have knowledge of $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{H}}^{\leftarrow}$. The realization $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ characterizes $\tilde{\mathbf{H}}$ completely. Next, it is shown that it also contains all information to construct $\tilde{\mathbf{H}}^{\leftarrow}$.

Lemma 5. *Given a realization $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ of the Hambo transform of a system $G(z)$. Then realizations of the proper part of the system $\widetilde{zG(z)}(\lambda)$ are given by*

(1) $$\begin{bmatrix}\tilde{A}_1 & \tilde{B}_1\\ \tilde{C}_1 & \tilde{D}_1\end{bmatrix}=\begin{bmatrix}\tilde{A} & \tilde{B}\\ A_b^{\mathrm{T}}\tilde{C}+C_b^{\mathrm{T}}X_C\tilde{A} & A_b^{\mathrm{T}}\tilde{D}+C_b^{\mathrm{T}}X_C\tilde{B}\end{bmatrix}$$

with X_C the solution to $D_b^{\mathrm{T}}X_C\tilde{A}+B_b^{\mathrm{T}}\tilde{C}=X_C$.

(2) $$\begin{bmatrix}\tilde{A}_2 & \tilde{B}_2\\ \tilde{C}_2 & \tilde{D}_2\end{bmatrix}=\begin{bmatrix}\tilde{A} & \tilde{B}A_b^{\mathrm{T}}+\tilde{A}X_BB_b^{\mathrm{T}}\\ \tilde{C} & \tilde{D}A_b^{\mathrm{T}}+\tilde{C}X_BB_b^{\mathrm{T}}\end{bmatrix}$$

with X_B the solution to $\tilde{A}X_BD_b^{\mathrm{T}}+\tilde{B}C_b^{\mathrm{T}}=X_B$.

The state-space matrices associated with $\widetilde{zG(z)}(\lambda)$ bear the necessary information to construct the infinite Hankel matrix $\tilde{\mathbf{H}}^{\leftarrow}$. Note that this implies that once a state-space realization of $\tilde{G}(\lambda)$ is known, one does not need to compute the parameters $M_k^{\leftarrow}$ according to (16), because the realizations of Lemma 5 can be used to derive them.

One can now write down factorizations of the matrices $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{H}}^{\leftarrow}$ in terms of extended controllability and observability matrices. Define for instance the matrices $\tilde{\Gamma}$ and $\tilde{\Gamma}_1$ to represent the extended observability matrices associated with the pairs $(\tilde{A},\tilde{C})$ and $(\tilde{A}_1,\tilde{C}_1)$, respectively. Similarly, define the extended controllability matrices $\tilde{\Delta}$ and $\tilde{\Delta}_1$ on the basis of $(\tilde{A},\tilde{B})$ and $(\tilde{A}_1,\tilde{B}_1)$. It then holds that $\tilde{\mathbf{H}}=\tilde{\Gamma}\tilde{\Delta}$ and $\tilde{\mathbf{H}}^{\leftarrow}=\tilde{\Gamma}_1\tilde{\Delta}_1$. Note that $\tilde{\Delta}_1=\tilde{\Delta}$.

According to Algorithm 1, one can now obtain a realization (A,B,C) of the system $G(z)$ according to

$$B=\tilde{\Delta}\mathbf{T}_2\mathbf{e}_1,\quad C=\mathbf{e}_1^{\mathrm{T}}\mathbf{T}_1^{\mathrm{T}}\tilde{\Gamma},\tag{27}$$

$$A=\tilde{\Gamma}^{\dagger}\tilde{\Gamma}_1\tilde{\Delta}_1\tilde{\Delta}^{\dagger}=\tilde{\Gamma}^{\dagger}\tilde{\Gamma}_1.\tag{28}$$

Because of the particular structure of $\mathbf{T}_1$ and $\mathbf{T}_2$, the state space matrices can be calculated by taking matrix inner products between known stable, strictly proper transfer functions, which can be formulated in terms of Sylvester equations.

One can interpret Eqs. (27) and (28) as expressions to recover a minimal state-space representation (A,B,C) for $G(z)$ from the state-space matrices $(\tilde{A},\tilde{B},\tilde{C})$ representing the strictly proper part of $G(z)$. In other words, they represent the inverse Hambo transform operation. Hence, the following proposition.

Proposition 6 (Inverse Hambo transform). *Given a minimal state-space realization $(\tilde{A},\tilde{B},\tilde{C})$ of the strictly-proper part of a Hambo transform $\tilde{G}(\lambda)$ of a system*

$G \in RH_2$, then a minimal state-space realization (A, B, C) of $G(z)$ is given by

$$A = \tilde{X}_o^{-1} X_A^o, \quad B = X_B, \quad C = X_C, \tag{29}$$

where $\tilde{X}_o$ is the observability Gramian of the pair $(\tilde{A}, \tilde{C})$ and X_B, X_C and X_A are the solutions to the following set of equations:

$$\tilde{A} X_B D_b^T + \tilde{B} C_b^T = X_B, \tag{30}$$

$$D_b^T X_C \tilde{A} + B_b^T \tilde{C} = X_C, \tag{31}$$

$$\tilde{A}^T X_A^o \tilde{A} + \tilde{C}^T (A_b^T \tilde{C} + C_b^T X_C \tilde{A}) = X_A^o. \tag{32}$$

Note that the expression for A in (29) simplifies to $A = X_A^o$ if the system $(\tilde{A}, \tilde{B}, \tilde{C})$ is in output balanced form, i.e. when $\tilde{X}_o = I$. Also, note that the matrices A and $\tilde{A}$ have the same dimension, which is consistent with the fact that McMillan degree is invariant under the Hambo transformation.

It should be mentioned that the particular form of expressions (29) and (32) depends on the factorization $\tilde{\mathbf{H}}^{\leftarrow} = \tilde{\Gamma}_1 \tilde{\Delta}_1$ used in the derivation. If the factorization $\tilde{\mathbf{H}}^{\leftarrow} = \tilde{\Gamma}_2 \tilde{\Delta}_2$ were to be used, representing the observability and controllability matrices associated with the second realization in Lemma 5, one would have found an alternative expression for the matrix A. In that case, it holds that $A = X_A^c \tilde{X}_c^{-1}$ with $\tilde{X}_c$ the controllability Gramian of the pair $(\tilde{A}, \tilde{B})$ and X_A^c the solution to

$$\tilde{A} X_A^c \tilde{A}^T + (\tilde{B} A_b^T + \tilde{A} X_B B_b^T) \tilde{B}^T = X_A^c. \tag{33}$$

The expression for A simplifies to $A = X_A^c$ if $(\tilde{A}, \tilde{B}, \tilde{C})$ is in input balanced form.

It now is easy to formulate the following minimal realization algorithm for finite expansions.

Algorithm 2. *Let L_k for $k = 1, \ldots, N$ be the first N expansion coefficient vectors of a system G in RH_2 having McMillan degree n. A minimal state-space realization (A, B, C) of G is obtained as follows*:

(1) *Compute the Markov parameters M_k for $k = 1, \ldots, N-1$ from the corresponding expansion coefficient vectors L_k according to formula* (15).
(2) *Check whether* $\text{rank}(\tilde{\mathbf{H}}_{N_1,N_2}) = n$ *with $N_1 + N_2 = N - 1$. If not then the algorithm fails.*
(3) *Compute from $\tilde{\mathbf{H}}_{N_1,N_2}$ a minimal realization $(\tilde{A}, \tilde{B}, \tilde{C})$, of McMillan degree n, by applying a Ho–Kalman algorithm.*
(4) *Compute (A, B, C) from $(\tilde{A}, \tilde{B}, \tilde{C})$ by applying the inverse Hambo transform using* (29)–(32).

It is interesting to note how the solution of the minimal realization problem has produced a set of expressions to compute the Hambo inverse transform. Along similar lines of reasoning, a dual set of equations can be derived to compute a minimal state-space realization of the Hambo transform $\tilde{G}(\lambda)$ on the basis of a minimal state-space realization of $G(z)$.

Proposition 7 (Hambo transform). *For a system $G \in RH_2$, with minimal state-space realization (A, B, C), a minimal state-space realization of its Hambo transform is given by*

$$\tilde{A} = X_o^{-1} X_{\tilde{A}}^o, \quad \tilde{B} = X_{\tilde{B}}, \quad \tilde{C} = X_{\tilde{C}}, \quad \tilde{D} = X_{\tilde{D}}, \tag{34}$$

where X_o is the observability Gramian of the pair (A, C) and $X_{\tilde{B}}$, $X_{\tilde{C}}$, $X_{\tilde{A}}$ and $X_{\tilde{D}}$ are the solutions to the following set of equations:

$$A X_{\tilde{B}} A_b + B C_b = X_{\tilde{B}}, \tag{35}$$

$$A_b X_{\tilde{C}} A + B_b C = X_{\tilde{C}}, \tag{36}$$

$$A_b X_{\tilde{D}} A_b^T + (B_b D + A_b X_{\tilde{C}} B) B_b^T = X_{\tilde{D}}, \tag{37}$$

$$A^T X_{\tilde{A}}^o A + C^T (D_b C + C_b X_{\tilde{C}} A) = X_{\tilde{A}}^o. \tag{38}$$

As before the expressions for the matrix $\tilde{A}$ can alternatively be written as $\tilde{A} = X_{\tilde{A}}^c X_c^{-1}$, with X_c the controllability Gramian of the pair (A, B) and $X_{\tilde{A}}^c$ the solution to

$$A X_{\tilde{A}}^c A^T + (B D_b + A X_{\tilde{B}} B_b) B^T = X_{\tilde{A}}^c. \tag{39}$$

Also, the expressions for $\tilde{A}$ again simplify when (A, B, C) is in input or output balanced form.

An interesting corollary result of Proposition 7 is as follows.

Corollary 8. *Given a system $G(z) \in RH_2$ with minimal state-space realization (A, B, C), its expansion coefficients satisfy*

$$L_k = \tilde{C} \tilde{A}^{k-1} B, \tag{40}$$

with $\tilde{A}$ and $\tilde{C}$ as given by Eq. (34).

This result might suggest that a realization of G can be obtained through application of a Ho–Kalman algorithm to the sequence $\{L_k\}$ followed by application of formulas (30) and (32). The problem, however, is that the pair $(\tilde{A}, B)$ is not reachable, for a large class of systems. Hence, application of a Ho–Kalman algorithm to the sequence $\{L_k\}$ might produce a matrix $\tilde{A}$ that has a dimension smaller than the McMillan degree of G. However, as shown in the next section, Corollary 8 turns out to be a key result in the solution of the minimal partial realization problem.

The formulas for computing the Hambo transform and its inverse, as given in Propositions 7 and 6 are useful spinoff results of the analysis involved in solving the realization problem, for the case where a finite number of expansion coefficients are given. They provide us with compact expressions to compute the transforms directly in terms of the state-space matrices of minimal realizations, in an efficient and reliable manner. Previously, simple formulas were available only for the Laguerre transform and its inverse (Nurges, 1987; Heuberger, 1991; Fischer & Medvedev, 1998) while the computation of the Hambo transform, and especially its inverse, still required quite elaborate computations (Heuberger & Van den Hof, 1996).

6. Minimal partial realization

In this section, an algorithm is given that provides a solution to the minimal partial realization problem for expansions in the Hambo bases as formulated in the introduction. A solution to this problem is called a *minimal partial realization* of the sequence $\{L_k\}_{k=1,\dots,N}$. In Tether (1970), it was shown that a unique solution (modulo similarity transformation) to the classical minimal partial realization problem is obtained through application of the Ho–Kalman algorithm, provided that a certain rank condition is satisfied by the sequence of Markov parameters $\{g_k\}_{k=1,\dots,N}$.

A similar result can be derived for the generalized problem. Given the expansion coefficients L_k for $k=1,\dots,N$ one can calculate the Markov parameters M_k for $k=0,\dots,N-1$ as described in Section 3. This would perhaps suggest that the problem can be solved by means of Algorithm 2 under condition that the sequence $\{M_k\}_{k=1,\dots,N-1}$ satisfies the realizability condition given by Tether (1970). This condition is, however, not sufficient to guarantee that the resulting realization $(\tilde{A},\tilde{B},\tilde{C},M_0)$ constitutes a valid Hambo transform. We require a realizability criterion that is specifically tuned to our problem.

The key to find such a realizability condition is provided by Corollary 8. This result shows that for a system $G\in RH_2$ the sequences $\{L_k\}$ and $\{M_k\}$ are realized by state-space realizations that share the state transition matrix $\tilde{A}$. This leads us to consider the sequence of concatenated matrices

$$K_k=[M_k|L_k|L_{k+1}] \tag{41}$$

for $k=1,\dots,N-1$. Parameter L_{k+1} is included in view of the fact that parameter M_k is obtained on the basis of L_k and L_{k+1}.

The following lemma provides the conditions under which the minimal partial realization problem can be solved.

Lemma 9. *Let* $\{L_k\}_{k=1,\dots,N}$ *be an arbitrary sequence of* $n_b\times 1$ *vectors and let* M_k *and* K_k *for* $k=1,\dots,N-1$ *be derived from* L_k *via relations* (15) *and* (41). *Then, there exists a unique minimal realization* (*modulo similarity transformation*) $(\tilde{A},\tilde{B},\tilde{C})$ *with McMillan degree n, and an* $n\times 1$ *vector B such that*

(a) $M_k=\tilde{C}\tilde{A}^{k-1}\tilde{B}$ *for* $k=1,\dots,N-1$, *and* $L_k=\tilde{C}\tilde{A}^{k-1}B$ *for* $k=1,\dots,N$,
(b) *the infinite sequences* $[M_k\ L_k]:=\tilde{C}\tilde{A}^{k-1}[\tilde{B}\ B]$ *satisfy relation* (15) *for all* $k\in\mathbb{N}$,

if and only if there exist positive N_1,N_2 *such that* $N_1+N_2=N-1$ *and*

$$\operatorname{rank}\hat{\mathbf{H}}_{N_1,N_2}=\operatorname{rank}\hat{\mathbf{H}}_{N_1+1,N_2}=\operatorname{rank}\hat{\mathbf{H}}_{N_1,N_2+1}=n, \tag{42}$$

where $\hat{\mathbf{H}}_{i,j}$ *is the Hankel matrix built from the matrices* $\{K_k\}$ *with block-dimensions* $i\times j$.

Note that condition (42) is in fact equal to the condition given by Tether (1970) applied to the sequence $\{K_k\}_{k=1,\dots,N-1}$. Also, note that it can only be checked for $N>2$. On the basis of Lemma 9 and its proof we can formulate the following proposition that also provides an algorithm to solve the minimal partial realization problem, i.e. given a sequence of expansion coefficients, when does there exist a system with minimal degree that matches these coefficients.

Proposition 10. *Let* $\{L_k\}_{k=1,\dots,N}$ *be an arbitrary sequence of* $n_b\times 1$ *vectors, then there exists a minimal realization* (A,B,C) *of McMillan degree n, such that* $\{L_k\}_{k=1,\dots,N}$ *are the first N expansion coefficients of* $G(z)=C[zI-A]^{-1}B$, *if*

(1) *there exist positive* N_1,N_2 *such that* $N_1+N_2=N-1$ *and condition* (42) *of Lemma* 9 *holds,*
(2) *the minimal realization* $(\tilde{A},[\tilde{B}\ X_2\ X_3],\tilde{C},\tilde{D})$, *resulting from application of the Ho–Kalman algorithm to the sequence* $\{K_k\}_{k=1,\dots,N-1}$, *is stable.*

Furthermore, the matrices A,B,C *are derived by application of the inverse Hambo transform to the realization* $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ *using Eqs.* (29)–(32).

Remark 11. The requirement that $\tilde{A}$ is stable assures that the Ho–Kalman algorithm yields a valid Hambo transform of a stable system. It should be mentioned that it is possible to give a formal definition of the Hambo operator transform that applies to rational transfer functions that are in L_2, and which therefore may be unstable. Such a setup lies beyond the scope of this paper.

However, it is straightforward to extend the algorithm provided by Proposition 10 to the case where $\tilde{A}$ has no poles on the unit circle. If $\tilde{G}$ has unstable poles, it has to be separated in a stable and unstable part. The unstable part is transformed by mirroring it to a stable function and after transformation, mirroring the transform back to an unstable function. Hence, the only situation which is actually not covered by this algorithm is when $\tilde{A}$ has poles on the unit circle, if $\tilde{A}$ has poles that are reciprokes of poles of G_b, the resulting system may be non-causal.

If condition (42) is *not* fulfilled one can—as in the classical case—investigate the set of all minimal realizations that matches the given sequence of the Markov parameters and choose a component of this set to proceed. The problem of parameterizing these extensions will be the subject of future research.

7. The underlying interpolation problem

It is well known that the classical problem of minimal partial realization from the first N Markov parameters is equivalent to the problem of constructing a stable strictly proper real-rational transfer function of minimal degree that interpolates to the first $N-1$ derivatives of $G(z)$ evaluated at infinity (Anderson & Antoulas, 1990). Similarly, the least-squares approximation of a stable transfer function $G(z)$ in terms of a finite set of rational basis functions interpolates to the function $G(z)$ and/or its derivatives in the points $1/\xi_i$ with ξ_i being the poles of the basis functions involved (Walsh, 1956).

In the basis construction considered in this paper the error function of an Nth order approximation $\hat{G}(z)$ takes on the form

$$E(z)=\sum_{k=N+1}^{\infty} L_k^{\mathrm{T}} V_k(z)=G_{\mathrm{b}}^N(z)\sum_{k=1}^{\infty} L_{N+k}^{\mathrm{T}} V_1(z)G_{\mathrm{b}}(z)^{k-1}.$$

Due to the repetition of the all-pass function $G_{\mathrm{b}}(z)$ in V_k, the error $E(z)$ will have as a factor the function $G_{\mathrm{b}}^N(z)$. This means that $E(z)$ has zeros of order N at each of the points $1/\xi_i$ and subsequently $\hat{G}(z)$ interpolates to $(\mathrm{d}^{k-1}G)/(\mathrm{d}z^{k-1})$ in $z=1/\xi_i$ for $k=1,\dots,N$ and $i=1,\dots,n_{\mathrm{b}}$. This interpolation property in fact holds true for any model of which the first N expansion coefficient vectors match those of the system, in particular, for a model found by solving the partial realization problem.

In view of the interpolating property of the basis function expansion it is not surprising that there exists a one-to-one correspondence between the expansion coefficient vector sequence $\{L_k\}_{k=1,\dots,N}$ and the interpolation data $\{(\mathrm{d}^{k-1}G)/(\mathrm{d}z^{k-1})(1/\xi_i)\}_{k=1,\dots,N}$. An explicit expression for this relation can be derived by exploiting the linear transformation that links the set of basis function vectors V_k and the set of vectors that consists of single-pole transfer functions as given by

$$S_k^{\mathrm{T}}(z)=\left[\frac{1}{(z-\xi_1)^k},\dots,\frac{1}{(z-\xi_{n_{\mathrm{b}}})^k}\right]^{\mathrm{T}},$$

with ξ_i the poles of the basis generating function $G_{\mathrm{b}}(z)$. If the poles ξ_i are assumed to be distinct one can write

$$\begin{bmatrix} V_1(z)\\ V_2(z)\\ \vdots\end{bmatrix}=\begin{bmatrix} T_{11} & 0 & \cdots\\ T_{21} & T_{22} & \\ \vdots & & \ddots\end{bmatrix}\begin{bmatrix} S_1(z)\\ S_2(z)\\ \vdots\end{bmatrix}$$

with $T_{kl}\in\mathbb{R}^{n_{\mathrm{b}}\times n_{\mathrm{b}}}$. The coefficient vectors L_k are obtained according to (7) as

$$L_k=\frac{1}{2\pi i}\oint V_k(1/z)G(z)\frac{\mathrm{d}z}{z}.$$

When substituting $V_k(1/z)=\sum_{l=1}^{k} T_{kl}S_l(1/z)$ and applying Cauchy's integral formula one finds the following expression for the coefficients L_k:

$$L_k=\sum_{l=1}^{k} T_{kl}\,\mathrm{diag}\left\{\frac{-1}{(-\xi_i)^l}\right\}_{i=1,\dots,n_{\mathrm{b}}}\sum_{m=0}^{l-1}\begin{pmatrix} l-1\\ m\end{pmatrix}\frac{1}{m!}\left[\frac{\mathrm{d}^mG}{\mathrm{d}z^m}(1/\xi_1)\quad\cdots\quad\frac{\mathrm{d}^mG}{\mathrm{d}z^m}(1/\xi_{n_{\mathrm{b}}})\right]^{\mathrm{T}}.$$

In matrix/vector notation, the relation between the interpolation data and expansion coefficients can be expressed as

$$\begin{bmatrix} L_1\\ L_2\\ \vdots\end{bmatrix}=\begin{bmatrix} T_{11} & 0 & \cdots\\ T_{21} & T_{22} & \\ \vdots & & \ddots\end{bmatrix}\Pi\begin{bmatrix} F_0\\ F_1\\ \vdots\end{bmatrix}$$

with

$$F_m^{\mathrm{T}}=\frac{1}{(m)!}\left[\frac{1}{\xi_1^m}\frac{\mathrm{d}^mG}{\mathrm{d}z^m}(1/\xi_1),\dots,\frac{1}{\xi_{n_{\mathrm{b}}}^m}\frac{\mathrm{d}^mG}{\mathrm{d}z^m}(1/\xi_{n_{\mathrm{b}}})\right]^{\mathrm{T}},$$

and Π a matrix that is given by

$$\Pi=\begin{bmatrix} \Lambda^{-1} & 0 & \cdots\\ 0 & -\Lambda^{-2} & \\ \vdots & & \ddots\end{bmatrix}\begin{bmatrix} \binom{0}{0}I & 0 & \cdots\\ \binom{1}{0}I & \binom{1}{1}I & \\ \vdots & & \ddots\end{bmatrix}\tag{43}$$

with $\Lambda^{-k}=\mathrm{diag}\{1/\xi_i^k\}_{i=1,\dots,n_{\mathrm{b}}}$. Since $1/\xi_i$ exists only for $\xi_i\neq 0$, we must make the additional assumption that $\xi_i\neq 0$, $\forall i$. Actually, the corresponding elements in Π simplify considerably when $\xi_i=0$ for some i but we will not consider that situation here. Eq. (43) shows that there exists a direct correspondence between the first N coefficient vectors L_k and the first N vectors F_k that contain the data $(\mathrm{d}^{k-1}G)/(\mathrm{d}z^{k-1})$ for $k=1,\dots,N$ evaluated at the points $1/\xi_i$.

A similar relation can be derived that shows the correspondence between the generalized Markov parameter sequence $\{M_{k-1}\}_{k=1,\dots,N}$ and the interpolation data

$\{(\mathrm{d}^{k-1}G)/(\mathrm{d}z^{k-1})\}_{k=1,\ldots,N}$, e.g. starting from Eq. (11). Another option is to use expression (15) to compute the Markov parameters $\{M_{k-1}\}$ from the coefficients $\{L_k\}$, for $k=1,\ldots,N$. One can hence solve the following interpolation problem, by means of the algorithm of Proposition 10.

Problem 12 (Interpolation problem). *Given the interpolation conditions*

$$\frac{\mathrm{d}^{k-1}G}{\mathrm{d}z^{k-1}}(1/\xi_i)=c_{i,k}, \quad c_{i,k}\in\mathbb{C}$$

for $i=1,\ldots,n_\mathrm{b}$ *and* $k=1,\ldots,N$, *with* $\xi_i\neq 0$ *distinct points inside the unit disc, find the* RH_2 *transfer function of least possible degree that interpolates these points.*

The problem is solved for $N>2$ by constructing an all-pass function G_b with balanced realization $(A_\mathrm{b},B_\mathrm{b},C_\mathrm{b},D_\mathrm{b})$ such that the eigenvalues of A_b are ξ_i. From this all-pass function one can then obtain all the parameters that are necessary to compute the set of Markov parameters M_k that correspond to the interpolation data. Proposition 10 gives the desired transfer function, provided that the necessary conditions are satisified. Note that the ξ_i should come in conjugate pairs to ensure that the resulting transfer function has real-valued coefficients.

The relation between the Markov parameters M_k and the derivatives of $G(z)$ evaluated at $1/\xi_i$ has been treated in a similar context in Audley and Rugh (1973) on the representation of systems in the so-called H-matrix form. The H-matrix is not to be mistaken for the Hankel operator discussed earlier but it is closely connected to it. It takes on a Toeplitz instead of a Hankel matrix form but the basic elements of the H-matrix for the basis considered in this paper are still the Markov parameters M_k. Audley and Rugh provided an algorithm to realize a transfer function of minimal degree from a finite dimensional H-matrix representation by directly solving the underlying interpolation problem.

8. Approximate realization

The classical partial realization algorithm can be applied as a system identification method, as in e.g. Zeiger and McEwen (1974) and Kung (1978), building a Hankel matrix with (possibly noise corrupted) expansion coefficients and by applying rank reduction through singular value truncation. This approach can be applied similarly to the generalized situation using Algorithm 2. In Section 9, an example is given in which this method is compared with the classical approximate realization method. Aside from this identification context, the approximate realization procedure can also be applied as a model reduction method.

In comparison with the classical case, approximate realization in the generalized case has one additional difficulty. It is due to the fact that not every system in $RH_{2,0}^{n_\mathrm{b}\times n_\mathrm{b}}$ is the Hambo transform of a system in H_2. A necessary condition [4] is that it commutes with the transfer matrix $N(\lambda)$. This follows directly from the variable substitution property of the Hambo transform according to (13). Naturally, this commutative property of Hambo transforms is not true for general approximate realizations $\{\tilde{A},\tilde{B},\tilde{C}\}$ obtained with Algorithm 2, respectively realizations $(\tilde{A},[X_1|X_2|X_3],\tilde{C})$ obtained through the algorithm of Proposition 10. Although formulas (29)–(32) for the inverse transform still can be applied, the resulting system will not have a one-to-one correspondence with the approximate realization. In the exact realization setting, this problem does not arise. The full implications of this phenomenon, such as the topological properties of the approximation, are not fully understood yet and will be the subject of further research. The problem can be circumvented, however, by using the approximate realization $(\tilde{A},X_2,\tilde{C})$ for $\{L_k\}$ resulting from the algorithm of Proposition 10 and considering the approximation

$$\hat{G}(z)=\sum_{k=1}^{\infty}(\tilde{C}\tilde{A}^{k-1}X_2)^\mathrm{T}V_k(z)$$

which (under the condition that $\tilde{A}$ is stable) is finite dimensional with McMillan degree smaller or equal to $n_\mathrm{b}\dim(\tilde{A})$. A minimal realization for $\hat{G}(z)$ can subsequently be obtained by application of Algorithm 2. For the example in the next section the first approach and Algorithm 2 are used.

The formulas for the inverse Hambo transformation (Eqs. (29)–(32)) make it possible to transfer not only the realization problem to the transform domain but also the whole identification procedure itself. This idea has been applied before for the Laguerre and two-parameter Kautz case in Fischer and Medvedev (1998) and Diaz, Fischer, and Medvedev (1998), where a subspace identification method is used for the extraction of the state-space matrices of the transformed system from expansions of the measured data in terms of these basis functions. A state-space representation in the time domain is then obtained by applying inversion formulas. This idea can straightforwardly be extended to the general case using the formulas provided in this paper. It should be noted, however, as explained above, that in general the outcome of the identification procedure does not immediately result in a valid operator transform, not even in the Laguerre case. Therefore, care should be taken when the inverse transformation formulas are applied in such a context.

[4] It can be shown that for Hambo transforms of systems in $RH_{2,0}$ this is even a necessary and sufficient condition.

9. Example

In this example, we compare the application of the generalized approximate realization method suggested in Section 8 with the classical approximate realization method of Kung (1978). We consider a sixth order SISO transfer function $G(z)$, given by

$$G(z) = 10^{-3} \frac{-0.564z^5 + 43.9z^4 - 21.7z^3 - 1.04z^2 - 95.7z + 75.2}{z^6 - 3.35z^5 + 4.84z^4 - 4.44z^3 + 3.11z^2 - 1.48z + 0.318}.$$

The impulse response of $G(z)$, as shown in Fig. 1, reveals that the system incorporates a mix of fast and slow dynamics.

Ten simulations are carried out in which the response of the system $G(z)$ to a Gaussian white-noise input with unit standard deviation is determined. An independent Gaussian noise disturbance with standard deviation 0.05 is added to the output. This amounts to a signal-to-noise ratio (in terms of RMS values) of about 17 dB. The length of the input and output data signals is taken to be 1000 samples. For each of the ten data sets two basis function models of the form

$$\hat{G}(z) = \sum_{k=1}^{N} \hat{L}_k^{\mathrm{T}} V_k(z), \tag{44}$$

are estimated using the least-squares method described in Van den Hof et al. (1995). The first model is a 40th order FIR model. Hence, in this case $V_k(z) = z^{-k}$ and $N = 40$. The second model uses a generalized basis that is generated by a second-order all-pass function with poles 0.5 and 0.9. For this model 20 coefficient vectors are estimated. Hence, the number of estimated coefficients is equal for both models.

We now apply the approximate realization method using the estimated expansion coefficients of both models, for all ten simulations. In either case a sixth order model is computed, through truncation of the SVD of the finite Hankel matrix. In Fig. 2, step response plots of the resulting models are shown. It is seen that approximate realization using the standard basis results in a model that fits only the first samples of the response. This is a known drawback of this method. Employing the generalized basis, with poles 0.5 and 0.9 results in models that better capture the transient behavior. Apparently, a sensible choice of basis can considerably improve the performance of the Kung algorithm.

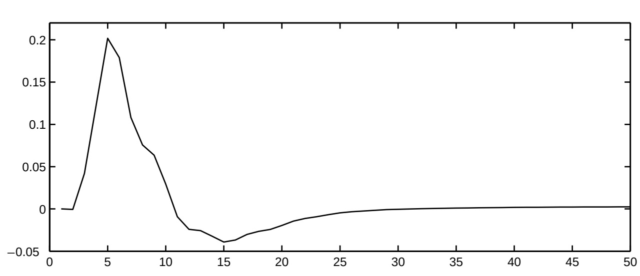

Fig. 1. Impulse response of the example system.

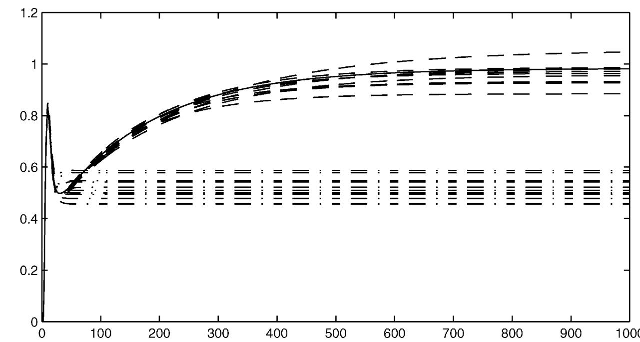

Fig. 2. Step responses of the example system (solid) and the models obtained in ten simulations with approximate realization using the standard basis (dash-dotted) and the generalized basis (dashed).

10. Conclusions

In this paper, an algorithm is derived that solves the minimal partial realization problem for expansions in terms of generalized orthonormal basis functions, generated according to the Hambo basis construction. The realization problem is solved by linking it to the classical realization problem formulated in the Hambo operator transform domain. As corollary results expressions are obtained for directly computing the operator transform and its inverse in terms of minimal state-space representations. The presented algorithm yields an original solution to a general minimal rational interpolation problem, and can also be used in an approximate sense, e.g. for the purpose of model reduction or in a system identification setting.

Appendix

Proof of Lemma 1.

$$G_{\mathrm{b}}(1/z)V_1^{\mathrm{T}}(z) = \sum_{l=0}^{\infty} g_{\mathrm{b}_l} z^l \sum_{k=1}^{\infty} B_{\mathrm{b}}^{\mathrm{T}} A_{\mathrm{b}}^{\mathrm{T}^{k-1}} z^{-k}.$$

Put the term for $l = 0$ aside for a moment. Making use of the fact that $A_{\mathrm{b}}A_{\mathrm{b}}^{\mathrm{T}} + B_{\mathrm{b}}B_{\mathrm{b}}^{\mathrm{T}} = I$ (this follows from (4)) the remainder of the sums can be written as

$$\sum_{l=1}^{\infty}\sum_{k=1}^{\infty} C_{\mathrm{b}}A_{\mathrm{b}}^{l-1} A_{\mathrm{b}}^{\mathrm{T}^{k-1}} z^{l-k} - \sum_{l=1}^{\infty}\sum_{k=1}^{\infty} C_{\mathrm{b}}A_{\mathrm{b}}^{l} A_{\mathrm{b}}^{\mathrm{T}^{k}} z^{l-k}.$$

It is obvious that most terms in this expression cancel out. It can be simplified to

$$\sum_{l=0}^{\infty} C_b A_b^l z^l + \sum_{k=1}^{\infty} C_b A_b^{T^k} z^{-k}.$$

With $C_b A_b^T = -D_b B_b^T$ (this follows from (4)) this becomes

$$\sum_{l=0}^{\infty} C_b A_b^l z^l - \sum_{k=1}^{\infty} D_b B_b A_b^{T^{k-1}} z^{-k}.$$

The last term is exactly equal to minus the term for $l=0$ which we set aside. So we end up with the desired result. □

Proof of Proposition 2. Because $y = Gu$, it follows from (7) that

$$\mathscr{Y}(m) = \langle V_m, Gu\rangle_M = \left\langle V_m, G\sum_{j=1}^{\infty} \mathscr{U}^T(j)V_j \right\rangle_M$$
$$= \sum_{j=1}^{\infty} \langle V_m, V_j G\rangle_M \mathscr{U}(j).$$

Using the shift structure in the basis this can be written as

$$\mathscr{Y}(m) = \sum_{j=1}^{\infty} \langle V_1 G_b^{m-1}, V_1 G_b^{j-1} G\rangle_M \mathscr{U}(j). \tag{45}$$

The adjoint of the transfer function $G_b(z)$ is equal to $G_b^T(1/z)$, which by the all-pass property is also equal to the inverse of $G_b(z)$. By these facts the last equation can be expressed as

$$\mathscr{Y}(m) = \sum_{j=1}^{\infty} \langle V_1(z), V_1(z) G_b^{j-m}(z) G(z)\rangle_M \mathscr{U}(j). \tag{46}$$

The outcome of the inner product term is equal to zero for $j > m$. This follows directly from the fact that the elements of $V_1(z)$ are orthogonal to the space spanned by the functions V_k for $k > 1$ which is equal to the shift-invariant subspace $G_b(z)H_2$. □

Proof of Proposition 4. We partition the matrix $\tilde{\mathbf{H}}$ into blocks of dimension $n_b \times n_b$ with the i,jth block denoted as $\tilde{\mathbf{H}}_{(i,j)}$. It holds that $\tilde{\mathbf{H}}_{(i,j)} = \mathbf{v}_i \mathbf{H}\mathbf{u}_{j-1}^T$. The vector $\mathbf{H}\mathbf{u}_{j-1}^T$ corresponds to the output of the system G in response to the input $U_{j-1} \in \ell_2^{n_b}(-\infty, 0]$, restricted to the space of future signals $\ell_2^{n_b}[1,\infty)$. This output can be expressed as

$$\mathbf{P}_{H_2^{n_b}} G(z) U_{j-1}(1/z)^T = \mathbf{P}_{H_2^{n_b}} G(z) G_b^{-j} V_1(z)^T. \tag{47}$$

The last equality follows from Eqs. (9) and (8). We then have that

$$\tilde{\mathbf{H}}_{(i,j)} = \langle V_i(z), \mathbf{P}_{H_{2-}^{n_b}} G(z)\, G_b(z)^{-j}\, V_1(z)\rangle_M$$
$$= \langle G_b(z)^{i-1} V_1(z), G(z) G_b(z)^{-j}\, V_1(z)\rangle_M.$$

Because $G_b(z)$ is inner this expression simplifies to

$$\tilde{\mathbf{H}}_{(i,j)} = \langle G_b(z)^{i+j-1}\, V_1(z), G(z) V_1(z)\rangle_M, \tag{48}$$

which is equal to M_{i+j-1} as was established earlier, see Proposition 2. □

Proof of Lemma 5. Using the variable substitution property of the Hambo operator transform, reflected in expression (13), it follows that the system $zG(z)$ has a Hambo transform that is equal to $N^T(1/\lambda)\tilde{G}(\lambda)$, or equivalently $\tilde{G}(\lambda)N^T(1/\lambda)$. These two forms lead to two different realizations for the proper part of $\widetilde{zG(z)}(\lambda)$. The first one, $N^T(1/\lambda)\tilde{G}(\lambda)$ can be written as

$$\sum_{t=0}^{\infty}\sum_{k=0}^{\infty} n_k^T \tilde{g}_t \lambda^{k-t}, \tag{49}$$

where n_k and $\tilde{g}_t$ represent the kth and tth pulse response parameters of the transfer matrices $N(\lambda)$ and $\tilde{G}(\lambda)$, respectively. To evaluate the proper part of $\widetilde{zG(z)}(\lambda)$ we make an orthogonal projection of (49) onto the space $H_{2,0}^{n_b\times n_b}$. We then find

$$\mathbf{P}_{H_{2,0}} N^T(1/\lambda)\tilde{G}(\lambda) = \sum_{t=0}^{\infty}\sum_{k=0}^{t} n_k^T \tilde{g}_t \lambda^{k-t}$$
$$= \sum_{k=0}^{\infty}\sum_{t=k}^{\infty} n_k^T \tilde{g}_t \lambda^{k-t}. = \sum_{t'=0}^{\infty}\sum_{k=0}^{\infty} n_k^T \tilde{g}_{t'+k} \lambda^{-t'}$$

with $t' = t-k$. This is equal to

$$\left(\sum_{k=0}^{\infty} n_k^T \tilde{g}_k\right) + \sum_{t'=1}^{\infty}\left(\sum_{k=0}^{\infty} n_k^T \tilde{C}\tilde{A}^k\right)\tilde{A}^{t'-1}\tilde{B}\lambda^{-t'}. \tag{50}$$

Clearly, this system has realization $(\tilde{A}_1, \tilde{B}_1, \tilde{C}_1, \tilde{D}_1)$ with $\tilde{A}_1 = \tilde{A}$, $\tilde{B}_1 = \tilde{B}$, $\tilde{C}_1 = \sum_{k=0}^{\infty} n_k^T \tilde{C}\tilde{A}^k$ and $\tilde{D}_1 = \sum_{k=0}^{\infty} n_k^T \tilde{g}_k$. The result now follows by observing that

$$\sum_{k=0}^{\infty} n_k^T \tilde{C}\tilde{A}^k = A_b^T \tilde{C} + \sum_{k=1}^{\infty} C_b^T D_b^{T^{k-1}} B_b^T \tilde{C}\tilde{A}^k$$
$$= A_b^T \tilde{C} + C_b^T X_C \tilde{A} \tag{51}$$

and

$$\sum_{k=0}^{\infty} n_k^T \tilde{g}_k = A_b^T \tilde{D} + \sum_{k=1}^{\infty} C_b^T D_b^{T^{k-1}} B_b^T \tilde{C}\tilde{A}^{k-1}\tilde{B}$$
$$= A_b^T \tilde{D} + C_b^T X_C \tilde{B} \tag{52}$$

with X_C the solution to $D_b^T X_C \tilde{A} + B_b^T \tilde{C} = X_C$. The second realization for the proper part of $\widetilde{zG(z)}(\lambda)$ follows similarly when starting from the form $\tilde{G}(\lambda)N^T(1/\lambda)$. □

Proof of Proposition 6. Because $(\tilde{A}, \tilde{B}, \tilde{C})$ is minimal it follows that a left-inverse of $\tilde{\Gamma}$ is given by $\tilde{X}_o^{-1}\tilde{\Gamma}^T$.

Hence, it follows from (28) that $A=\tilde{X}_o^{-1}\tilde{\Gamma}^T\Gamma_1$. The product $\tilde{\Gamma}^T\Gamma_1$ is equal to the solution of Eq. (32). From (27) we have $B=\tilde{\Delta}\mathbf{T}_2\mathbf{e}_1$ and $C=\mathbf{e}_1^T\mathbf{T}_1^T\tilde{\Gamma}$. The form of $\mathbf{T}_1$ and $\mathbf{T}_2$ implies that

$$\mathbf{e}_i^T\mathbf{T}_1^T=[B_b^T \quad B_b^TD_b \quad B_b^TD_b^2 \quad \cdots], \tag{53}$$

$$\mathbf{e}_i^T\mathbf{T}_2^T=[C_b \quad C_bD_b \quad C_bD_b^2 \quad \cdots]. \tag{54}$$

It is then seen that $\tilde{\Delta}\mathbf{T}_2\mathbf{e}_1$ is equal to the solution of (30) and $\mathbf{e}_1^T\mathbf{T}_1^T\tilde{\Gamma}$ is equal to the solution of (31). □

Proof of Proposition 7. The derivation of this result is completely dual to the derivation of the equations in algorithm 2, except for the $\tilde{D}$ term. That is, in this case we use that $\tilde{\mathbf{H}}=\mathbf{T}_1\mathbf{H}\mathbf{T}_2^T$. A full-rank decomposition of $\tilde{\mathbf{H}}$ can then be obtained from a full-rank decomposition of $\mathbf{H}$:

$$\mathbf{H}=\Gamma\Delta \to \tilde{\mathbf{H}}=(\mathbf{T}_1\Gamma)(\Delta\mathbf{T}_2^T)=\tilde{\Gamma}\tilde{\Delta}.$$

The $\tilde{B}$ and $\tilde{C}$ matrices can be extracted as the first block-row and block-column (with block dimension n_b) of $\tilde{\Gamma}$ and $\tilde{\Delta}$, respectively. It then follows that

$$\tilde{C}=\langle V_1(z),(zI-A^T)^{-1}C^T\rangle_M,$$

$$\tilde{B}=\langle (zI-A)^{-1}B, z^{-1}U_0(z)\rangle_M,$$

which corresponds to the Sylvester equations (35) and (36). $\tilde{A}$ is obtained through $\tilde{A}=\Gamma^\dagger\tilde{\mathbf{H}}^{G_b}\Delta^\dagger$, where $\tilde{\mathbf{H}}^{G_b}$ is the matrix $\tilde{\mathbf{H}}$ with its first block-column removed. To compute $\tilde{A}$ we need to know the standard basis equivalent of $\tilde{\mathbf{H}}^{G_b}$ which is denoted by $\mathbf{H}^{G_b}$. Then with $\mathbf{T}_1\mathbf{H}^{G_b}\mathbf{T}_2^T=\tilde{\mathbf{H}}^{G_b}$ and $\tilde{\Gamma}^\dagger=\Gamma^\dagger\mathbf{T}_1^T$ and $\tilde{\Delta}^\dagger=\mathbf{T}_2\Delta^\dagger$ it follows that

$$\tilde{A}=\Gamma^\dagger\mathbf{H}^{G_b}\Delta^\dagger.$$

Formally, $\tilde{\mathbf{H}}^{G_b}$ is the Hankel matrix associated with the transfer function $\lambda\tilde{G}(\lambda)$. By Eq. (14) we know that this corresponds, in the z-domain, to the strictly proper part of $(G_b(z))^{-1}G(z)=G_b(1/z)G(z)$ (hence the notation $\tilde{\mathbf{H}}^{G_b}$). Similar to the derivation of the realizations in Lemma 5 we can derive, on the basis of a state-space realization (A,B,C,D) of $G(z)$ the following possible realizations of the proper part of $G_b(1/z)G(z)$.

(1) $(A,\ B,\ D_bC+C_bX_{\tilde{C}}A,\ D_bD+C_bX_{\tilde{C}}B)$,
(2) $(A,\ BD_b+AX_{\tilde{B}}B_b,\ C,\ DD_b+CX_{\tilde{B}}B_b)$

with $X_{\tilde{B}}$ and $X_{\tilde{C}}$ defined as in Proposition 7. This then gives us implicit full-rank decompositions of $\mathbf{H}^{G_b}$. More precisely, with Γ_1, Δ_1, Γ_2 and Δ_2 defined in the obvious manner, as in the derivation of Algorithm 2, we then have that

$$\tilde{A}=\Gamma^\dagger\Gamma_1\Delta_1\Delta^\dagger=\Gamma^\dagger\Gamma_1=X_o^{-1}\Gamma^T\Gamma_1, \tag{55}$$

$$\tilde{A}=\Gamma^\dagger\Gamma_2\Delta_2\Delta^\dagger=\Delta_2\Delta^\dagger=\Delta_2\Delta^TX_c^{-1}, \tag{56}$$

with X_o and X_c the observability and controllability matrices associated with (A,B,C). It is straightforward to see that $\Gamma^T\Gamma_1=X_{\tilde{A}}^o$ and $\Delta_2\Delta^T=X_{\tilde{A}}^c$, with $X_{\tilde{A}}^o$ and $X_{\tilde{A}}^c$ the solutions to the given Sylvester equations. For the derivation of $\tilde{D}$ we observe that it follows directly from Eq. (13) that $\tilde{D}=\sum_{k=0}^\infty g_kA_b^k=DI+\sum_{k=1}^\infty g_kA_b^k$. Pre-multiplication with A_b and post-multiplication with A_b^T, and using that $A_bA_b^T=I-B_bB_b^T$ (follows from (4)) yields

$$A_b(\tilde{D}-DI)A_b^T=\tilde{D}-DI-A_b\sum_{k=1}A_b^{k-1}B_bCA^{k-1}BB_b^T,$$

which is equal to $\tilde{D}-DI-A_bX_{\tilde{C}}BB_b^T$. Note that a dual expression involving $X_{\tilde{B}}$ can be derived in the same manner. Also, note that the expressions for $\tilde{A}$ and $\tilde{D}$ have a remarkable symmetry. In fact, with (A,B,C) minimal, the Sylvester equation for $\tilde{A}$ can also be written as $\tilde{A}=\sum_{k=0}^\infty g_{b_k}A^k$, where g_{b_k} represent the pulse response parameters of $G_b(z)$. □

Proof of Corollary 8. Since $L_k=\langle V_k(z),G(z)\rangle_M$ we can also write

$$\begin{bmatrix} L_1\\ L_2\\ L_3\vdots\end{bmatrix}=\mathbf{T}_1\begin{bmatrix} C\\ CA\\ CA^2\\ \vdots\end{bmatrix}B=\mathbf{T}_1\Gamma B=\tilde{\Gamma}B=\begin{bmatrix}\tilde{C}\\ \tilde{C}\tilde{A}\\ \tilde{C}\tilde{A}^2\\ \vdots\end{bmatrix}B$$

with $\Gamma,\tilde{\Gamma}$ and $(\tilde{A},\tilde{C})$ as defined in (the proof of) Proposition 7. □

Proof of Lemma 9. *Sufficiency*: Suppose that $N>2$ and that condition (42) is satisfied for some n. Then the classical minimal partial realization criterion of Tether implies that application of the Ho–Kalman algorithm to $\hat{\mathbf{H}}_{N_1,N_2}$ yields the unique (modulo similarity transformation) minimal realization $(\tilde{A},[X_1|X_2|X_3],\tilde{C})$ with McMillan degree n, such that $[M_k|L_k|L_{k+1}]=\tilde{C}\tilde{A}^{k-1}[X_1|X_2|X_3]$ for $k=1,\dots,N-1$. It therefore holds that

$$\tilde{\Gamma}_{N-2}\tilde{A}X_2=\tilde{\Gamma}_{N-2}X_3 \tag{57}$$

with $\tilde{\Gamma}_k$ defined as $\tilde{\Gamma}_k=[\tilde{C}^T \ \tilde{A}^T\tilde{C} \ \cdots \ \tilde{C}^T\tilde{A}^{T^{k-1}}]^T$. By construction, $\tilde{\Gamma}_{N_1}$ is a full-rank factor of $\hat{\mathbf{H}}_{N_1,N-2}$. Because $N_1\leqslant N-2$ it follows that $\tilde{\Gamma}_{N-2}$ is full column rank. Hence (57) implies that $\tilde{A}X_2=X_3$. Consequently, it holds that $L_k=\tilde{C}\tilde{A}^{k-1}X_2$ for $k=1,\dots,N$. We now show that the sequence $\hat{M}_k=\tilde{C}\tilde{A}^{k-1}X_1$, $k\in\mathbb{N}$ corresponds to the coefficient sequence $\hat{L}_k=\tilde{C}\tilde{A}^{k-1}X_2$, $k\in\mathbb{N}$ through Eq. (15). We know that this is true by construction for $k=1,\dots,N-1$. Because $\{\hat{M}_k\}$ and $\{\hat{L}_k\}$ share the state transition matrix $\tilde{A}$ there exists a sequence α_i, $i = 1, \ldots, N-1$ such that

$$\hat{M}_{N+p} = \sum_{i=1}^{N-1} \alpha_i \hat{M}_{i+p}, \quad \hat{L}_{N+p} = \sum_{i=1}^{N-1} \alpha_i \hat{L}_{i+p} \tag{58}$$

for $p \in \mathbb{N}_0$. Let us denote by $\{\hat{\hat{M}}_k\}$ the parameters obtained from $\{\hat{L}_k\}$, for $k \in \mathbb{N}$, through Eq. (15). It is not difficult to see that it follows from Eq. (58) that $\hat{\hat{M}}_k = \sum_{i=1}^{N-1} \alpha_i \hat{\hat{M}}_{i+p}$, for $p \geqslant 0$. Hence we find that $\hat{M}_k = \hat{\hat{M}}_k$ for $k \geqslant N$ as well. We now have that the sequence $M_k = \tilde{C}\tilde{A}^{k-1} X_1$ corresponds to the sequence $L_k = \tilde{C}\tilde{A}^{k-1} X_2$ via relation (15). Hence, if $\tilde{A}$ is stable then the sequence $\{M_k\}_{k \in [0,\infty)}$ represents a valid Hambo transform. Hence with $\tilde{B} = X_1$ and $B = X_2$, (a) and (b) are satisfied. It remains to be seen that $(\tilde{A}, \tilde{B}, \tilde{C})$ is the minimal and unique (modulo similarity) realization for which (a) and (b) hold true. Suppose that there exists another realization $(\hat{\tilde{A}}, \hat{\tilde{B}}, \hat{\tilde{C}})$ with $\dim \hat{\tilde{A}} = \hat{n} \leqslant n$ that is not similar to $(\tilde{A}, \tilde{B}, \tilde{C})$ and an $\hat{n} \times 1$ vector $\hat{B}$ for which (a) and (b) hold. This would imply that $K_k = \hat{\tilde{C}}\hat{\tilde{A}}^{k-1} [\hat{\tilde{B}} | \hat{B} | \hat{\tilde{A}}\hat{B}]$ for $k = 1, \ldots, N-1$ which contradicts the classical Tether condition.

Necessity: Consider any realization $(\tilde{A}, \tilde{B}, \tilde{C})$ and vector B that satisfy (a) and (b). Then the Hankel matrices $\hat{\hat{\mathbf{H}}}_{N_1+i, N_2+j}$, with $i, j > 0$ built from the parameters $\hat{K}_k = [\hat{M}_k | \hat{L}_k | \hat{L}_{k+1}]$ have rank equal to rank $\hat{\hat{\mathbf{H}}}_{N_1, N_2} = n$. □

Proof of Proposition 10. This is an immediate consequence of Lemma 9, since the stability of $\tilde{A}$ assures that the system with realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is a valid Hambo transform. □

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