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Direct and Indirect Continuous-Time Identification in Dynamic Networks

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***Abstract*— Many systems can be modelled as networks of interconnected continuous-time transfer functions. The coefficients of the transfer functions in the network are often directly related to physical properties of the system. Thus, the question we consider is: under what conditions is it possible to obtain a consistent estimate of a continuous-time transfer function embedded in a dynamic network? We consider both direct and indirect continuous-time approaches. We show that a discrete-time model of a continuous-time data generating system may have a different interconnection structure (due to aliasing) and may have algebraic loops (due to the intersample behaviour). Subsequently we present an instrumental variable based method to directly identify a continuous-time transfer function embedded in a network.**

I. INTRODUCTION

Many physical systems are naturally modeled as continuous-time dynamic networks. Moreover, the continuous-time transfer functions are often directly related to physical properties of the system under investigation, such as resistances, capacitances, permeabilities of materials, diameters of pipes, etc. Thus, by identifying a continuous-time transfer function embedded in the continuous-time dynamic network, estimates of the physical properties of the system are obtained. In this paper we consider the question: under what conditions can a consistent estimate of a continuous-time transfer function be obtained when it is embedded in a dynamic network? The underlying objective is to obtain estimates of physical parameters of a physical system. The paper is split into two parts. First an *indirect* continuous-time identification approach is investigated, and secondly a *direct* continuous-time identification approach is investigated. Throughout the paper we consider a general framework for dynamic networks that are made up of measured variables, process noise, sensor noise and known external variables.

In the first part of this paper the relationship between the continuous-time and discrete-time representations of a dynamic network is addressed. If a transfer function has the same frequency response in both representations (for -$\omega_s/2 < \omega < \omega_s/2$ where ω_s is the sampling frequency), then a discrete-time estimate of the transfer function can be converted into a continuous-time transfer function from which the physical parameters of the system can be estimated. This is referred to as the *indirect continuous-time identification approach* [1]. Two effects of discretization are investigated in this paper. The effect of discretization on the interconnection structure of the discrete-time representation, and the effect of discretization on the presence of algebraic loops in the discrete-time representation. If the discrete-time network has a different interconnection structure than the continuous-time network, this can lead to an unexpected bias when identifying particular transfer functions embedded in the network. Knowledge (or assumptions) on the presence of algebraic loops is essential when choosing an appropriate identification method, since some methods (such as the Direct Method and Joint IO methods [2]) will result in biased estimates if there are algebraic loops present.

In the second part of this paper, we consider directly identifying a continuous-time model (thus skipping the intermediate step of first identifying a discrete-time model). There are several advantages to such an approach including dealing with non-uniformly sampled data, less sensitivity to high sampling rates, and identifying stiff systems [1], [3]. It is in part due to these advantages that *direct continuous-time identification* is becoming a more prominent topic in the identification literature (see for instance [1], [4], [5], [6]).

The paper proceeds as follows. In Section II we define a continuous-time dynamic network model, then in Section III we discuss the indirect continuous-time identification approach applied to dynamic networks, and in Section IV we present an instrumental variable method for direct continuous-time dynamic network identification.

II. CONTINUOUS-TIME DYNAMIC NETWORKS

In this section a continuous-time dynamic network model is briefly presented. The framework is a continuous-time version of that presented in [2]. Each *internal variable* is assumed to be such that it can be written as:

$$w_j(t) = \sum_{k\in\mathcal{N}_j} G_{jk}(p)w_k(t) + r_j(t) + v_j(t)$$

with $G_{jk}(s)$ a continuous-time transfer function, p is the differential operator, i.e. $pu(t) = \frac{\mathrm{d}}{\mathrm{d}t}u(t)$, and

- $\mathcal{N}_j$ is the set of indices of internal variables with direct causal connections to w_j, i.e. $i \in \mathcal{N}_j$ if $G^0_{ji} \neq 0$
- the r's denote the variables that are driven by a known source and are referred to as *external variables*
- and the v's denote *process noise*, which includes any unmodeled disturbances that affect the system.

The process noise is modeled as a continuous-time stochastic process with rational power spectral density. It may be that

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the process noise and/or external excitation term is not present at some nodes. All the internal variables can be written in one equation as:

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G_{12}^0 & \cdots & G_{1L}^0 \\ G_{21}^0 & 0 & \ddots & G_{2L}^0 \\ \vdots & \ddots & \ddots & \vdots \\ G_{L1}^0 & G_{L2}^0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_L \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_L \end{bmatrix}$$
$$= G^0(p)w(t) + r(t) + v(t). \quad (1)$$

Equation (1) defines a dynamic network. The internal variables can also be expressed in an input/output form:

$$w(t) = F^0(p)(r(t) + v(t)) \quad (2)$$

where the internal variables are expressed only in terms of external and noise variables and $F^0 = (I - G^0)^{-1}$.

We suppose that measurements of the internal variables are obtained using *analog to digital converters* (ADC)s. An analog to digital converter consists of four components: a sensor, an anti-aliasing filter, a sampler, and a digitizer [7]. Adopting the notation of [7], the discrete-time version of a continuous-time variable will be denoted with a superscript $\star$. We assume that every ADC measures with some error which we call *sensor noise*. The (discrete-time) output of a ADC is then denoted:

$$\tilde{w}_k^\star(n) = w_k^{aa}(nT) + s_k(nT), \quad n = 0, 1, 2, \ldots \quad (3)$$

where n is used to denote discrete-time, T denotes the sampling period, $w_k^{aa}(t) = A_k(p)w_k(t)$ where $A_k(s)$ is the transfer function of the anti-aliasing filter, and s_k is the sensor noise. A superscript $\star$ denotes a discrete-time signal, and $\tilde{w}_k$ denotes that w_k is measured with (sensor) noise. If anti-aliasing filters are not present, then $G_{aa}(s) = 1$. The sensor noise $s_k(nT)$, $n = 0, 1, 2, \ldots$, is modeled as a discrete-time stochastic process with rational power spectral density.[1] Equations (1) and (3) define the continuous-time data generating system. Throughout the paper we assume the dynamic network (1) is *well-posed* in the sense that all principal minors of $(I - G)$ are non-zero and *stable* in the sense that $(I - G)^{-1}$ is stable.

External variables, r_k, may be digital signals that have been converted into analog signals by a *digital to analog converter* (DAC). A DAC is equipped with a hold circuit which determines the intersample behaviour of the output of the DAC. For instance, some DACs are equipped with *zero-order-hold* circuits, which means that the output of the DAC is held constant in between sampling instants [7]. Mathematically, the continuous-time signal $r_k(t)$ that is the output of a DAC is:

$$r_k(t) = \sum_{n=0}^{\infty} r_k^\star(n) g_k^{\text{hold}}(t - nT) \quad (4)$$

where $g_k^{\text{hold}}(t)$ is the impulse response of the hold circuit of the DAC. For a DAC equipped with a zero-order-hold, $g_k^{\text{hold}}(t) = \text{rect}(t)$, where $\text{rect}(t)$ is a rectangular pulse of length T. Note, however, we do not neccesarily assume that all external variables are generated by DACs.

[1] A sampled sequence of a continuous-time stochastic process can be well approximated in this way [6].

III. Indirect Continuous-Time Identification

In the indirect approach, first a discrete-time transfer function is estimated from the sampled data. Then a continuous-time transfer function is constructed based on the discrete-time transfer function. In order to be able to (easily) obtain an estimate of the continuous-time transfer function $G_{ji}^0(s)$ from the transfer function $G_{ji}^0(z)$ estimated from the (sampled) data, it is essential that $G_{ji}(z)$ is an accurate representation of the dynamics of $G_{ji}^0(s)$. Thus, it is essential to understand the effect of discretizing (1). We address two effects: first the effect of aliasing on the interconnection structure of the discrete-time representation of the continuous-time data generating system, and secondly the effect of the intersample behavior on the presence of direct feed-though terms in the discrete-time dynamic network. Both these effects are important when attempting to identify a transfer function embedded in a dynamic network using the indirect continuous-time identification approach.

A. Effect of Discretization on the Interconnection Structure

In the following proposition the relationship between a discrete-time dynamic network and its continuous-time counterpart is presented.

Proposition 1: Consider a dynamic network with no process noise present, and external excitation present at every node. Assume that each external variable is a discrete-time signal that has been converted into a continuous-time signal by a DAC. The discrete-time representations of the data set in input/output form and network form respectively are:

$$w^\star(n) = F^\star(q)r^\star(n), \text{ and} \quad (5)$$
$$w^\star(n) = G^\star(q)w^\star(n) + D^\star(q)r^\star(n) \text{ with} \quad (6)$$
$$\tilde{w}^\star(n) = w^\star(n) + s^\star(n)$$

where $F^\star(q)$, $G^\star(q)$, and $D^\star(q)$ are matrices of discrete-time transfer functions and $s^\star(n)$ denotes sensor noise. The discrete-time transfer functions $F^\star$ and $G^\star$ are related to their continuous-time counterparts as defined in (1) and (2) by the following equations:

$$F^\star(z) = \mathcal{Z}\Big[\Big(\mathcal{L}^{-1}\big[A(s)F(s)G_{\text{hold}}(s)\big]\Big)^\star\Big], \quad (7)$$
$$G^\star(z) = I - D^\star(z)F^{\star^{-1}}(z) \quad (8)$$

where $\mathcal{Z}[\,\cdot\,]$ and $\mathcal{L}[\,\cdot\,]$ denote the Z and Laplace transforms respectively, $A(s)$ is a diagonal matrix with the (k, k)th element equal to the anti-aliasing filter of the kth ADC, $G_{\text{hold}}(s)$ is a diagonal matrix with the (k, k)th element equal to the transfer function of the hold circuit of the kth DAC, $D^\star$ is a diagonal matrix of the diagonal entries of $F^\star$. □

The proof can be found in the Appendix. The main point of Proposition 1 is that $G^\star(z)$ is not constructed simply by discretizing each element of $G(s)$. In fact, a zero element in $G(s)$ is not neccessarily zero in $G^\star(z)$. Thus, the discrete-time dynamic network may have a different interconnection structure (there may be extra links present) than the corresponding continuous-time network.

The reason for this difference is due to the effect of *aliasing*. A feature of the continuous-time to discrete-time transformation by sampling and digitizing, i.e. the operation $\mathcal{Z}[(\mathcal{L}^{-1}[\ \cdot\ ])^\star]$ is that a product (or quotient) of two continuous-time transfer functions (say $B(s)$ and $C(s)$) cannot be separated/factored in the discrete-time domain, i.e.

$$\mathcal{Z}\big[\big(\mathcal{L}^{-1}[B(s)C(s)]\big)^\star\big] \neq \mathcal{Z}\big[\big(\mathcal{L}^{-1}[B(s)]\big)^\star\big]\,\mathcal{Z}\big[\big(\mathcal{L}^{-1}[C(s)]\big)^\star\big]. \quad (9)$$

The reason is due to aliasing. The effect of aliasing is easier to see in frequency domain [6]:

$$\begin{aligned}\mathcal{F}\big[\big(\mathcal{L}^{-1}[B(s)C(s)]\big)^\star\big] &= \Big(B(j\omega)C(j\omega)\Big) * \Big(\frac{1}{T}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_s)\Big)\\ &= B(j\omega)C(j\omega) + \underbrace{\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} B(j\omega - j\omega_s n)C(j\omega - j\omega_s n)}_{\text{component due to aliasing}} \qquad (10)\end{aligned}$$

where $\mathcal{F}[\ \cdot\ ]$ is the Fourier Transform and ω_s is the sampling frequency. From (10) if $B(j\omega)C(j\omega)$ is non-zero for frequencies greater than $\omega_s/2$ then the component due to aliasing will be non-zero in the range $-\omega_s/2 < \omega < \omega_s/2$.

The matrix $F(s) = (I - G(s))^{-1}$ contains many products of transfer functions. Thus, when performing the inverse calculation in discrete-time, i.e. obtaining $G^\star(z)$ from $F^\star(z)$ by (8), there will be no cancellations of transfer functions resulting in zero entries in $G^\star(z)$. The reasoning is illustrated in the following example.

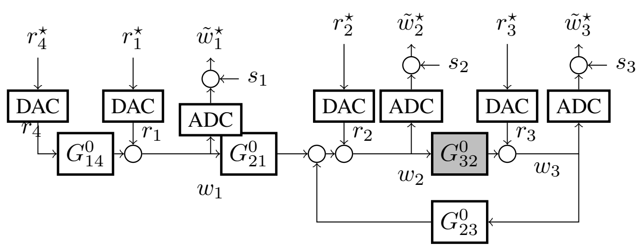

Fig. 1. Closed loop data generating system with no process noise and continuous-time external variables that are constructed from discrete-time external variables (using DACs).

Example 1: Consider the system shown in Fig. 1. Suppose that the ADCs are not equipped with anti-aliasing filters ($A(s) = I$), and all the DACs are equipped with zero-order-hold circuits ($G_{\text{hold}}(s) = \text{diag}[\text{Rect}(s), \ldots, \text{Rect}(s)]$). The continuous-time dynamic network is

$$\begin{bmatrix} w_1(t)\\ w_2(t)\\ w_3(t)\\ w_4(t)\end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & G_{14}(p)\\ G_{21}(p) & 0 & G_{23}(p) & 0\\ 0 & G_{32}(p) & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} w_1(t)\\ w_2(t)\\ w_3(t)\\ w_4(t)\end{bmatrix} + \begin{bmatrix} r_1(t)\\ r_2(t)\\ r_3(t)\\ r_4(t)\end{bmatrix}$$

By Proposition 1 discrete-time representation of the system:

$$\begin{bmatrix} w_1^\star\\ w_2^\star\\ w_3^\star\\ w_4^\star\end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & G_{14}(q)\\ G_{21}(q) & 0 & G_{23}(q) & G_{24}(q)\\ G_{31}(q) & G_{32}(q) & 0 & G_{34}(q)\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} w_1^\star\\ w_2^\star\\ w_3^\star\\ w_4^\star\end{bmatrix} + \begin{bmatrix} r_1^\star\\ D_2(q)r_2^\star\\ D_3(q)r_3^\star\\ r_4^\star\end{bmatrix}$$

where the argument n has beeen dropped. Note that the resulting interconnection structure of the discrete-time network is not the same as that of the continuous-time network. The expression for the transfer function $G_{23}(z)$ is equal to

$$G_{23}(z) = \frac{\mathcal{Z}\Big[\big(\mathcal{L}^{-1}[G_{23}(s)\text{Rect}(s)S(s)]\big)^\star\Big]}{\mathcal{Z}\Big[\big(\mathcal{L}^{-1}[S(s)]\big)^\star\Big]}$$

where $S(s) = (\frac{1}{1-G_{23}(s)G_{32}(s)})\text{Rect}(s)$. If (9) were to hold with equality, then the product in the numerator could be "split" resulting in a cancellation with the denominator, and $G_{23}(z)$ would be the discrete-time version of $G_{23}(s)$. $\square$

There are two cases when the elements of $G^\star(z)$ are (approximately) the discrete versions of the corresponding elements of $G(s)$: when the sampling rate is fast enough such that the system does not have significant frequency content at frequencies greater than half the sampling frequency and/or when anti-aliasing filters are used to take the measurements.

First consider the effect of a fast sampling rate. Suppose that all the dynamics in the network are low-pass. By increasing the sampling rate, the aliasing terms in (10) become smaller with the result that

$$\mathcal{Z}\big[\big(\mathcal{L}^{-1}[B(s)C(s)]\big)^\star\big] \approx \mathcal{Z}\big[\big(\mathcal{L}^{-1}[B(s)]\big)^\star\big]\,\mathcal{Z}\big[\big(\mathcal{L}^{-1}[C(s)]\big)^\star\big]. \quad (11)$$

Since all the products of transfer functions can be split, the operations to obtain $G^\star(z)$ from $F^\star(z)$ "undo" the operations in (2) where $F(s)$ is obtained from $G(s)$.

If the external variables and the process noise variables only have power in the frequency range $[-\omega_s/2, \omega_s/2]$ then the dynamics in (10) that result in aliasing are not excited. Thus, the approximation (11) can be quite good (leading to the same result as above). Note that this means ADCs equipped with zero-order-holds cannot be used (the rect(t) function has frequency content everywhere).

A second alternative is that anti-aliasing filters are used to take the measurements. Thus, all frequencies outside of $[-\omega_s/2, \omega_s/2]$ in (10) are filtered out. The result is that

$$\begin{aligned}&\mathcal{Z}\big[\big(\mathcal{L}^{-1}[A_k(s)B(s)C(s)]\big)^\star\big] \approx\\ &\qquad \mathcal{Z}\big[\big(\mathcal{L}^{-1}[A_k(s)B(s)]\big)^\star\big]\cdot\mathcal{Z}\big[\big(\mathcal{L}^{-1}[A_k(s)C(s)]\big)^\star\big]\end{aligned}$$

where $A_k(s)$ is the transfer function of the anti-aliasing filter. From (7) and (8), with this approach the dynamics of the anti-aliasing filters become part of the equations.

In this section we have illustrated that if there is aliasing occuring then the interconnection structure of the discrete-time dynamic network is not the same as that of the corresponding continuous-time data generating system. This is a side-effect of aliasing that is unique to dynamic networks.

From the perspective of identification in dynamic networks this can have unexpected consequences. Suppose that anti-aliasing filters are not used to take the measurements. Then, although the interconnection structure of the continuous-time data generating system is known, this interconnection structure is not the same as that of the equivalent discrete-time dynamic network. Thus, if the interconnection structure of the continuous-time data generating system is imposed on the discrete-time representation, a bias will result. This bias is in addition to the usual bias that results from aliasing.

B. Effect of Discretization on the Presence of Delays

We show that the inter-sample behavior of the external and noise variables affects the presence of direct feed-through terms in the discrete-time representation of the continuous-time system. The presence of direct-feedthrough terms (and by extension algebraic loops) is an important feature of a discrete-time dynamic network when selecting an appropriate method to identify a module embedded in the network.

The main reasoning is based on a result from [8]. Consider the following equation:

$$w_k(t) = F_{kk}(p)r_k(t), \tag{12}$$

with the following discrete-time representation:

$$w_k^\star(n) = F_{kk}^\star(q)r_k^\star(n). \tag{13}$$

Consider the direct feed-through term of $F_{kk}^\star(z)$, i.e. $f_{kk}^\star(0)$. The result of [8] says that the whether $f_{kk}^\star(0)$ is zero or not depends (in part) on the inter-sample behavior of the continuous-time variable $r_k(t)$ in (12). Formally, consider the following proposition is taken from [8].

Proposition 2: Consider a continuous-time causal SISO system of (12). Consider the discrete-time representation of the system, (13). The direct feed-through term $f_{kk}^\star(0)$ is dependent on the inter-sample behavior of $r(t)$:

- For a piece-wise constant (zero-order-hold) $r(t)$,
$$f_{kk}^\star(0) = f_{kk}(0).$$
- For a band-limited $r(t)$ (i.e. $r(t)$ has no power above the frequency $w_s = \frac{\pi}{T}$),
$$f_{kk}^\star(0) = \int_0^T \operatorname{sinc}\Big(\frac{\pi\tau}{T}\Big) f_{kk}(\tau)\mathrm{d}\tau.$$

For a proof see [8]. By Proposition 2, if every internal variable in the network is exclusively driven by an external variable that is generated by a DAC equipped with a zero-order-hold, then the delay structure of $F^\star(z)$ is the same as that of $F(s)$ (and consequently if there is no aliasing, generically the delay structure of $G^\star(z)$ is the same as that of $G(s)$). However, in a realistic continuous-time data generating system, there will be continuous-time noise variables $v_k(t)$ that excite the system (that are not piece-wise constant). Thus, by Proposition 2 many of the non-zero entries of the matrix $F^\star(z)$ will have direct feed-through terms. Since $F^\star = (I - G^\star)^{-1}$, it follows that the discrete-time representation may have algebraic loops.

Recall that the objective adressed in this paper is to identify a transfer function embedded in a continuous-time dynamic network. In the literature, several methods have been proposed to identify discrete-time transfer functions embedded in a discrete-time network [2]. For instance the Direct, Joint IO, Two-Stage, and Instrumental Variable (IV) Methods. To apply these methods the interconnection structure of the network must be known. In order to obtain consistent estimates using the Direct and Joint IO methods, there must not be any algebraic loops involving the output of the transfer function of interest [2].

Thus, by Propositions 1 and 2 the indirect continuous-time identification approach will work well if (1) there is no aliasing occurring and (2) the Two-Stage Method or an IV based method are used to identify the module embedded in the discrete-time dynamic network (that represents a continuous-time dynamic network). In this situation the interconnection structure of the discrete-time dynamic network is known (since it matches the known interconnection structure of the continuous-time network) and even though there may be algebraic loops present in the network, by using the Two-Stage or IV Method these do not pose a problem.

IV. Direct Continuous-Time Identification

In this section we present a method to directly identify a continuous-time transfer function (denoted $G_{ji}^0(s)$) embedded in a dynamic network. We extend the Basic CLIVC method of [9], [10] such that it can be used (a) in the presence of sensor noise, (b) for identification in interconnection structures more complex than a closed loop. In [11] a frequency domain approach is presented to identify continuous-time closed-loop systems where they also consider the presence of sensor noise, but there is still a requirement that a (sensor noise free) periodic, external reference signal is present. The method we develop in this section is the continuous-time counterpart to that presented in [12]

First, a set of internal variables must be chosen to form the data set. The internal variable w_j is chosen as the 'output'. Choose the set of 'inputs' as all internal variables that have a direct causal connection to w_j (i.e. choose w_k, $k \in \mathcal{N}_j$ as inputs). The transfer functions to be identified are parameterized as rational functions in p:

$$G_{jk}(p,\theta) = \frac{B_{jk}(p,\theta)}{A_{jk}(p,\theta)}, \tag{14}$$

for all $k \in \mathcal{N}_j$, where B_{jk} and A_{jk} are polynomials in p:

$$B_{jk}(p,\theta) = b_0^{jk}p^{n_b} + b_1^{jk}p^{n_b-1} + \cdots + b_{n_b}^{jk},$$
$$A_{jk}(p,\theta) = p^{n_a} + a_1^{jk}p^{n_a-1} + \cdots + a_{n_a}^{jk}.$$

where the parameters are a_n^{jk}, $n = 1,\ldots,n_a$, $k \in \mathcal{N}_j$ and b_n^{jk}, $n = 0,\ldots,n_b$, $k \in \mathcal{N}_j$, i.e.

$$\theta = [a_1^j \;\cdots\; a_{n_a}^j \; b_0^{jk_1} \;\cdots\; b_{n_b}^{jk_1} \;\cdots\; b_0^{jk_d} \;\cdots\; b_{n_b}^{jk_d}]^T$$

where $\{k_1,\ldots,k_d\} = \mathcal{N}_j$. For notational convenience, we assume all polynomials $B_{jk}(\theta)$ and $A_{jk}(\theta)$, $k \in \mathcal{N}_j$ have the same orders, denoted n_b, and n_a respectively. The internal variable w_j can be expressed as

$$\begin{aligned} w_j(t) &= \sum_{k\in\mathcal{N}_j} G_{jk}(p)^0 w_k(t) + v_j(t) \\ &= \frac{1}{\breve{A}_j^0(p)} \sum_{k\in\mathcal{N}_j} \breve{B}_{jk}^0(p) w_k(t) + v_j(t) \end{aligned} \tag{15}$$

where

$$\breve{A}_j(p) = \prod_{n\in\mathcal{N}_j} A_{jn}(p) \text{ and } \breve{B}_{jk}(p) = \prod_{n\in\mathcal{N}_j\backslash k} B_{jk}(p)A_{jn}(p).$$

Consequently, the following differential equation holds:

$$\breve{A}_j^0(p)w_j(t) = \sum_{k\in\mathcal{N}_j} \breve{B}_{jk}^0(p)w_k(t) + \breve{A}_j^0(p)v_j(t). \qquad (16)$$

Moreover, (16) holds at each time instant $t = 0, T, 2T, \ldots$. Suppose that the ADCs are not equipped with anti-aliasing filters ($G_{aa} = 1$). Then, the differential equation relating the measured values of the internal variables $\tilde{w}_j$ and $\tilde{w}_k$, $k \in \mathcal{N}_j$ at times $t = 0, T, 2T, \ldots$ is

$$\begin{aligned}\breve{A}_j^0(p)\tilde{w}_j(t_n) &= \sum_{k\in\mathcal{N}_j} \breve{B}_{jk}^0(p)\big(\tilde{w}_k(t_n) - s_k(t_n)\big) \\ &\quad + \breve{A}_j^0(p)\big(v_j(t_n) + s_j(t_n)\big) \\ &= \sum_{k\in\mathcal{N}_j} \breve{B}_{jk}^0(p)\tilde{w}_k(t_n) + \breve{v}_j(t_n) \qquad (17)\end{aligned}$$

where t_n denotes the time instant $t = nT$, $n = 0, 1, \ldots$ and

$$\breve{v}_j(t_n) = \sum_{k\in\mathcal{N}_j} -\breve{B}_{jk}(p)s_k(t_n) + \breve{A}_j(p)\big(v_j(t_n) + s_j(t_n)\big). \quad (18)$$

The notation in (17) and (18) is adopted from [9], however, it is sloppy because the continuous-time operator p cannot be applied to a discrete-time sequence. Expression (17) can be rearranged collecting all coefficients on the right-hand side:

$$\begin{aligned}\tilde{w}_j^{(n_a)}(t_n) &= \sum_{k\in\mathcal{N}_j} \breve{B}_{jk}(p)\tilde{w}_k(t_n) + \big(1 - \breve{A}_j(p)\big)\tilde{w}_j(t_n) + \breve{v}_j(t_n) \\ &= \tilde{\phi}_j^T(t_n)\breve{\theta}^0 + \breve{v}_j(t_n) \qquad (19)\end{aligned}$$

where superscript (m) denotes the mth order derivative, $\breve{\theta}^0$ denotes the coefficients of the data generating system,

$$\breve{\theta}^0 = [\breve{a}_1^{j^0} \ \cdots \ \breve{a}_{\breve{n}_a}^{j^0} \ \breve{b}_0^{jk_1{}^0} \ \cdots \ \breve{b}_{\breve{n}_b}^{jk_1{}^0} \ \cdots \ \breve{b}_0^{jk_d{}^0} \ \cdots \ \breve{b}_{\breve{n}_b}^{jk_d{}^0}]^T$$

where $\{k_1, \ldots, k_d\} = \mathcal{N}_j$, and

$$\begin{aligned}\tilde{\phi}_j^T(t_n) = \big[&-\tilde{w}_j^{(n_a-1)}(t_n) \ \cdots \ -\tilde{w}_j(t_n) \ \tilde{w}_{k_1}^{(n_b)}(t_n) \\ &\cdots \ \tilde{w}_{k_1}(t_n) \cdots \ \tilde{w}_{k_d}^{(n_b)}(t_n) \ \cdots \ \tilde{w}_{k_d}(t_n)\big] \quad (20)\end{aligned}$$

where $\{k_1, \ldots, k_d\} = \mathcal{N}_j$. If $\breve{\theta}$ is known, it is possible to calculate the value of θ in (14). The IV estimate of $\breve{\theta}^0$ is

$$\hat{\theta}_{IV} = \text{sol}\left\{\frac{1}{N}\sum_{n=0}^{N-1} Z(t_n)\big(\tilde{w}_j^{(n_a)}(t_n) - \tilde{\phi}_j^T(t_n)\breve{\theta}\big) = 0\right\}, \quad (21)$$

where $Z(t_n) = [z^{(n_z)}(t_n) \ z^{(n_z-1)}(t_n) \ \cdots \ z(t_n)]^T$ and $z(t_n)$ is a vector of instrumental variables. The choice of the instrumental variables is critical with respect to the consistency of the estimates. We consider all external and (measurements of) internal variables, except $\tilde{w}_k$, $k \in \mathcal{N}_j \cup \{j\}$ as potential candidate instrumental variables. Let $\mathcal{I}_j$ and $\mathcal{X}_j$ denote the sets of indices of internal and external variables respectively chosen as instrumental variables. Then,

$$z(t_n) = [r_{\ell_1}(t_n) \ \cdots \ r_{\ell_n}(t_n) \ \tilde{w}_{m_1}(t_n) \ \cdots \ \tilde{w}_{m_n}(t_n)]^T \quad (22)$$

where $\mathcal{X}_j = \{\ell_1, \ldots, \ell_n\}$ and $\mathcal{I}_j = \{m_1, \ldots, m_n\}$.

Proposition 3: Consider a dynamic network as defined in (1). Choose the sets $\mathcal{I}_j$ and $\mathcal{X}_j$ of instrumental variables such that $\mathcal{I}_j \cap \{\mathcal{N}_j \cup \{j\}\} = \emptyset$. Consider the estimate $\hat{\theta}_{IV}$ of (21) where $n_z \geq \lceil \text{length}\big(\tilde{\phi}_j(t)\big)/\text{length}\big(z(t)\big)\rceil$. The estimate $\hat{\theta}_{IV}$ is consistent if the following conditions are satisfied:

(a) If v_j is present, there is no path from w_j to any w_ℓ, $\ell \in \mathcal{I}_j$
(b) The matrix $\bar{\mathbb{E}}\Big[Z(t_n)\tilde{\phi}_j(t_n)]\Big]$ has full column rank.
(c) Each sensor noise s_ℓ, $\ell \in \mathcal{I}_j$ is uncorrelated to all s_k, $k \in \mathcal{N}_j$.
(d) If v_j is present, then it is uncorrelated to all v_m with a path to w_j.
(e) The parameterization is flexible enough, i.e. there exists a θ such that $G_{jk}(s,\theta) = G_{jk}(s)$, $\forall k \in \mathcal{N}_j$. □

The proof follows along the exact same lines as the proof of Proposition 3 in [12]

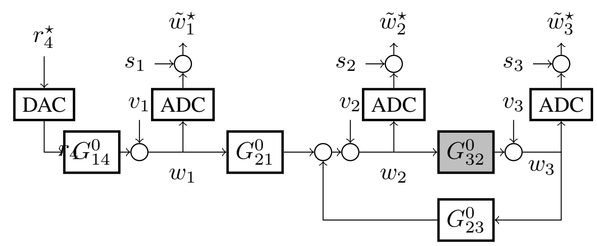

Fig. 2. Closed loop data generating system

Example 2: Consider the data generating system shown in Fig. 2. Suppose that the objective is to obtain a consistent estimate of G_{32}^0. Thus, $\{j\} = \{3\}$, and $\mathcal{N}_3 = \{2\}$. Choose $\tilde{w}_1$ as the instrumental variable ($\mathcal{I}_3 = \{1\}$, and $\mathcal{X}_3 = \emptyset$), then $\{\mathcal{N}_3 \cup \{3\}\} \cap \mathcal{I}_3 = \emptyset$ as required. Since there is no path from w_3 to w_1, Condition (a) of Proposition 3 holds. If the remaining conditions of Proposition 3 hold, then consistent estimates of G_{jk}^0, $k \in \mathcal{N}_j$ are obtained by solving (21). □

In order to calculate $\hat{\theta}_{IV}$ in (21) it is necessary to calculate the time derivatives of w_k $k \in \mathcal{N}_j \cup \{j\}$ and the instrumental variables. The *state-variable filter* approach is to approximate the derivative as [4], [1]

$$w_k^{(m)}(t_n) = p^n w_k(t_n) \approx \frac{p^m}{(p+\lambda)^{m_s}} w_k(t_n) \qquad (23)$$

where $m_s \geq m$. The parameter λ determines the bandwidth of the approximation. Within the bandwidth determined by λ, the filter $\frac{p^m}{(p+\lambda)^{m_s}}$ is equal to the derivative operator. Other methods for approximating the derivative exist (see [4] for instance). For many of the methods the approximation of the derivative improves with increasing sampling rate and low pass dynamics [13], [14]. Thus, it is beneficial to use the fastest sampling rate possible.

V. Conclusion

In this paper we present methods for consistently identifying a continuous-time transfer function embedded in a continuous-time dynamic network. Our framework includes the presence of sensor noise and process noise. In the first part of this paper we investigate the indirect continuous-time identification approach. In the second part of the paper, investigate the direct continuous-time identification apprioach. We present a continuous-time IV method to consistently estimate a transfer function embedded in a dynamic network.

Appendix

Proof of Proposition 1. Because there is an external variable present at each node, and they are all generated by DACs, it follows from (4) that (2) can be expressed as:

$$\begin{bmatrix} w_1(t) \\ \vdots \\ w_L(t) \end{bmatrix} = f(t) * \begin{bmatrix} g_1^{\text{hold}}(t) & & \\ & \ddots & \\ & & g_L^{\text{hold}}(t) \end{bmatrix} * \begin{bmatrix} \sum_{m=0}^{\infty} r_1^\star(m) * \delta(t-mT) \\ \vdots \\ \sum_{m=0}^{\infty} r_L^\star(m) * \delta(t-mT) \end{bmatrix},$$

where $*$ denotes convolution, $\delta(t)$ is an impulse function, and $f(t) = \mathcal{L}^{-1}[F(s)]$. Let $f_c(t) = \mathcal{L}^{-1}[F(s)\text{diag}(G_1^{\text{hold}}(s), \dots, G_L^{\text{hold}}(s)))]$. Then the above equation can be expressed as:

$$\begin{aligned} w(t) &= \int_0^t f_c(t-\tau) \sum_{m=0}^{\infty} r^\star(m) * \delta(\tau - mT) \mathrm{d}\tau \\ &= \sum_{m=0}^{\infty} f_c(t-mT) r^\star(m), \end{aligned} \tag{24}$$

where $w(t) = [w_1(t) \ \cdots \ w_L(t)]^T$ and $r^\star(m) = [r_1^\star(m) \ \cdots \ r_L^\star(m)]^T$. The continuous-time output of the anti-aliasing filters of the ADCs is:

$$\tilde{w}(t) = \sum_{m=0}^{\infty} a(t-mT) * f_c(t-mT) r^\star(m) + s(t) \tag{25}$$

where $a(t)$ is a diagonal matrix with the impulse response of the k anti-aliasing filter at the kth entry, and s is a vector of sensor noise variables. Let $F_{ac}(s) = A(s)F_c(s)$, then by (3) and (25) the discrete-time representation of $\tilde{w}(t)$ is

$$\tilde{w}^\star(n) = \sum_{m=0}^{\infty} f_{ac}^\star(n-m) r^\star(m) + s(n), \tag{26}$$

where $f_{ac}^\star(n) = \big(f_{ac}(t)\big)^\star = \big(\mathcal{L}^{-1}[F_{ac}(s)]\big)^\star$. Interestingly, (26) is a discrete-time convolution, thus it can be expressed:

$$\tilde{w}^\star(n) = F_c^\star(q) r^\star(n) + s(n) \tag{27}$$

where $F_c^\star(q) = \mathcal{Z}[f_c^\star(n)] = \mathcal{Z}\big[\big(\mathcal{L}^{-1}[F_c(s)]\big)^\star\big]$. Equation (27) is of the required form in (5) with

$$F_{ac}^\star(z) = \mathcal{Z}\Big[\Big(\mathcal{L}^{-1}\big[A(s)F(s)G_{\text{hold}}(s)\big]\Big)^\star\Big]$$

which proves (7).

The discrete-time dynamic network equations can be obtained from (27). Consider the following steps:

$$F_c^{\star^{-1}}(q) w^\star(n) = r^\star(n). \tag{28}$$

In order to ensure that $G^\star$ has zeros on the diagonal, first set the main diagonal of the matrix on the left in (28) to all ones by multiplying by $D^\star$, where $D^\star$ is a diagonal matrix of all the diagonal elements of $F_c^\star$:

$$D^\star(q) F_c^{\star^{-1}}(q) w^\star(n) = D^\star(q) r^\star(n). \tag{29}$$

Thus, by construction, $D^\star(q)F_c^{\star^{-1}}(q)$ can be expressed as $I + (D^\star(q)F_c^{\star^{-1}}(q) - I)$ where the second matrix in the sum has zeros on the diagonal. Thus, (29) can be expressed as:

$$w^\star(n) = (I - D^\star(q)F_c^{\star^{-1}}(q)) w^\star(n) + D^\star(q) r^\star(n). \tag{30}$$

Let $G^\star(q) = (I - D^\star(q)F_c^{\star^{-1}}(q))$. Then (30) can be expressed as

$$w^\star(n) = G^\star(q) w^\star(n) + D^\star(q) r^\star(n) \tag{31}$$

which is of the form (6), and (8) is proved.

References

[1] H. Garnier and L. Wang, Eds., *Identification of Continuous-Time Models from Sampled Data*, ser. Advances in Industrial Control. London: Springer-Verlag, 2008.

[2] P. M. J. Van den Hof, A. Dankers, P. S. C. Heuberger, and X. Bombois, "Identification of dynamic models in complex networks with prediction error methods - basic methods for consistent module estimates," *Automatica*, vol. 49, pp. 2994–3006, Oct. 2013.

[3] G. P. Rao and H. Unbehauen, "Identification of continuous-time systems," *Control Theory and Applications, IEE Proceedings -*, vol. 153, no. 2, pp. 185–220, March 2006.

[4] H. Garnier, M. Mensler, and A. Richard, "Continuous-time model identification from sampled data: Implementation issues and performance evaluation," *International Journal of Control*, vol. 76, no. 13, pp. 1337–1357, 2003.

[5] E. K. Larsson, M. Mossberg, and T. Soderstrom, "An overview of important practical aspects of continuous-time arma system identification," *Circuits, Systems and Signal Processing*, vol. 25, no. 1, pp. 17–46, 2006.

[6] R. Pintelon and J. Schoukens, *System Identification, A Frequency Domain Approach*, 2nd ed. Hoboken, New Jersey, USA: IEEE Press, John Wiley and Sons, Inc., 2012.

[7] M. S. Fadali and A. Visioli, *Digital Control Engineering*, 2nd ed. Waltham, MA, USA: Academic Press, 2013.

[8] H. D. J. Laurijsse, "System identification of individual control area dynamics in interconnected power systems," Master's thesis, Eindhoven University of Technology, 2014.

[9] M. Gilson, H. Garnier, P. Young, and P. M. J. Van den Hof, "Optimal instrumental variable methods for closed-loop continuous time model identification," in *Identification of Continuous-Time Models from Sampled Data*, ser. Advances in Industrial Control, H. Garnier and L. Wang, Eds. London: Springer-Verlag, 2008, ch. 5, pp. 133–160.

[10] C. Cheng and J. Wang, "Identification of continuous-time systems in closed-loop operation subject to simple reference changes," in *Proceedings of the International Symposium on Advanced Control of Industrial Processes (ADCONIP)*, Hangzhou, China, May 2011, pp. 363–368.

[11] R. Pintelon, J. Schoukens, and Y. Rolain, "Frequency-domain approach to continuous-time system identification: Some practical aspects," in *Identification of Continuous-Time Models from Sampled Data*, ser. Advances in Industrial Control, H. Garnier and L. Wang, Eds. London: Springer-Verlag, 2008, ch. 8, pp. 215–248.

[12] A. Dankers, P. M. J. Van den Hof, X. Bombois, and P. S. C. Heuberger, "Errors in variables identification in dynamic networks by an instrumental variable approach," in *Proceedings of 19th IFAC World Congress*, Cape Town, South Africa, 2014, accepted.

[13] H. Van Hamme, R. Pintelon, and J. Schoukens, "Identification and discrete time modeling of continuous time systems: a general framework," in *Identification of Continuous-Time Systems: Methodology and Computer Implementation*, N. K. Sinha and G. P. Rao, Eds. Dordrecht, The Netherlands: Kluwer Academic Publishers, 1991, ch. 2, pp. 17–77.

[14] D. Marelli and M. Fu, "A continuous-time linear system identification method for slowly sampled data," *IEEE Transactions on Signal Processing*, vol. 58, no. 5, pp. 2521–2533, May 2010.