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Predictor Input Selection for Direct Identification in Dynamic Networks

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***Abstract*— In the literature methods have been proposed which enable consistent estimates of modules embedded in complex dynamic networks. In this paper the network extension of the so called closed-loop Direct Method is investigated. Currently, for this method the variables which must be included in the predictor model are not considered as a user choice. In this paper it is shown that there is some freedom as to which variables to include in the predictor model as inputs, and still obtain consistent estimates of the module of interest. Conditions on this choice of predictor inputs are presented.**

I. Introduction

Obtaining models of complex dynamic networks from data is becoming an increasingly important area of research. In many fields of science and engineering such as power systems, biological systems, flexible mechanical structures, economic systems, etc., it is becoming possible to collect data at various locations, or of different variables that have dynamic interrelations (i.e. form a dynamic network).

A dynamic network consists of *modules* that are *embedded* according to an *interconnection structure* [1]. There is a significant number of applications where the interconnection structure of the network is a priori known. For example in power systems or flexible mechanical structures, the interconnections between measurement locations are known, however the dynamics of the interconnections are unknown. To estimate the dynamics of a particular module in the network it is necessary to collect data that is generated by the network by taking measurements of various variables in the network.

Although it may be possible to take measurements at many different locations in the network, it may be expensive or inconvenient to do so. Therefore, it may be attractive to use the minimum number of required measurement locations in order to identify a particular module embedded in a network. Secondly, it may be unsafe, or practically unfeasable to measure some variables in the network. Therefore it would be preferable if it is not neccesary to measure these variables in order to obtain estimates of the dynamics of interest.

The question addressed in this paper is: given a dynamic network with known interconnection structure, which variables must be included as inputs in the predictor model in order to guarantee that it is possible to obtain consistent estimates of a particular module of interest that is embedded in the network? Conditions are presented that the set of predictor inputs must satisfy. In this paper the conditions are derived for the Direct Prediction-Error Method as described in [2], [3].

This problem could also be interpreted as determining which variables should be measured (where should sensors be placed) in order to obtain consistent estimates of a particular module in the network. By this interpretation, conditions on the sensor placement scheme are presented such that the Direct Method results in consistent estimates of the module of interest.

There is a growing interest in dynamic network identification, including the case where the interconnection structure is not known a priori ([4], [5], [6], [7], [3] and references therein). If the interconnection structure is not known, then all variables must be included as predictor inputs (no choices based on the interconnection structure can be made). Suppose the interconnection structure is known, then it becomes possible to choose the set of predictor inputs which is optimal in some sense.

The results of this paper are complementary to the results in [8] where the conditions were derived that the predictor inputs must satisfy when using the Two-Stage Prediction-Error Method. The advantage of the method described in this paper, compared with [8] is that an external reference signal is not required. Moreover, the results presented in this paper are strict generalizations of the results in [2], [3].

In Section II the background material is presented, Section III contains the main result and Section IV contains algorithms to check the conditions.

II. Background

In this section first the dynamic networks considered in this paper are formally defined, then the prediction-error framework and Direct Method are briefly presented, and finally some definitions from graph theory are presented.

A. Dynamic Networks and Problem Setup

The networks considered in this paper are built up of L elements, related to L scalar internal variables w_j, $j = 1, \dots, L$. It is assumed that each internal variable is such that it can be written as:

$$w_j(t) = \sum_{i \in \mathcal{N}_j} G^0_{ji}(q) w_i(t) + r_j(t) + v_j(t) \qquad (1)$$

with G^0_{jk} a proper rational transfer function,, q is the delay operator (i.e. $q^{-1}u(t) = u(t-1)$), and

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- $\mathcal{N}_j$ is the set of indices of node variables with direct causal connections to w_j, i.e. $i \in \mathcal{N}_j$ if $G^0_{ji} \neq 0$,
- v_j is an unmeasured disturbance term that is a realization of a stationary stochastic process with rational spectral density: $v_j = H^0_j(q)e_j$ where e_j is a white noise process, and H^0_j is a monic, stable, and minimum phase filter, and
- r_j is a external excitation term that is known to the user.

It may be that the disturbance term and/or external excitation term is not present at some nodes. The sets of all indices of the external excitation terms and disturbance terms that are present are denoted $\mathcal{R}$ and $\mathcal{V}$ respectively. All the internal variables can be written in one equation as:

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} 0 & G^0_{12} & \cdots & G^0_{1L} \\ G^0_{21} & 0 & \ddots & G^0_{2L} \\ \vdots & \ddots & \ddots & \vdots \\ G^0_{L1} & G^0_{L2} & \cdots & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} + \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_L \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_L \end{bmatrix}$$
$$= G^0 w + r + H^0 e \tag{2}$$

where, if an external excitation signal is not present at node i then the ith entry of r is 0.

A *path* from $w_i \rightarrow w_j$ will be understood to mean that there are transfer functions such that $G_{jn_1}G_{n_1n_2}\cdots G_{n_ki}$ is non-zero. A *loop* is a path from $w_j \rightarrow w_j$.

A directed graph of the network can be constructed:

1. Let all w_k, $k = \{1,\ldots,L\}$ be nodes.
2. Let all v_k, $k \in \mathcal{V}$ and r_m, $m \in \mathcal{R}$ be nodes.
3. For all $i, j \in \{1,\ldots,L\}$ if $G_{ji} \neq 0$, then include a directed edge from node w_i to node w_j.
4. For all $k \in \mathcal{V}$ add a directed edge from v_k to w_k.
5. For all $k \in \mathcal{R}$ add a directed edge from v_k to w_k.

To characterize the suitability of the equations (2) in describing a physical system, the property of well-posedness is used [9]. The dynamic networks considered are assumed to satisfy the following general conditions.

Assumption 1:

(a) The network is well-posed in the sense that all minors of $\lim_{z\rightarrow\infty}(I - G^0(z))$ are non-zero. [1]
(b) $(I - G^0)^{-1}$ is stable.
(c) All r_m, $m \in \mathcal{R}$ are uncorrelated to all v_k, $k \in \mathcal{V}$.[2]

B. Prediction Error Identification

The prediction-error framework is an identification framework that is based on the one-step-ahead predictor model. See [10] for a detailed description and analysis.

Let w_j denote the variable which is to be predicted. Let w_k, $k \in \mathcal{D}_j$ and r_k, $k \in \mathcal{P}_j$ denote the *predictor inputs* (the set of internal and external variables that will be used to predict w_j). The one-step-ahead predictor for w_j is [10]:

$$\hat{w}_j(t|t-1,\theta) = \sum_{k\in\mathcal{D}_j} H_j^{-1}(q,\theta)G_{jk}(q,\theta)w_k(t)$$
$$+\sum_{k\in\mathcal{P}_j} H_j^{-1}(q,\theta)F_{jk}(q,\theta)r_k(t) + \big(1 - H_j^{-1}(q,\theta)\big)w_j(t). \tag{3}$$

where $H_j(q,\theta)$ is the noise model and $F_{jk}(q,\theta)$ models the dynamics between r_k, $k \in \mathcal{P}_j$ and w_j. From (1) if $\mathcal{D}_j = \mathcal{N}_j$, then $\mathcal{P}_j$ should be chosen as $\{j\}$, and $F_{jj}(q,\theta) = 1$. Although currently a parameterization including $F_{jk}(\theta)$ may seem to add unnecessary complexity to the predictor, the importance will become apparent later in the paper. Note that this is a multi-input, single-output (MISO) predictor. The prediction error is:

$$\varepsilon_j(t,\theta) = w_j(t) - \hat{w}_j(t|t-1,\theta)$$
$$= H_j(\theta)^{-1}\big(w_j - \sum_{k\in\mathcal{D}_j} G_{jk}(\theta)w_k - \sum_{k\in\mathcal{P}_j} F_{jk}(\theta)r_k\big) \tag{4}$$

where arguments q and t have been dropped for notational clarity. The unknown parameters, θ, are estimated by minimizing the sum of squared (prediction) errors (SSE):

$$V_j(\theta) = \frac{1}{N}\sum_{t=0}^{N-1} \varepsilon_j^2(t,\theta). \tag{5}$$

where N is the data length. Under standard (weak) assumptions the estimated parameter vector $\hat{\theta}_N$ converges in the number of data N as [10]

$$\hat{\theta}_N \rightarrow \theta^* \text{ with probability 1 as } N \rightarrow \infty.$$

where

$$\theta^* = \arg\min_\theta \bar{\mathbb{E}}[\varepsilon_j^2(t,\theta)] \quad \text{and} \quad \bar{\mathbb{E}} := \lim_{N\rightarrow\infty}\frac{1}{N}\sum_{t=0}^{N-1}\mathbb{E},$$

and $\mathbb{E}$ is the expected value operator. The function $\bar{\mathbb{E}}[\varepsilon_j^2(t,\theta)]$ is denoted $\bar{V}_j(\theta)$. If $G_{jk}(q,\theta^*) = G^0_{jk}$ the module transfer is said to be estimated *consistently*.

C. Direct Method

As in identification in closed-loops, identification in networks may that the problem that the "output" disturbance v_j is correlated to the predictor inputs w_k, $k \in \mathcal{D}_j$. In the closed-loop identification literature several methods have been developed to deal with this problem [11]. One of those methods is the Direct Method which is defined by the following algorithm.

Algorithm 1: Direct Method.

1. Choose a set of internal and external variables to include as inputs to the predictor (i.e. choose $\mathcal{D}_j$ and $\mathcal{P}_j$).
2. Construct the predictor (3).
3. Obtain estimates $G_{jk}(q,\theta^*)$ by minimizing the sum of squared prediction errors (5).

Step 1 of the algorithm is usually an implicit choice dependent on the network structure [2], [4]. However, in this paper it is explicitly considered a user choice.

[1]This condition is adopted from [9] and imposes weak restrictions on allowable feed-through terms in the network but still allows for the occurrence of algebraic loops. Moreover, it ensures that both G^0 and $(I - G^0)^{-1}$ only contain proper transfer functions

[2]Throughout this paper r uncorrelated to v will mean that the cross-correlation function $R_{rv}(\tau)$ is zero for all τ.

The main idea behind the Direct Method as presented in [2] is that if (a) there is a delay in every loop in the network, and (b) the noise v_j can be exactly whitened to e_j, and (c) $\mathcal{D}_j$ is chosen as $\mathcal{N}_j$, then the estimates obtained using Algorithm 1 are consistent estimates of G^0_{jk}, $k \in \mathcal{N}_j$. Formally, the proposition that is proved in [2] is as follows.

Proposition 1: Consider a dynamic network as defined in Section II-C that satisfies Assumption 1. Consistent estimates of G^0_{ji}, $i \in \mathcal{N}_j$ can be obtained using Algorithm 1 if the following conditions are satisfied:

(a) For both the network and the parameterized model, every loop $w_j \to w_j$ has a delay.
(b) $\mathcal{D}_j = \mathcal{N}_j$, $\mathcal{P}_j = \{j\}$.
(c) v_j is present and uncorrelated to v_k, $k \in \mathcal{V} \setminus j$
(d) The power spectral density of $[w_j\ r_j\ w_{n_1}\ \cdots\ w_{n_n}]^T$, $n_* \in \mathcal{N}_j$ is positive definite for all $\omega \in [-\pi, \pi]$.
(e) The data generating system (2) is in the set of possible models, i.e. there exists a θ_0 such that $G_{ji}(\theta_0) = G^0_{ji}$, $i \in \mathcal{N}_j$, $F_{jj}(\theta_0) = 1$, and $H_j(\theta_0) = H^0_j$.

Notice that all transfer functions G_{jk}, $k \in \mathcal{N}_j$ are consistently estimated. However, the objective was to only obtain consistent estimates of G^0_{ji}. As is shown in this paper, Condition (b) can be made less restrictive, with the result that only G^0_{ji} is estimated consistently, and no guarantees are made about the other transfer functions that are estimated.

D. Some Useful Results From Graph Theory

A graph G is made up of nodes which are interconnected by edges. The set of nodes of G is denoted $V(G)$.

Definition 1 (A-B path): Given a directed graph G and sets of nodes A and B. Denote the nodes in the graph x_i. A path $P = x_0 x_1 \cdots x_k$, where the x_i are all distinct, is an *A-B path* if $V(P) \cap A = \{x_0\}$, and $V(P) \cap B = \{x_k\}$ [12].

Definition 2 (A-B Separating Set): Consider a directed graph G. Given $A, B \subset V(G)$, a set $X \subseteq V(G)$ is an *A-B separating set* if the removal of the nodes in X results in a graph with no A-B paths [12].

Lemma 1: Consider a directed graph with adjacency matrix A. Then for $k \geq 1$, the (j,i)th entry A^k is zero if there is no path of length k from $i \to j$. [12]

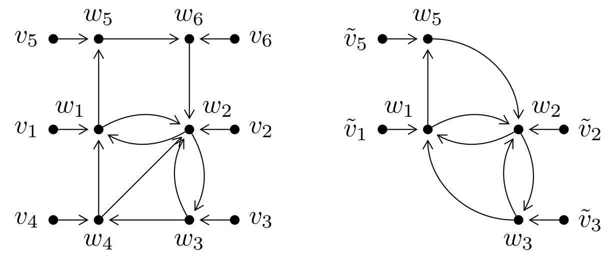
III. Predictor Input Selection

In this section conditions are presented that the set of predictor inputs must satisfy to allow a consistent estimate of G^0_{ji} using Algorithm 1. This enables the user to choose a set of variables from a given data set such that the conditions are satisfied. Equivalently, it enables the user to place sensors in order to collect the required data.

First a property of dynamic networks is investigated. Then, some properties of the noise terms are discussed. Both these properties lead up to the statement of the main result.

A. Network Property

A property of the network equations is that w_j can be expressed in many ways using different sets of internal variables.



(a) (b)

Fig. 1. Graphs of the networks in Examples 1 and 2.

Example 1: Consider the network described by:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = \begin{bmatrix} 0 & G^0_{12} & 0 & G^0_{14} & 0 & 0 \\ G^0_{21} & 0 & G^0_{23} & G^0_{24} & 0 & G^0_{26} \\ 0 & G^0_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & G^0_{43} & 0 & 0 & 0 \\ G^0_{51} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G^0_{65} & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}$$

A graph of the network is shown in Figure 1. The variable w_2 can be expressed in terms of w_1, w_4, and w_6:

$$w_2 = G^0_{21} w_1 + G^0_{24} w_4 + G^0_{26} w_6 + v_2, \qquad (6)$$

or in terms of w_1, w_3, and w_5:

$$w_2 = G^0_{21} w_1 + G^0_{24} G^0_{43} w_3 + G^0_{26} G^0_{65} w_5 + G^0_{24} v_4 + G^0_{26} v_6 + v_2,$$

or in terms of w_1 and w_4:

$$w_2 = (G^0_{21} + G^0_{26} G^0_{65} G^0_{51}) w_1 + G^0_{24} w_4 + G^0_{26} G^0_{65} v_5 + G^0_{26} v_6 + v_2$$

or in terms of w_1 and w_5

$$w_2 = \frac{1}{1 - G^0_{24} G^0_{43} G^0_{32}} \Big(G^0_{21} w_1 + G^0_{26} G^0_{65} w_5 + G^0_{24} v_4 + G^0_{24} G^0_{43} v_3 + G^0_{26} v_6 + v_2 \Big). \qquad \square$$

Let $\mathcal{D}_j$ denote the set of indices of internal variables which are chosen to describe w_j since this will end up being the same set $\mathcal{D}_j$ in (3) (in Example 1 for (6) $\mathcal{D}_j = \{1, 4, 6\}$). From the example, it can be seen that for different sets $\mathcal{D}_j$, the transfer functions between the variables also change. In other words, the transfer function between w_1 and w_2 is not a constant, but depends on the choice of $\mathcal{D}_j$. Note that only proper mappings from $w_k \to w_j$, $k \in \mathcal{D}_j$ are considered. This phenomenon was also investigated in [8].

For a general network, w_j can be causally expressed in terms of w_k, $k \in \mathcal{D}_j$ using the following notation and equations. Let $\mathcal{Z}_j$ denote the set of indices k, such that $k \notin \{j\} \cup \mathcal{D}_j$. Let $\mathcal{Z}_j$ denote the set of indices k, such that $k \notin \{j\} \cup \mathcal{D}_j$. Let $w_\mathcal{D}$ denote the vector $[w_{k_1}\ w_{k_2}\ \cdots]^T$, $k_* \in \mathcal{D}_j$. Let $r_\mathcal{D}$ denote the vector $[r_{k_1}\ r_{k_2}\ \cdots]^T$, $k_* \in \mathcal{D}_j$, where the ℓth entry is zero if r_ℓ is not present in the network (i.e. $\ell \notin \mathcal{R}$). The vectors $w_\mathcal{Z}$, $v_\mathcal{D}$, $v_\mathcal{Z}$ and $r_\mathcal{Z}$ are defined analogously. The ordering of the elements of $w_\mathcal{D}$, $v_\mathcal{D}$, and $r_\mathcal{D}$ is not important, as long as it is the same for all these vectors (the same holds for $w_\mathcal{Z}$, $v_\mathcal{Z}$, and $r_\mathcal{Z}$). The transfer function matrix between $w_\mathcal{D}$ and w_j is denoted $G^0_{j\mathcal{D}}$.

The other transfer function matrices are defined analogously. Using this notation, the network equations (2) are

$$\begin{bmatrix} w_j \\ w_{\mathcal{D}} \\ w_{\mathcal{Z}} \end{bmatrix} = \begin{bmatrix} 0 & G^0_{j\mathcal{D}} & G^0_{j\mathcal{Z}} \\ G^0_{\mathcal{D}j} & G^0_{\mathcal{D}\mathcal{D}} & G^0_{\mathcal{D}\mathcal{Z}} \\ G^0_{\mathcal{Z}j} & G^0_{\mathcal{Z}\mathcal{D}} & G^0_{\mathcal{Z}\mathcal{Z}} \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \\ w_{\mathcal{Z}} \end{bmatrix} + \begin{bmatrix} v_j \\ v_{\mathcal{D}} \\ v_{\mathcal{Z}} \end{bmatrix} + \begin{bmatrix} r_j \\ r_{\mathcal{D}} \\ r_{\mathcal{Z}} \end{bmatrix},$$

The variables $w_{\mathcal{Z}}$ can be eliminated from the equations:

$$\begin{aligned} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} &= \begin{bmatrix} 0 & G^0_{j\mathcal{D}} \\ G^0_{\mathcal{D}j} & G^0_{\mathcal{D}\mathcal{D}} \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} + \begin{bmatrix} G^0_{j\mathcal{Z}} \\ G^0_{\mathcal{D}\mathcal{Z}} \end{bmatrix} (I - G^0_{\mathcal{Z}\mathcal{Z}})^{-1} \begin{bmatrix} G^0_{\mathcal{Z}j} & G^0_{\mathcal{Z}\mathcal{D}} \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} \\ &\quad + \begin{bmatrix} G^0_{j\mathcal{Z}} \\ G^0_{\mathcal{D}\mathcal{Z}} \end{bmatrix} (I - G^0_{\mathcal{Z}\mathcal{Z}})^{-1} (v_{\mathcal{Z}} + r_{\mathcal{Z}}) + \begin{bmatrix} v_j + r_j \\ v_{\mathcal{D}} + r_{\mathcal{D}} \end{bmatrix} \\ &= \begin{bmatrix} \breve{G}^0_{jj} & \breve{G}^0_{j\mathcal{D}} \\ \breve{G}^0_{\mathcal{Z}j} & \breve{G}^0_{\mathcal{D}\mathcal{D}} \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} + \begin{bmatrix} I & 0 & \breve{G}^0_{j\mathcal{Z}} \\ 0 & I & \breve{G}^0_{\mathcal{D}\mathcal{Z}} \end{bmatrix} \begin{bmatrix} v_j + r_j \\ v_{\mathcal{D}} + r_{\mathcal{D}} \\ v_{\mathcal{Z}} + r_{\mathcal{Z}} \end{bmatrix}. \end{aligned} \quad (7)$$

where a new notation is introduced in the second equality. Note that by Assumption 1a the inverse $(I - G^0_{\mathcal{Z}\mathcal{Z}})^{-1}$ exists, and that all transfers in the last line are proper (causal). Lastly, the diagonal entries of $\breve{G}^0$ must be removed. Let $D^0_{\mathcal{D}\mathcal{D}} = \text{diag}(G^0_{\mathcal{D}\mathcal{D}})$. Then:

$$\begin{aligned} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} = \begin{bmatrix} \breve{G}^0_{jj} & \\ & D^0_{\mathcal{D}\mathcal{D}} \end{bmatrix}^{-1} \Bigg(&\begin{bmatrix} 0 & \breve{G}^0_{j\mathcal{D}} \\ \breve{G}^0_{\mathcal{Z}j} & \breve{G}^0_{\mathcal{D}\mathcal{D}} - D^0_{\mathcal{D}\mathcal{D}} \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} \\ &+ \begin{bmatrix} I & 0 & \breve{G}^0_{j\mathcal{Z}} \\ 0 & I & \breve{G}^0_{\mathcal{D}\mathcal{Z}} \end{bmatrix} \begin{bmatrix} v_j + r_j \\ v_{\mathcal{D}} + r_{\mathcal{D}} \\ v_{\mathcal{Z}} + r_{\mathcal{Z}} \end{bmatrix} \Bigg) \end{aligned} \quad (8)$$

Consequently, w_j has been causally expressed in terms of w_k, $k \in \mathcal{D}_j$, v_k, $k \in \mathcal{V}$ and r_k, $k \in \mathcal{R}$ as desired.

From (7) the stochastic terms of each w_k, $k \in \mathcal{D}_j$ are:

$$\begin{bmatrix} \tilde{v}_j \\ \tilde{v}_{\mathcal{D}} \end{bmatrix} = \begin{bmatrix} I & 0 & \breve{G}^0_{j\mathcal{Z}} \\ 0 & I & \breve{G}^0_{\mathcal{D}\mathcal{Z}} \end{bmatrix} \begin{bmatrix} v_j \\ v_{\mathcal{D}} \\ v_{\mathcal{Z}} \end{bmatrix}. \quad (9)$$

The properties of $\tilde{v}$ play an important role in the formulation of the main result. The power spectral density of $\tilde{v}$ is

$$\Phi_{\tilde{v}}(\mathcal{D}_j) = \begin{bmatrix} \Phi_{v_j} + \tilde{G}^0_{j\mathcal{Z}} \Phi_{v_{\mathcal{Z}}} \tilde{G}^{0*}_{j\mathcal{Z}} & \tilde{G}^0_{j\mathcal{Z}} \Phi_{v_{\mathcal{Z}}} \tilde{G}^{0*}_{\mathcal{D}\mathcal{Z}} \\ \tilde{G}^0_{\mathcal{D}\mathcal{Z}} \Phi_{v_{\mathcal{Z}}} \tilde{G}^{0*}_{j\mathcal{Z}} & \Phi_{v_{\mathcal{D}}} + \tilde{G}^0_{\mathcal{D}\mathcal{Z}} \Phi_{v_{\mathcal{Z}}} \tilde{G}^{0*}_{\mathcal{D}\mathcal{Z}} \end{bmatrix} \quad (10)$$

where $*$ denotes complex conjugate and Φ_{v_j}, $\Phi_{v_{\mathcal{Z}}}$ and $\Phi_{v_{\mathcal{D}}}$ are the power spectral densities of v_j, $v_{\mathcal{Z}}$ and $v_{\mathcal{D}}$ respectively. If each v_k, $k \in \mathcal{V}$ is assumed to be uncorrelated to each other, then Φ_v is diagonal. However, this does not imply $\Phi_{\tilde{v}}(\mathcal{D}_j)$ is diagonal. In summary, combining (8) and (9):

$$\begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} = \begin{bmatrix} 0 & \tilde{G}^0_{j\mathcal{D}} \\ \tilde{G}^0_{\mathcal{D}j} & \tilde{G}^0_{\mathcal{D}\mathcal{D}} \end{bmatrix} \begin{bmatrix} w_j \\ w_{\mathcal{D}} \end{bmatrix} + \begin{bmatrix} \tilde{G}^0_{j\mathcal{Z}} \\ \tilde{G}^0_{\mathcal{D}\mathcal{Z}} \end{bmatrix} r_{\mathcal{Z}} + \begin{bmatrix} \tilde{v}_j + r_j \\ \tilde{v}_{\mathcal{D}} + r_{\mathcal{D}} \end{bmatrix} \quad (11)$$

where again a new notation has been introduced, and $\tilde{v}$ has power spectral density (10) and it should be emphasized that the transfer functions (11) are functions of $\mathcal{D}_j$. The relation between (2) and (11) are illustrated in the following example.

Example 2: Consider the network of Example 1. Choose $\mathcal{D}_j = \{1, 3, 5\}$, then (11) becomes:

$$\begin{bmatrix} w_2 \\ w_1 \\ w_3 \\ w_5 \end{bmatrix} = \begin{bmatrix} 0 & G^0_{21} & G^0_{23} + G^0_{24}G^0_{43} & G^0_{26}G^0_{65} \\ G^0_{21} & 0 & G^0_{14}G^0_{43} & 0 \\ G^0_{32} & 0 & 0 & 0 \\ 0 & G^0_{51} & 0 & 0 \end{bmatrix} \begin{bmatrix} w_2 \\ w_1 \\ w_3 \\ w_5 \end{bmatrix} + \begin{bmatrix} \tilde{v}_2 \\ \tilde{v}_1 \\ \tilde{v}_3 \\ \tilde{v}_5 \end{bmatrix} \quad (12)$$

where

$$\Phi_{\tilde{v}} = \begin{bmatrix} |H^0_2|^2 + |G^0_{24}H^0_4|^2 + |G^0_{26}H^0_6|^2 & G^0_{24}|H^0_4|^2 G^{0*}_{14} & 0 & 0 \\ G^0_{14}|H_4|^2 G^{0*}_{24} & |H^0_1|^2 + |G^0_{14}H^0_4|^2 & 0 & 0 \\ 0 & 0 & |H^0_3|^2 & 0 \\ 0 & 0 & 0 & |H^0_5|^2 \end{bmatrix}$$

The graph of (12) is shown in Fig. 1b. When comparing the graphs of Fig 1a and b, note that in Fig. 1b the vertices w_4 and w_6 have been removed, and edges coming in and out of those nodes have been reconnected. □

In the next section it is shown how the idea that (11) can be obtained from (2) is linked to predictor input selection.

B. Predictor Input Selection

In order to be able to identify G^0_{ji}, the term must become explicit somewhere in the expression for w_j. For instance, suppose that for the network shown in Example 1 the transfer G^0_{21} is to be estimated. The expression G^0_{21} only appears as the relationship between w_1 and w_2 for $\mathcal{D}_j = \{1, 4, 6\}$ and $\{1, 3, 5\}$. The following proposition presents conditions that $\mathcal{D}_j$ must satisfy in order to ensure that the transfer function between w_i and w_j is G^0_{ji}.

Proposition 2: Consider a dynamic network as defined in Section (II-A) that satisfies Assumption 1. The transfer function $\tilde{G}_{ji}(q, \mathcal{D}_j) = G^0_{ji}(q)$ if the following conditions on $\mathcal{D}_j$ are satisfied:

(a) $i \in \mathcal{D}_j$, $j \notin \mathcal{D}_j$,
(b) every loop $w_j \to w_j$ passes through a node w_k, $k \in \mathcal{D}_j$.
(c) every path $w_i \to w_j$ excluding the path G^0_{ji} passes through a node w_k, $k \in \mathcal{D}_j$, □

Before proceeding to the proof, note that all conditions are satisfied in Example 1 for the first two sets $\mathcal{D}_j = \{1, 4, 6\}$ and $\{1, 3, 5\}$. For the third choice, $\mathcal{D}_j = \{1, 4\}$, Condition (c) is not satisfied since the path $w_1 \to w_5 \to w_6 \to w_2$ does not pass through any nodes in $\mathcal{D}_j$. For the last set $\mathcal{D}_j = \{1, 5\}$, Condition (b) is not satisfied since there is a path $w_2 \to w_2$ which does not pass through any nodes in $\mathcal{D}_j$.

The following lemma is used in the proof of Proposition 2. It is proved in [2], or can be proved using Mason's rules.

Lemma 2: Consider a dynamic network with transfer matrix G_0 that satisfies all conditions of Assumption 1. Let $\mathfrak{G}^0_{mn}$ be the (m, n)th entry of $(I - G_0)^{-1}$. If there is no path from w_n to w_m then $\mathfrak{G}^0_{mn} = 0$. □

Next proceed with the proof of Proposition 2.

Proof: From (8) and (11),

$$\tilde{G}^0_{ji}(q, \mathcal{D}_j) = \frac{1}{1 - \breve{G}^0_{jj}(q, \mathcal{D}_j)} \breve{G}^0_{ji}(q, \mathcal{D}_j)$$

The following reasoning will show how Conditions (c) - (b) ensure that $\breve{G}^0_{ji}(q, \mathcal{D}_j) = G^0_{ji}(q)$ and $\breve{G}^0_{jj}(q, \mathcal{D}_j) = 0$, resulting in $\tilde{G}^0_{ji}(q, \mathcal{D}_j) = G^0_{ji}(q)$.

Consider first the term $\breve{G}^0_{jj}(q, \mathcal{D}_j)$. From (7):

$$\breve{G}^0_{jj} = G^0_{j\mathcal{Z}}(I - G^0_{\mathcal{Z}\mathcal{Z}})^{-1} G^0_{\mathcal{Z}j} = \sum_{k_1 \in \mathcal{Z}_j} \sum_{k_2 \in \mathcal{Z}_j} G^0_{jk_1} \mathfrak{G}^0_{k_1k_2} G^0_{k_2j} \quad (13)$$

where $\mathfrak{G}^0_{k_1k_2}$ is the (k_1, k_2) entry of $(I - G^0_{\mathcal{Z}\mathcal{Z}})^{-1}$. By Lemma 2 if there is no path $w_{k_2} \to w_{k_1}$ that passes only through nodes w_k, $k \in \mathcal{Z}_j$, then the transfer $\mathfrak{G}_{k_1k_2}$ is zero. By Condition (b) there is no path $w_j \to w_j$ that passes only through nodes w_k, $k \in \mathcal{Z}_j$. Thus at least one of G_{jk_1}, $\mathfrak{G}_{k_1k_2}$, or G_{k_2j} in (13) is equal to zero for each $k_1, k_2 \in \mathcal{Z}_j$. Therefore, $\breve{G}_{jj}(q, \mathcal{D}_j) = 0$.

From (7) the expression for $\breve{G}^0_{ji}(q, \mathcal{D}_j)$ is:

$$\breve{G}^0_{ji} = G^0_{ji} + G^0_{j\mathcal{Z}}(I - G^0_{\mathcal{Z}\mathcal{Z}})^{-1}G^0_{\mathcal{Z}i} \tag{14}$$

By the same reasoning it follows that by Condition (c) the second term in (14) is 0, and thus, $\breve{G}^0_{ji} = G^0_{ji}$ as desired. ■

Suppose a set $\mathcal{D}_j$ is chosen such that it satisfies the conditions of Proposition 2. If the variables w_k, $k \in \mathcal{D}_j$ are included as inputs to the predictor (3) would Algorithm 1 result in consistent estimates of G^0_{ji}? Unfortunately, not always. The problem lies in the fact that the fundamental assumptions guaranteeing consistency of estimates in Algorithm 1 may be violated in (11).

In the following text, first the fundamental mechanism that assures consistent estimates using Algorithm 1 is stated.

Proposition 3: Consider a dynamic network as defined in Section II-A that satisfies Assumption 1. Algorithm 1 leads to consistent estimates if there exists an $H_j(\theta^*)$ such that

(a) $H_j^{-1}(\theta^*)\tilde{v}_j(t)$ is white, and

(b) $\bar{\mathbb{E}}\big[H_j^{-1}(q,\theta^*)\tilde{v}_j(t) \cdot \Delta G_{jk}(q,\theta,\mathcal{D}_j)w_k(t)\big] = 0, \forall k \in \mathcal{D}_j$,

and for all θ, where $\Delta G_{jk}(q, \theta, \mathcal{D}_j) = \tilde{G}^0_{jk}(q, \mathcal{D}_j) - G_{jk}(q, \theta)$. □

The full proof can be extracted from the reasoning in [3], [2]. However, consider the following sketch. In light of (11) the prediction error (4) can be expressed as:

$$\varepsilon_j(\theta) = H_j^{-1}(\theta)\big(\sum_{k\in\mathcal{D}_j} \Delta G_{jk}(\theta)w_k + \sum_{k\in\mathcal{R}\setminus\mathcal{D}_j} \Delta F_{jk}(\theta)r_k + \tilde{v}_j\big)$$

If the conditions of Proposition 3 hold, then it can be shown that $\bar{V}_j(\theta) \geq \bar{\mathbb{E}}[\tilde{e}_j(t)^2]$, where $\tilde{e}_j$ is the whitened version of $\tilde{v}_j$. Secondly, it can be shown that when $\bar{V}_j(\theta) = \bar{\mathbb{E}}[\tilde{e}_j(t)^2]$ it must hold that $G_{jk}(\theta) = \tilde{G}^0_{jk}(\mathcal{D}_j)$, $k \in \mathcal{D}_j$.

The following example will illustrate how the conditions of Proposition 3 can be violated in (11).

Example 3: Consider a network described by:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 & G^0_{12} & G^0_{13} \\ G^0_{21} & 0 & G^0_{23} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{15}$$

where all transfer functions are strictly proper and Φ_v is diagonal. Suppose G^0_{21} is to be estimated. Choose $\mathcal{D}_2 = \{1\}$. This choice of $\mathcal{D}_2$ satisfies the conditions of Proposition 2. Rewrite (15) in terms of only w_k, $k \in \mathcal{D}_2$ as in (11):

$$\begin{bmatrix} w_2 \\ w_1 \end{bmatrix} = \begin{bmatrix} 0 & G^0_{21} \\ G^0_{12} & 0 \end{bmatrix} \begin{bmatrix} w_2 \\ w_1 \end{bmatrix} + \begin{bmatrix} \tilde{v}_2 \\ \tilde{v}_1 \end{bmatrix}.$$

with

$$\Phi_{\tilde{v}} = \begin{bmatrix} |H^0_1|^2 + |G^0_{13}H^0_3|^2 & G^0_{13}|H^0_3|^2 G^{0\,*}_{23} \\ G^0_{23}|H^0_3|^2 G^{0\,*}_{13} & |H^0_2|^2 + |G^0_{23}H^0_3|^2 \end{bmatrix} \tag{16}$$

By the spectral factorization theorem, $\tilde{v}_2$ can be expressed as $\tilde{H}_2(q,\theta^*)\tilde{e}_2(t)$, where $\tilde{e}_2(t)$ is white and θ^* is unique. This shows that Condition (a) of Proposition 3 is satisfied.

Since θ^* is unique, Condition (b) must also be satisfied for this particular θ^*. For this example, Condition (b) is:

$$\bar{\mathbb{E}}[\tilde{e}_2(t) \cdot \Delta G_{21}(q,\theta,\mathcal{D}_2)w_1(t)] = \bar{\mathbb{E}}\Big[\tilde{e}_2(t) \cdot \frac{\Delta G_{21}(q,\theta,\mathcal{D}_2)}{1 - G^0_{12}(q)G^0_{21}(q)}\Big(\tilde{v}_1(t) + G^0_{12}(q)\tilde{v}_2(t)\Big)\Big]. \tag{17}$$

Since both G^0_{12} and G^0_{21} have delays, it follows that the second term in (17) is a function of $\tilde{v}_2(t-k)$, $k \geq 1$. Since $\tilde{e}_2(t)$ is constructed to be white, $\bar{\mathbb{E}}[\tilde{e}_2(t)\tilde{v}_2(t-k)] = 0$, $k \geq 1$. Consequently, the second term of (17) is zero.

However, by (16) $\tilde{v}_1$ and $\tilde{v}_2$ are correlated, therefore, the first term in (17) will not equal zero, violating Condition (b) of Proposition 3. Consequently, consistent estimates cannot be guaranteed using Algorithm 1 for this network.

Note that if either G^0_{13} or G^0_{23} were 0 then the conditions of Proposition 3 would be satisfied because $\Phi_{\tilde{v}}$ in (16) would be diagonal. □

From Example 3 it appears that the conditions of Proposition 2 are not sufficient to guarantee consistent estimates using the Direct Method. The problem lies in the disturbance terms. In statistics the variable v_3 in Example 3 is referred to as a *confounding* variable [13].

In the main result of the paper, it is shown that if $\tilde{v}_j$ is uncorrelated to the other $\tilde{v}_k$, $k \in \mathcal{D}_j$ then it is possible to obtain consistent estimates using Algorithm 1. From the expression of $\Phi_{\tilde{v}}$ in (10), it can be seen that $\tilde{v}_j$ is uncorrelated to $\tilde{v}_{\mathcal{D}}$ if $\tilde{G}^0_{\mathcal{D}\mathcal{Z}}\Phi_{v_\mathcal{Z}}\tilde{G}^{0^*}_{j\mathcal{Z}} = 0$. If $\Phi_{v_\mathcal{Z}}$ is diagonal then this equation will hold as long as there is no node in $\mathcal{Z}_j$ from which there is a path to both w_j and any other node w_k, $k \in \mathcal{D}_j$. It is now possible to formally state the main result of this paper.

Proposition 4: Consider a dynamic network as defined in Section II-A that satisfies Assumption 1. Assume Φ_v is diagonal. Let $\{w_k\}$, $k \in \mathcal{D}_j$ and $\{r_k\}$ be the set of internal and external variables respectively that are included as inputs to the predictor (3). Suppose that $\mathcal{P}_j$ is chosen such that $k \in \mathcal{P}_j$ if there is a path from $r_k \to w_j$ that passes only through w_n, $n \in \mathcal{Z}_j$. Consistent estimates of G^0_{ji} are obtained using Algorithm 1 if the following conditions hold:

(a) There is a delay in every loop $w_j \to w_j$.

(b) The set $\mathcal{D}_j$ satisfies the conditions of Proposition 2.

(c) There is no node w_k, $k \in \mathcal{Z}_j$ from which there is both a path to w_j and a path to any other w_n, $n \in \mathcal{D}_j$.

(d) Power spectral density of $[w_{k_1} \ \cdots \ w_{k_n} r_{\ell_1} \ \cdots \ r_{\ell_m}]^T$, $k_* \in \mathcal{D}_j$, $\ell_* \in \mathcal{R}_d$ is positive definite for all $\omega \in [-\pi, \pi]$.

(e) The parameterization is chosen flexible enough, i.e. there exists a parameter θ^* such that $G_{jk}(\theta^*) = \tilde{G}_{jk}(\mathcal{D}_j)$, $k \in \mathcal{D}_j$, $F_{jk}(\theta^*) = \tilde{G}_{jk}(\mathcal{D}_j)$, $k \in \mathcal{P}_j$, and $H(\theta^*) = \tilde{H}_j(\mathcal{D}_j)$ where $\tilde{G}_{jk}(\mathcal{D}_j)$ and $\tilde{H}_j(\mathcal{D}_j)$ are defined in (11).

Proof: By Condition (b) and Proposition 2 w_j can be expressed in terms of w_k, $k \in \mathcal{D}_j$ as:

$$w_j = G^0_{ji}w_i + \sum_{k\in\mathcal{D}_j\setminus i} \tilde{G}^0_{jk}(\mathcal{D}_j)w_k + \sum_{k\in\mathcal{R}\setminus\mathcal{D}_j} \tilde{G}^0_{jk}(\mathcal{D}_j)r_k + \tilde{v}_j$$

where some of the transfer functions in the second summation may be zero. From (7) the transfer functions in the second summation are

$$\tilde{G}^0_{jk}(\mathcal{D}_j) = [G^0_{j\mathcal{Z}}(I - G^0_{\mathcal{Z}\mathcal{Z}})^{-1}]_{jk} = \sum_{\ell \in \mathcal{Z}_j} G^0_{j\ell}\mathfrak{G}^0_{\ell k} = 0,$$

where $[\cdot]_{jk}$ denotes the (j,k) entry of the matrix in square brackets. By Lemma 2 if there is no path from $w_k \to w_\ell$ then $\mathfrak{G}^0_{\ell k}$ is zero. Thus if there is no path from r_k to w_j either $\mathfrak{G}^0_{\ell k}$ or $G^0_{j\ell}$ (or both) is zero. On the other hand, if there is such a path, then by construction, $k \in \mathcal{P}_j$. Consequently, w_j can be expressed as:

$$w_j = G^0_{ji} w_i + \sum_{k \in \mathcal{D}_j \setminus i} \tilde{G}^0_{jk}(\mathcal{D}_j) w_k + \sum_{k \in \mathcal{P}_j} \tilde{G}^0_{jk}(\mathcal{D}_j) r_k + \tilde{v}_j \quad (18)$$

which can be considered as the 'data generating system'.

Next it is shown that (18) satisfies all conditions of Proposition 1, meaning that consistent estimates of G^0_{ji} can be obtained using Algorithm 1.

By Condition (a) every loop that passes through w_j in the network has a delay. By (11) and Lemma 2 it follows that every loop that passes through w_j in (11) has a delay.

By Condition (c) and (10) it follows that $\tilde{v}_j$ is uncorrelated to all other noise sources $\tilde{v}_k$, $k \in \mathcal{D}_j$. The remaining conditions of Proposition 1 are also satisfied. ■

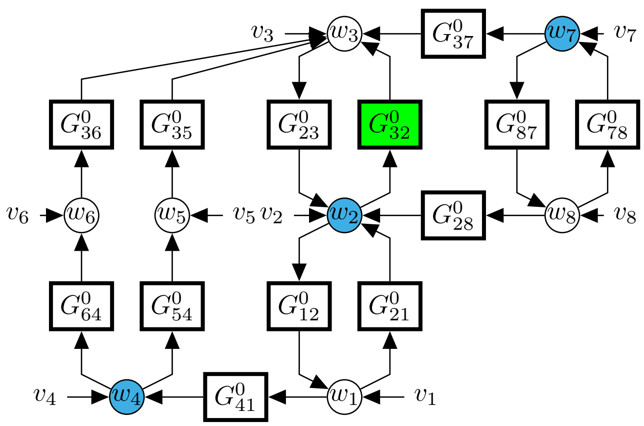


Fig. 2. Network that is analyzed in Example 4. Each rectangle represents a transfer function, and for notational convenience labels of the w_i's have been placed inside each summation, which denotes that the output of the sum is the variable w_i.

Example 4: Consider the dynamic network shown in Fig. 2. Suppose the objective is to obtain consistent estimates of G^0_{32} (denoted in green) using Algorithm 1. The set $\mathcal{D}_j$ must be chosen so that it satisfies the Conditions of Proposition 4. Choose $\mathcal{D}_3 = \{2, 4, 7\}$ (denoted in blue).

Another possible choice for $\mathcal{D}_3 = \{2, 5, 6, 7\} = \mathcal{N}_3$. This choice of $\mathcal{D}_3$ always satisfies Condition b. □

IV. Algorithms

Condition (b) can be reformulated using the notions separating sets in graph theory. The advantage is that tools from graph theory can be used to check the conditions [12], [14].

Let the node w_j be split into two nodes, w_j^+ to which all paths coming into w_j are connected and w_j^- to which all paths leaving w_j are connected. w_j^+ is connected to w_j^- with the path $G_{j^+j^-} = 1$. The Conditions (a) - (c) of Proposition 2 can be re-expressed as

1. $\mathcal{D}_j \setminus \{i\}$ is a $\{w_i\}$-$\{w_j\}$ separating set for the network with path G^0_{ji} removed,
2. $\mathcal{D}_j$ is a $\{w_j^-\}$-$\{w_j^+\}$ separating set for the network with path $G_{j^+j^-}$ removed,

These conditions can be formulated as a single condition. The set $\mathcal{D}_j$ is a $\{w_i, w_j^-\}$-$\{w_j^+\}$ separating set for the network with edges G^0_{ji} and $G_{j^+j^-}$ removed.

Condition (c) of Proposition 4 can be reformulated as follows. Consider the graph of (11). Switch the direction of all paths coming into w_j^+ (this is the effect of the conjugated term $\tilde{G}^0_{j\mathcal{D}}$ in (10)). For this new graph, there must be no path of length greater than 1 from w_j^+ to any w_k, $k \in \mathcal{D}_j$. Whether such a path exists can be checked using Lemma 1.

V. Conclusion

Conditions on the predictor inputs have been presented such that it is possible to obtain consistent estimates of the dynamics of a particular module embedded in a dynamic network using the Direct Prediction-Error Method. This enables the user to design sensor placement schemes.

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