Identifiability in Dynamic Acyclic Networks with Partial Excitations and Measurements

Xiaodong Cheng, Shengling Shi, Ioannis Lestas, and Paul M.J. Van den Hof

***Abstract*—This paper deals with dynamic networks in which the causality relations between the vertex signals are represented by linear time-invariant transfer functions (modules). Considering an acyclic network where only a subset of its vertices are measured and a subset of the vertices are excited, we explore conditions under which all the modules are identifiable on the basis of measurement data. Two sufficient conditions are presented in the paper, where the former one can be applied to perform an identifiability analysis for a full network, while the latter one, based on the concept of *tree/anti-tree covering*, induces a synthesis approach to allocate actuators and sensors in an acyclic network.**

I. Introduction

Dynamic networks have appeared in a wide range of technological applications including biochemical reaction systems, decentralized manufacturing processes, and smart power grids. In recent years, considerable attention from the systems and control domain has been devoted to address data-driven modeling problems in dynamic networked systems with large-scale interconnection structure, see some of the representative works in [1], [3], [6], [10]–[12], [15], [17], [18], [25], [27], [30], [33]. From an identification perspective, the modeling problems in a network setting can be viewed as an extension of classical system identification on the basis of open-loop data or closed-loop data, towards an identification problem with data collected from distributed dynamical systems operating under a network interconnection. This leads to a framework where signals are regarded as vertices in a network, and the causal dependence among the signals, typically represented by proper transfer functions (modules), are considered as directed edges.

Different data-driven modeling problems can be formulated in such a framework. For example, how to detect the topology of a network from measurement data [7], [14], [15], [21], [32]? How to consistently estimate selected local dynamics within a network [8], [9], [16], [19], [25], [29]? What are the conditions required to identify the dynamics of an entire network or a subnetwork [3], [5], [12], [23], [27], [28], [30]? In this paper, we focus on the last problem, where *identifiability* of a full network based on a subset of external excitation signals and measured vertex signals is of particular interest. This concept plays a key role in data-driven modeling of dynamic networks, as it essentially reflects if we are able to acquire a unique network model on the basis of measurement data. In the literature, there are two classes of network identifiability, namely, *global identifiability* [27], [30] that requires *all* the models in the set to be distinguishable, and *generic identifiability* [2], [3], [12], [31], which means that *almost all* models in the model set can be distinguished. Recently, the concept of *local generic identifiability* has been introduced in [13], which can be viewed as a weaker notion than generic identifiability, but lacking of a graph-theoretical characterization.

In this paper, we will focus on global and generic identifiability notions. In the relevant study, two problem settings of dynamic networks are commonly considered. A number of conditions have been derived for (generic) identifiability of a full dynamic network, see e.g., [3], [5], [12], [27], [30], which are based on network topology i.e., how the vertices are interconnected. While the identifiability analysis in [3], [12], [27] is performed under the assumption that all the vertices are excited by sufficiently rich external signals, these works lead to attractive path-based conditions for checking network identifiability. Depending on whether *generic identifiability* or *global identifiability* is considered, these conditions are interpreted in terms of the existence of *vertex disjoint paths* [3], [12] or *constrained vertex disjoint paths* [27] from the out-neighbors of each vertex to the measured vertices in a network. In contrast, the result in [28], [30] is developed for the setting where only a subset of vertices are affected by external signals including noises and excitation signals manipulated by users, while all the vertices are measured. Moreover, network identifiability is defined as a property of a parameterized model set, instead of a property of a single network as done in [3], [12], [27]. In [28], [30], network identifiability is characterized by the rank property of a certain transfer matrix that is determined by the presence and location of external signals, the correlation structure of disturbances, and the topology of parametrized modules. This rank condition therein has been further studied in [31] leading to generic identifiability of a network model set, which is equivalent to a vertex-disjoint path condition dual to the ones in [3], [12]. With the same network setting as in [28], [30], [31], the work in [4], [5] provides a new characterization for generic identifiability using the concept of *disjoint pseudo-tree covering*, from which a graphical tool for synthesizing allocation of external excitation signals can be effectively addressed.

While different measurement and excitation schemes are

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considered, most works in the existing literature require all the vertices to be either simultaneously excited or simultaneously measured. The work in [2] relaxes this requirement and investigates the identifiability condition with partial excitation and partial measurement, although these conditions require a priori knowledge on network dynamics, and graph-based results therein are provided only for special networks with tree and cycle topologies. In contrast, [24] deals with general network topology, where we provide a sufficient condition for generic identifiability of a single module in a network.

In line with the network setting of [2], [24], the current paper aims to derive general conditions for identifiability in full dynamic networks, where only partial excitation and partial measurement signals are available. First, we introduce the concept of transpose networks and show that identifiability of the model sets of a transpose network and its corresponding original network are equivalent, which builds a connection between the two settings in [3], [12], [27] and [5], [30], [31]. Although similar duality discussions have been seen in [5], [12], the analysis therein is performed under setting that the vertices in a network are fully excited or measured. In this paper, we further generalize the duality analysis to dynamic networks with the partial excitation and measurement by using transpose networks.

The main contribution of this paper is to provide an insight into the identifiability analysis of full networks with *acyclic topology* in the most general setting of *partial excitation and partial measurement*, which has not been addressed in the existing literature so far. We present sufficient conditions to determine both global and generic identifiability of acyclic networks. The first condition takes advantage of the hierarchical structure in acyclic networks to formulate rank conditions for every vertex, which can also be reinterpreted as path-based conditions and hence lead to a vertex-wise check for identifiability of a given acyclic network. To further solve problems of actuator and sensor allocation, we thereby present the second condition for generic identifiability of model sets of acyclic networks on the basis of a graphical concept, called *disjoint tree/anti-tree covering*. This condition can lead to an algorithmic procedure that admits generic identifiability of the model set of an acyclic network requiring as few as possible excited and measured vertices.

The rest of this paper is organized as follows. In Section II, we recap some basic notations used in graph theory and introduce the dynamic network model. Section III presents the results on transpose networks, and Section IV provides the main results of the paper on identifiability in acyclic networks, where a graph-theoretic approach is presented to the allocation of actuators and sensors. Finally, concluding remarks are made in Section V. The proofs of the technical results are presented in the appendix.

II. Preliminaries and Dynamic Network Setup

A. Notation

Denote $\mathbb{R}$ as the set of real numbers and $\mathbb{R}(q)$ as the rational function field over $\mathbb{R}$ with the variable q. I_n is the identity matrix of dimension n. The cardinality of a set $\mathcal{V}$ is represented by $|\mathcal{V}|$. A_{ij} denotes the (i,j) entry of a matrix A, and more generally, $[A]_{\mathcal{U},\mathcal{V}}$ denotes the submatrix of A that consists of the rows and columns of A indexed by two positive integer sets $\mathcal{U}$ and $\mathcal{V}$, respectively. Furthermore, $[A]_{\mathcal{U},\star}$ and $[A]_{\star,\mathcal{V}}$ represent the matrix containing the rows of A indexed by the set $\mathcal{U}$ and the matrix containing the columns of A indexed by the set $\mathcal{V}$, respectively.

The normal rank of a transfer matrix $A(q)$ is denoted by $\mathrm{Rank}(A(q))$, and $\mathrm{Rank}(A(q)) = r$ if the rank of $A(q)$ is equal to r for almost all values of q. Furthermore, let $A(q,\theta)$ be a parameterized transfer matrix with the parameters $\theta \in \Theta$, then the generic rank of $A(q,\theta)$ is defined as the maximum normal rank of $A(q,\theta)$ for all $\theta \in \Theta$ [26].

B. Graph Theory

The topology of a dynamic network is characterized by a graph $\mathcal{G}$ that consists of a finite and nonempty vertex set $\mathcal{V} := \{1, 2, ..., L\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. A directed graph is such that each element in $\mathcal{E}$ is an ordered pair of elements of $\mathcal{V}$. If $(i,j) \in \mathcal{E}$, we say that the edge is incident from vertex i to vertex j, and the vertex i is the *in-neighbor* of j, and j is the *out-neighbor* of i. Let $\mathcal{N}_j^-$ and $\mathcal{N}_j^+$ be the sets that collect all the in-neighbors and out-neighbors of vertex j, respectively.

A graph $\mathcal{G}$ is called *simple*, if $\mathcal{G}$ does not contain self-loops (i.e., $\mathcal{E}$ does not contain any edge of the form (i,i), $\forall\ i \in \mathcal{V}$), and there exists only one directed edge from one vertex to its each out-neighbor. In a simple graph, a directed *path* connecting vertices i_0 and i_n is a sequence of edges of the form (i_{k-1}, i_k), $k = 1, \cdots, n$, and every vertex appears at most once on the path. If there is a directed path from vertex i to j, we say j is *reachable* from i. Two directed paths are *vertex-disjoint* if they do not share any common vertex, including the start and the end vertices. In a simple directed graph $\mathcal{G}$, we denote $b_{\mathcal{U}\to\mathcal{Y}}$ as the maximum number of mutually vertex-disjoint paths from $\mathcal{U} \subseteq \mathcal{V}$ to $\mathcal{Y} \subseteq \mathcal{V}$.

Let $\mathcal{U}$ and $\mathcal{Y}$ be two vertex subsets in a directed graph $\mathcal{G}$. A set of vertices $\mathcal{D}$ in $\mathcal{G}$ is a *disconnecting set* from $\mathcal{U}$ to $\mathcal{Y}$, if all the directed paths from $\mathcal{U}$ to $\mathcal{Y}$ in $\mathcal{G}$ pass through $\mathcal{D}$. Roughly speaking, removing all the vertices in a disconnecting set $\mathcal{D}$ from $\mathcal{U}$ to $\mathcal{Y}$ will lead to no directed paths from $\mathcal{U}$ to $\mathcal{Y}$. Note that it is allowed for $\mathcal{D}$ to share common vertices in the sets $\mathcal{U}$ and $\mathcal{Y}$.

In this paper, we concentrate on a class of simple graphs, which are *acyclic*, meaning that there does not exist a pair of vertices i, j which can reach each other via directed paths. In a simple acyclic graph $\mathcal{G}$, a *source* is a vertex without any in-neighbors, and likewise, a *sink* is a vertex without any out-neighbors.

C. Dynamic Network Model

Consider a dynamic network whose topology is captured by a simple directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{1, 2, ..., L\}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Following the basic setup in [2], [25], each vertex describes an internal variable $w_j(t) \in \mathbb{R}$, and a compact form of the overall network dynamics is

$$\begin{aligned} w(t) &= G(q)w(t) + Rr(t) + v(t), \\ y(t) &= Cw(t), \end{aligned} \tag{1}$$

where q^{-1} is the delay operator, and $v(t) \in \mathbb{R}^L$ is the vector of zero-mean, stationary stochastic process noises. All the internal signals are stacked in the vector $w(t) := [w_1(t)\ w_2(t)\ \cdots\ w_L(t)]^\top$. $G(q)$ is transfer matrix with zero diagonal elements, where the (i,j)-th entry, denoted by $G_{ij}(q) \in \mathbb{R}(q)$, indicates the transfer operator from vertex j to vertex i, and it is represented by an edge $(j,i) \in \mathcal{E}$ in graph $\mathcal{G}$.

Let $\mathcal{R} \subseteq \mathcal{V}$ and $\mathcal{C} \subseteq \mathcal{V}$ be the vertices that are excited and measured, respectively, and $K = |\mathcal{R}|$ and $N = |\mathcal{C}|$. The signals $r(t) \in \mathbb{R}^K$ and $y(t) \in \mathbb{R}^N$ are the external excitation and measurement signals with $R = [I_L]_{\star,\mathcal{R}} \in \mathbb{R}^{L\times K}$, $C = [I_L]_{\mathcal{C},\star} \in \mathbb{R}^{N\times L}$ binary matrices indicating which vertices are excited and measured, respectively. Particularly, if a vertex k is unexcited, i.e. $k \in \mathcal{V}\backslash\mathcal{R}$, then the k-th row of R is zero, and analogously, if a vertex is unmeasured, then the corresponding column of C is zero.

Assumption 1: Throughout the paper, we consider a dynamic network (1) with the following properties.

1) The network (1) is *well-posed* and stable, i.e. $(I - G(q))^{-1}$ is proper and stable.
2) All the entries of $G(q)$ are proper transfer operators.

The above assumptions are standard in the identification of dynamic networks, see e.g., [2], [12], [27], [30], which ensure the properness and stability of the mapping from r to w.

Based on the model setting in (1), a problem of interest concerns if all the dynamics of a network, i.e., parameterized transfer functions in $G(q)$, can be consistently identified from the external excitation signals r and the measured output y. Let $M = (G, R, C)$ be a network model of (1) and

$$\mathcal{M} := \{M(q,\theta) = (G(q,\theta), R, C), \theta \in \Theta\} \tag{2}$$

be the network model set with parameterized models $M(q,\theta)$, in which all the nonzero transfer functions in $G(q,\theta)$ are parameterized independently. The network model set $\mathcal{M}$ including the information of a dynamic network including its topology, and the locations of actuators and sensors. Then identifiability of the network model set is defined as follows [3], [12], [23], [24].

Definition 1: Denote the transfer matrix

$$T(q,\theta) := (I - G(q,\theta))^{-1}. \tag{3}$$

Given the network model set $\mathcal{M}$ in (2). Consider $\theta_0 \in \Theta$ and the following implication

$$CT(q,\theta_1)R = CT(q,\theta_0)R \Rightarrow G_{ji}(q,\theta_1) = G_{ji}(q,\theta_0) \tag{4}$$

for all parameter $\theta_1 \in \Theta$. Then the module G_{ji} is

- **identifiable** in $\mathcal{M}$ from the submatrix $[T]_{\mathcal{C},\mathcal{R}}$ if the implication (4) holds for all $\theta_0 \in \Theta$;
- **generic identifiable** in $\mathcal{M}$ from the submatrix $[T]_{\mathcal{C},\mathcal{R}}$ if the implication (4) holds for almost all $\theta_0 \in \Theta$.

The model set $\mathcal{M}$ is (generic) identifiable if G_{ji} is (generic) identifiable for all $(i,j) \in \mathcal{E}$.

In [27], the identifiability in Definition 1 is also referred to as "global identifiability". The term "almost all" means the exclusion of parameters that are in a subset of Θ with Lebesgue measure zero. We refer to [3], [12] for more details about the notion of generic identifiability.

Remark 1: Although we consider only excitation input r in the definition of identifiability, the disturbances v_e in (1) can also be taken into account as in [5]. Under some mild assumptions, disturbance inputs play a similar role as the excitation inputs. For the sake of simplicity we will not include such generalization in this paper.

Identifiability of a dynamic network model set reflects the ability to distinguish between models in the set $\mathcal{M}$ from measurement data, or more precisely, from the transfer matrix $[T]_{\mathcal{C},\mathcal{R}}$ as described in Definition 1. In this sense, network identifiability essentially depends on the presence and location of external excitation signals r and the selection of measured vertex signals y.

III. Dynamic Transpose Networks

In the context of network identifiability, dual settings of dynamic networks have been discussed in [5], [12], in which all vertices are either measured or excited. In this section, we present a generalization of the duality results for network identifiability with partially excited and partially measured vertices, which is based on the notion of transpose graphs. This generalization holds for general networks, including both cyclic and acyclic networks. The result in this section will be applied to derive the main results in Section IV.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a simple directed graph. The *transpose graph* of $\mathcal{G}$ is defined by $\mathcal{G}' := (\mathcal{V}, \mathcal{E}')$, where $\mathcal{E}'$ is obtained by reversing the direction of each edge in $\mathcal{E}$. Thus, the term 'transpose' is because the adjacency matrix of $\mathcal{G}'$ is the transpose of that of the original graph $\mathcal{G}$. In contrast to the network model (1) defined on $\mathcal{G}$, a dynamic transpose network model can be defined on the basis of $\mathcal{G}'$ as follows.

Definition 2 (Dynamic Transpose Network): Consider the dynamical model $M = (G, R, C)$ in (1) associated with a directed graph $\mathcal{G}$. A transpose network model of M is defined based on the transpose graph $\mathcal{G}'$ as

$$\tilde{M} = (\tilde{G}, \tilde{R}, \tilde{C}), \tag{5}$$

where $\tilde{G} = G^\top$, $\tilde{R} = C^\top$, and $\tilde{C} = R^\top$.

The transpose network in (5) is defined as the conceptually dual dynamical model of (1), and it has the same set of vertices with all of the edges reversed compared to the orientation of the corresponding edges in the original network (1). Furthermore, all the excited (measured) vertices in the original network become measured (excited) ones in the transpose network. The relation between the identifiability properties in the original and transpose networks can be derived (see the proof in Appendix A).

Lemma 1 (Duality in identifiability): Let $\mathcal{M}' := \{\tilde{M}(q,\theta) = (G(q,\theta)^\top, C^\top, R^\top), \theta \in \Theta\}$ be the parameterized model set of the transpose network (5). Then, the following statements hold.

1) A module G_{ij} is identifiable in $\mathcal{M}$ if and only if the module $\tilde{G}_{ji}$ is identifiable in $\mathcal{M}'$;
2) The network model set $\mathcal{M}$ is identifiable if and only if the network model set $\mathcal{M}'$ is identifiable.

In some circumstances where identifiability of the original network is difficult to analyze, the dual results in Lemma 1 can be applied, leading to a simpler analysis on the basis of the corresponding transpose network. We present the following example to illustrate our point.

Example 1: Consider the dynamic network in Fig. 1a, where a SIMO problem pertaining to the modules $G_{\mathcal{N}_1^+,1}$ is of interest. Here, the input w_1, and the outputs w_2, w_3 and w_4 are excited, while w_2 and w_5 are the measured.

The available tool in the current literature to analyze this problem is the disconnecting set-based approach in [24], by which it is required to determine identifiability of each module in $G_{\mathcal{N}_1^+,1}$ separately. For instance, to check the identifiability of G_{41} in the original network, according to [24, Thm.3] we need to find an excited vertex set $\bar{\mathcal{R}}$ and a measured vertex set $\bar{\mathcal{C}}$ such that a disconnecting set $\mathcal{D}$ from $\bar{\mathcal{R}} \cup \mathcal{N}_1^+ \cup \{4\}$ to $\bar{\mathcal{C}}$ is found to fulfill the constraints: $b_{\bar{\mathcal{R}}\to\mathcal{D}} = |\mathcal{D}|$ and $b_{\{4\}\cup\mathcal{D}\to\bar{\mathcal{C}}} = |\mathcal{D}| + 1$. In this example, we can choose $\bar{\mathcal{R}} = \{1,2,3\}$, $\bar{\mathcal{C}} = \{4,5,6,7\}$, and $\mathcal{D} = \{3,4,5\}$ for identifiability of G_{41}.

On the other hand, it follows from Lemma 1 that identifiability of $G_{\mathcal{N}_1^+,1}$ in the original network is equivalent to identifiability of $\tilde{G}_{1,\mathcal{N}_1^-}$ in its transpose network, shown in Fig. 1b. In the transpose network, identifying $\tilde{G}_{1,\mathcal{N}_1^-}$ becomes an MISO problem with $\tilde{w}_2$, $\tilde{w}_3$, $\tilde{w}_4$, and $\tilde{w}_5$ as the inputs and $\tilde{w}_1$ as the output:

$$[\tilde{T}]_{1,\tilde{\mathcal{R}}} = \tilde{G}_{1,\mathcal{N}_1^-}[\tilde{T}]_{\mathcal{N}_1^-,\tilde{\mathcal{R}}}, \tag{6}$$

where $\tilde{\mathcal{R}} := \{2,5,6,7\}$ and $\mathcal{N}_1^- = \{2,3,4,5\}$. The analysis of this MISO problem actually requires a simpler procedure than applying the analysis in [24]. Although vertex 5 in $\mathcal{N}_1^-$ is not measured, the mapping from $\tilde{\mathcal{R}}$ to vertex 5 in the transpose network can be obtained as follows.

We take the second row of the equation $(I-G)TR = R$, and permute it as

$$\begin{bmatrix} -G_{25} & 1 & 0 \end{bmatrix} \begin{bmatrix} [T]_{5,\tilde{\mathcal{R}}} \\ [T]_{2,\tilde{\mathcal{R}}} \\ \star \end{bmatrix} = [\tilde{R}]_{2,\star} = \mathbf{e}_2,$$

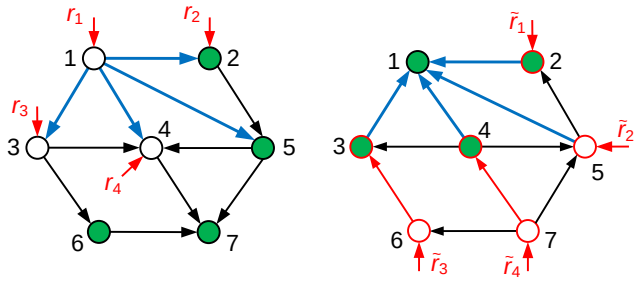
with $\mathbf{e}_2$ the second column vector in the identity matrix. The transfer function $[T]_{5,\tilde{\mathcal{R}}}$ thereby is represented as $[T]_{5,\tilde{\mathcal{R}}} = G_{25}^{-1}(\mathbf{e}_2 - [T]_{2,\tilde{\mathcal{R}}}) = [T]_{2,5}^{-1}(\mathbf{e}_2 - [T]_{2,\tilde{\mathcal{R}}})$. Therefore, in (6), the transfer matrix

$$[\tilde{T}]_{\mathcal{N}_1^-,\tilde{\mathcal{R}}} = \begin{bmatrix} [\tilde{T}]_{\{3,4\},\tilde{\mathcal{R}}} \\ [\tilde{T}]_{2,5}^{-1}(\mathbf{e}_2 - [\tilde{T}]_{2,\tilde{\mathcal{R}}}) \end{bmatrix}$$

can be acquired from measurement data. Since there exists a constrained set of four vertex disjoint paths from $\tilde{\mathcal{R}}$ to $\mathcal{N}_1^-$, $[\tilde{T}]_{\mathcal{N}_1^-,\tilde{\mathcal{R}}}$ is invertible [27]. Then, $\tilde{G}_{1,\mathcal{N}_1^-}$ can be uniquely obtained by multiplying the both sides of (6) by $[\tilde{T}]_{\mathcal{N}_1^-,\tilde{\mathcal{R}}}^{-1}$. This also implies identifiability of $G_{\mathcal{N}_1^+,1}$ in the original network from Lemma 1.

IV. Identifiability Analysis in Acyclic Networks

In this section, an identifiability problem is investigated for dynamic networks with acyclic topology. We present two sufficient conditions for identifiability in acyclic networks,



(a) Original Network (b) Transpose Network

Fig. 1: A dynamic network (a) and its transpose network (b). The filled green vertices are measured, while the excited vertices are indicated by the red arrows. Identifiability of G_{41} in the original network model set is equivalent to that of $\tilde{G}_{14}$ in the transpose network model set.

where the second condition leads to a synthesis approach for selecting locations of sensors and actuators to achieve network identifiability.

In the existing studies e.g., [12], [22], [27], [30], necessary conditions for identifiability for general networks have been provided in the setting of either full excitation or full measurement. A necessary condition for identifiability in the case of partial excitation and measurement can be immediately obtained from the existing results.

Proposition 1: Consider the network model set $\mathcal{M}$ in (2) with $\mathcal{R} \subseteq \mathcal{V}$ and $\mathcal{C} \subseteq \mathcal{V}$ the excited and measured vertices. If $\mathcal{M}$ is identifiable from $[T]_{\mathcal{C},\mathcal{R}}$, then

$$\text{Rank}\left([T]_{\mathcal{N}_j^-,\mathcal{R}}(q,\theta)\right) = |\mathcal{N}_j^-|, \tag{7a}$$

$$\text{and Rank}\left([T]_{\mathcal{C},\mathcal{N}_j^+}(q,\theta)\right) = |\mathcal{N}_j^+|, \tag{7b}$$

hold for each $j \in \mathcal{V}$ and for all $\theta \in \Theta$.

Note that the generic rank of a transfer matrix can be characterized using a graphical concept, called vertex-disjoint paths [12], [31]. Specifically, for any submatrix $T_{\mathcal{B},\mathcal{A}}$ of the parameterized transfer matrix $T(q,\theta)$ in (3),

$$\text{Rank}\left([T]_{\mathcal{B},\mathcal{A}}(q,\theta)\right) = b_{\mathcal{A}\to\mathcal{B}}, \tag{8}$$

holds for almost all $\theta \in \Theta$, where $b_{\mathcal{A}\to\mathcal{B}}$ is the maximum number of vertex-disjoint paths from $\mathcal{A} \subseteq \mathcal{V}$ to $\mathcal{B} \subseteq \mathcal{V}$. With the relation (8), the two rank conditions in (7) can be reinterpreted based on the underlying graph of the model set: If a network model set $\mathcal{M}$ in (2) is (generically) identifiable, then

$$b_{\mathcal{R}\to\mathcal{N}_j^-} = |\mathcal{N}_j^-|, \text{ and } b_{\mathcal{N}_j^+\to\mathcal{C}} = |\mathcal{N}_j^+|, \tag{9}$$

for each vertex $j \in \mathcal{V}$.

In the case that the vertices are either all excited or all measured, the conditions in (7) and (9) become sufficient for identifiability and generic identifiability, respectively, as proven in e.g., [12], [27], [30]. However, these conditions are not sufficient for determining identifiability of networks where not all the vertices are excited, and not all the vertices are measured.

A. Sufficient Condition for Identifiability in Acyclic Networks

By exploiting the structural property of acyclic graphs, we present a sufficient conditions for the identifiability analysis in dynamic networks with acyclic topology.

Notice that acyclic graphs do not contain directed cycles, and thus any path between two distinct vertices has a finite length. For simplicity, we refer to the transfer $G_{ji}(q)$ as a directed edge incident from i to j, and then the product of a sequence of transfer functions $G_{i_1 i_2} G_{i_2 i_3} \cdots G_{i_{n-2} i_{n-1}} G_{i_{n-1} i_n}$ is called a directed path from i_n to i_1 if each node i_k, $k = 1, 2, \cdots, n$, appears once on the path. The following property of the transfer matrix T of an acyclic network is then given.

Lemma 2: Consider the network model in (1) with the underlying acyclic graph $\mathcal{G}$. Then, the following statements hold.

1) $T_{ii} = 1$, for all vertices in $\mathcal{G}$;
2) $T_{ji} = 0$, if $j \notin \mathscr{R}_i$.
3) T_{ji} represents the sum of all the directed paths from i to j, for all $j \in \mathscr{R}_i$,

where the set $\mathscr{R}_i$ collects all the vertices that are reachable from vertex i.

For given sets of excited and measured vertices, identifiability of the parameterized modules in G essentially reflects if we can uniquely obtain G from the submatrix $[T]_{\mathcal{C},\mathcal{R}}$. Lemma 2 indicates that every nonzero entry T_{ji} in the matrix $[T]_{\mathcal{C},\mathcal{R}}$ corresponds to a set of directed paths from i to j. Based on that, we present an iterative procedure to check the identifiability of individual modules in an acyclic network, where we say a path is *unknown* at the k-th iteration if it contains at least one parameterized edge (module) whose identifiability has not been determined in the previous $k-1$ iterations.

Initially, all the paths containing parameterized edges are unknown, and we implement Proposition 2 to determine the identifiability of each parameterized module in the network, and those that are shown to be identifiable in this iteration will be considered as known in the subsequent iterations.

Proposition 2: Consider the network model set $\mathcal{M}$ in (2) with $\mathcal{R} \cup \mathcal{C} = \mathcal{V}$ and all the models satisfying Assumption 1. Let $\mathcal{P}$ be the set of unknown paths from an excited vertex i to a measured vertex j at the k-th iteration. Then, $G_{\mu\nu}$ is identifiable in $\mathcal{M}$ from $T_{\mathcal{C},\mathcal{R}}$ at the k-th iteration if the directed edge (ν, μ) appears in all the paths in $\mathcal{P}$.

The detailed proof of Proposition 2 is provided in Appendix C. The condition in Proposition 2 is implemented with an iterative procedure. At the first iteration, the identifiability of the parameterized module $G_{\mu\nu}$ requires to find find an excited vertex i and a measured vertex j such that all the paths from i to j pass through the edge (ν, μ). If $G_{\mu\nu}$ is shown to be identifiable, then it becomes a known module for the latter check. At the following iterations, we need to find an edge (ν', μ') such that all the paths that do not pass through (ν', μ') contain only known modules, the module $G_{\mu'\nu'}$ is identifiable and hence known for the subsequent iterations.

Special results of Proposition 2 are provided as follow.

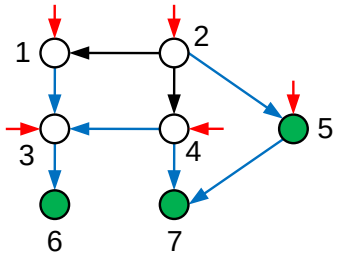


Fig. 2: An acyclic network, in which identifiability of all the modules can be verified by using Proposition 2.

Corollary 1: Consider the network model set $\mathcal{M}$ in (2) with $\mathcal{R} \cup \mathcal{C} = \mathcal{V}$ and all the models satisfying Assumption 1. Then the following statements hold.

1) If there exists a unique directed path from an excited vertex i to a measured vertex j, then all the modules in the path are identifiable in $\mathcal{M}$ from T_{ji}.
2) A tree network is identifiable if and only if the root is excited, and all the leaves are measured [2].

The first statement is obtained directly from Proposition 2. The second statement has been presented in [2], while it can be proved alternatively in an easy way by using the first statement, as there is always a unique directed path from the root to every leaf in a tree network. For some acyclic networks, the results in Proposition 2 and Corollary 1 provide an efficient analysis tool for identifiability of single modules or even a full network.

Example 2: With this example, we demonstrate the use of Proposition 2 and Corollary 1 in the acyclic network shown in Fig. 2, where vertices in $\mathcal{R} = \{1, 2, 3, 4\}$ are excited, and vertices in $\mathcal{C} = \{5, 6, 7\}$ are measured. We consider the identifiability from the submatrix $[T]_{\mathcal{C},\mathcal{R}}$.

Observe that T_{61} represents a unique path from vertex 1 to vertex 6, then both G_{31} and G_{63} are identifiable in $\mathcal{M}$ from T_{61} due to condition (1) in Corollary 1. Similarly, the mappings T_{52}, T_{64}, T_{74}, T_{75} represent unique paths, which leads to identifiability of modules G_{52}, G_{34}, G_{74}, and G_{75}, respectively.

Now, all the modules indicated by the blue color are identifiable and thus become known. It then follows from Proposition 2 that G_{42} is identifiable in $\mathcal{M}$ from T_{72}, since the edge $(2, 4)$ appears in the only unknown path from vertex 2 to vertex 7. With the identifiability of G_{42}, the module G_{12} is the only unknown module in $\mathcal{M}$, whose identifiability can be checked by the paths from 2 to 6. Therefore, the entire network is identifiable from $[T]_{\mathcal{C},\mathcal{R}}$.

The conditions in Proposition 2 and Corollary 1 require to find a specific pair of vertices for checking identifiability in each iteration. However, in general acyclic graphs, such conditions may not be used to check the identifiability of all the modules. Next, we further develop a condition for determining identifiability of the model set of an acyclic network in the general case. The following main result is provided to determine identifiability based on a vertex-wise inspection.

Theorem 1: Consider the network model set $\mathcal{M}$ in (2) with all the models satisfying Assumption 1. Let $\mathcal{R} \subseteq \mathcal{V}$ and $\mathcal{C} \subseteq \mathcal{V}$

be the sets of excited and measured vertices such that $\mathcal{R} \cup \mathcal{C} = \mathcal{V}$. The model set $\mathcal{M}$ is identifiable from $[T]_{\mathcal{C},\mathcal{R}}$ if the following conditions hold:

1) Every measured vertex j in the network satisfies

$$\text{Rank}\left([T]_{\mathcal{N}_j^-,\mathcal{R}}\right) = |\mathcal{N}_j^-|; \tag{10}$$

2) For each unmeasured (and excited) vertex j, there exist a set of measured vertices $\mathcal{C}_j$ and a set of excited vertices $\mathcal{R}_j$ such that $\widehat{\mathcal{N}}_j^- \subset \mathcal{C}_j$, $j \in \mathcal{R}_j$, and

$$\text{Rank}\left([T]_{\mathcal{C}_j,\mathcal{R}_j}\right) = |\mathcal{R}_j|, \tag{11}$$
$$\text{Rank}\left([T]_{\mathcal{C}_j,(\mathcal{R}_j\cup\mathcal{S}_j)\backslash j}\right) = |\mathcal{R}_j| - 1, \tag{12}$$

where $\widehat{\mathcal{N}}_j^- \subseteq \mathcal{N}_j^-$ collects all the unexcited in-neighbors of j, and $\mathcal{S}_j := \widehat{\mathcal{N}}_j^- \cup \left(\bigcup_{i\in\mathcal{N}_j^-} \mathcal{N}_i^+\right)$.

In Theorem 1, we leave out the dependence on the delay operator q and parameters θ in the transfer matrices for ease of notation, while the equations (10), (11), and (12) should hold for all $\theta \in \Theta$. The proof is given in Appendix D. The conditions in Theorem 1 are now discussed. The two conditions provide vertex-wise check for measured and unmeasured vertices in the network separately. Condition (1) requires a sufficient number of excitation signals to the in-neighbors of each measured vertex, which coincides with the rank condition for network identifiability in the full-measurement case [30]. Condition (2), on the other hand, is presented for checking the MISO problem $G_{j,\mathcal{N}_j^-}$ regarding each unmeasured vertex j in the network. To check condition (2), we require to find a set of excited vertices $\mathcal{R}_j$ and a set of measured vertices $\mathcal{C}_j$, which includes j and all the unexcited in-neighbors of j, respectively, and furthermore, $\mathcal{R}_j$ and $\mathcal{C}_j$ should satisfy the two rank requirements in (11) and (12).

We then illustrate the two constraints in (11) and (12) by using Fig. 3, where $\mathcal{R}_j = \{j, r_1, r_2, r_3, r_4, r_5\}$ and $\mathcal{C}_j = \{k, c_1, c_2, c_3, c_4, c_5\}$. From [3], [12], the full column rank of $[T]_{\mathcal{C}_j,\mathcal{R}_j}$ in (11) implies that there are $|\mathcal{R}_j| = 6$ vertex-disjoint paths from $\mathcal{R}_j$ to $\mathcal{C}_j$, indicated by the red dashed arrows, in which there is a directed path from j to a measured vertex $k \in \mathcal{C}_j$. Besides, the columns of matrix $[T]_{\mathcal{C}_j,(\mathcal{R}_j\cup\mathcal{S}_j)\backslash j}$ in (12) contains two compartments: $[T]_{\mathcal{C}_j,\mathcal{R}_j\backslash j}$ and $[T]_{\mathcal{C}_j,\mathcal{S}_j\backslash j}$, with $\mathcal{S}_j = \{j, c_1, c_2, n_1, n_2, n_3\}$ collecting all the unexcited in-neighbors of j, i.e. $\widehat{\mathcal{N}}_j^- = \{c_1, c_2\}$, and all the out-neighbors of j's in-neighbors $\bigcup_{i\in\mathcal{N}_j^-} \mathcal{N}_i^+ = \{j, n_1, n_2, n_3\}$. From (11) and (12), we have

$$\begin{aligned}\text{Rank}\left([T]_{\mathcal{C}_j,\mathcal{R}_j\backslash j}\right) &= |\mathcal{R}_j| - 1 \\ &= \text{Rank}\left(\left[[T]_{\mathcal{C}_j,\mathcal{R}_j\backslash j} \quad [T]_{\mathcal{C}_j,\mathcal{S}_j\backslash j}\right]\right).\end{aligned} \tag{13}$$

It is implied from (8) that there are no vertex-disjoint paths from $\mathcal{S}_j\backslash j$ to $\mathcal{C}_j$ in the graph that are vertex-disjoint with the ones from $\mathcal{R}_j\backslash j$ to $\mathcal{C}_j$. Further, it can be interpreted using the concept of disconnecting set. If (13) holds, there is disconnecting set $\mathcal{D}$ with $|\mathcal{D}| = |\mathcal{R}_j| - 1$, whose removal will lead to no paths from $\mathcal{R}_j\backslash j$ to $\mathcal{C}_j$ and from $\mathcal{S}_j\backslash j$ to $\mathcal{C}_j$, simultaneously. In Fig. 3, this set can be chosen as $\mathcal{D} = \{c_1, c_2, c_3, r_4, r_5\}$. Note that the unexcited in-neighbors of j, i.e. $\widehat{\mathcal{N}}_j^-$, are contained in $\mathcal{C}_j$, $\mathcal{S}_j$, and $\mathcal{D}$ simultaneously.

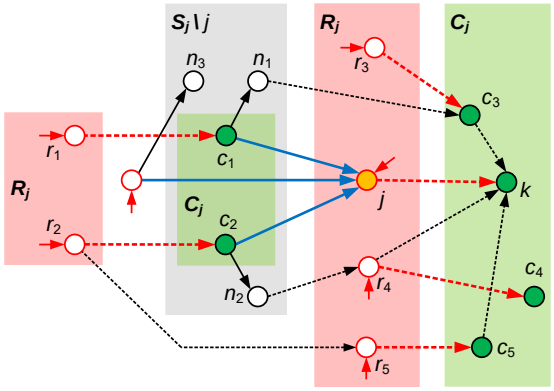


Fig. 3: An illustration of the second condition in Theorem 1, where the green circles indicate measured vertices, and dashed arrows represent paths.

The rank conditions in Theorem 1 can also be represented by means of vertex disjoint paths for generic identifiability of $\mathcal{M}$ as in [3], [12], [24], [31]. Specifically, the following graph-based result is obtained directly from Theorem 1.

Corollary 2: Consider the network model set $\mathcal{M}$ in (2) with $\mathcal{R}$ and $\mathcal{C}$ the sets of excited and measured vertices such that $\mathcal{R} \cup \mathcal{C} = \mathcal{V}$. The model set $\mathcal{M}$ is generically identifiable from $[T]_{\mathcal{C},\mathcal{R}}$ if the following conditions hold:

1) Every measured vertex j in the network satisfies

$$b_{\mathcal{R}\to\mathcal{N}_j^-} = |\mathcal{N}_j^-|; \tag{14}$$

2) For every edge unmeasured but excited vertex j, there exist a set of measured vertices $\mathcal{C}_j$ and a set of excited vertices $\mathcal{R}_j$ such that $\widehat{\mathcal{N}}_j^- \subset \mathcal{C}_j$, $j \in \mathcal{R}_j$, and

$$b_{\mathcal{R}_j\to\mathcal{C}_j} = |\mathcal{R}_j|, \tag{15}$$
$$b_{(\mathcal{R}_j\cup\mathcal{S}_i)\backslash j\to\mathcal{C}_j} = |\mathcal{R}_j| - 1, \tag{16}$$

where $\widehat{\mathcal{N}}_j^- \subseteq \mathcal{N}_j^-$ collects all the unexcited in-neighbors of j, and $\mathcal{S}_j := \widehat{\mathcal{N}}_j^- \cup \left(\bigcup_{i\in\mathcal{N}_j^-} \mathcal{N}_i^+\right)$.

It is implied from Theorem 1 and Corollary 2 that all the sources have to be excited and all the sinks have to be measured. This is actually necessary for the identifiability of a network model set in the partial excitation and measurement setting, see more details in [2], [3]. Moreover, we can show that if condition (2) in Corollary 2 is satisfied, then the following result holds, see the proof in Appendix E.

Proposition 3: Consider the network model set $\mathcal{M}$ in (2) with $\mathcal{R}$ and $\mathcal{C}$ the sets of excited and measured vertices. If (15) and (16) hold, then

$$b_{\mathcal{R}\to\mathcal{N}_j^-} = |\mathcal{N}_j^-|, \ \forall\ j \in \mathcal{R}, j \notin \mathcal{C}, \tag{17}$$

The condition (17) holds for all unmeasured and excited vertices in the network, and combining with (14) leads to one of the necessary condition in (9).

In the following example, we illustrate how to determine identifiability on the basis of network topology.

Example 3: This example demonstrates how to implement the conditions in Corollary 2 to check network identifiability

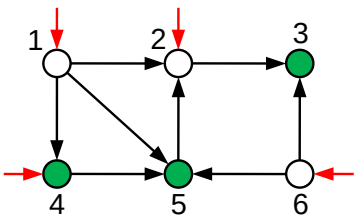


Fig. 4: An acyclic network, in which all the vertices satisfy the conditions in Theorem 1. Thus, the network model set is identifiable.

of the model set $\mathcal{M}$ of an acyclic network in Fig. 4. First, it is not hard to verify that each measured vertex in $\mathcal{C} = \{3,4,5\}$ satisfies (14), with $\mathcal{R} = \{1,2,4,6\}$. Then, to verify generic identifiability of $\mathcal{M}$, we implement condition (2) in Corollary 2 to check the unmeasured vertex 1, 2, and 6.

Note that for the source vertices $j = 1,6$, we have $\mathcal{N}_j^- = \varnothing$, which implies $\mathcal{S}_j = \varnothing$. Thus, the condition (2) in Corollary 2 becomes: $b_{\mathcal{R}_j \to \mathcal{C}_j} = |\mathcal{R}_j|$ and $b_{\mathcal{R}_j \setminus j \to \mathcal{C}_j} = |\mathcal{R}_j| - 1$, which are satisfied by simply choosing $\mathcal{R}_j = \{j\}$ and $\mathcal{C}_j = \{5\}$. Next, vertex 2 is checked, which has the in-neighbors $\mathcal{N}_2^- = \{1,5\}$ and $\widehat{\mathcal{N}}_2^- = \{5\}$ unexcited. Therefore, we have $\mathcal{S}_2 = \widehat{\mathcal{N}}_2^- \cup \left(\bigcup_{i \in \mathcal{N}_2^-} \mathcal{N}_i^+\right) = \{2,4,5\}$. Consider $\mathcal{R}_2 = \{2,4\}$ and $\mathcal{C}_2 = \{3,5\}$ that satisfy (15), i.e. $b_{\mathcal{R}_j \to \mathcal{C}_j} = 2$. It can be verified that $b_{(\mathcal{R}_2 \cup \mathcal{S}_2)\setminus 2 \to \mathcal{C}_2} = b_{\{4,5\} \to \{3,5\}} = 1$, i.e. condition (2) in Corollary 2 is also fulfilled. As a result, all the vertices in this network satisfy the conditions in Corollary 2, and thus the model set $\mathcal{M}$ is generically identifiable from $[T]_{\mathcal{C},\mathcal{R}}$.

To prove the result in Theorem 1, we utilize the hierarchical structure in an acyclic network that partitions the vertices into different layers. We refer to Appendix D for the detailed reasoning. The conditions in Theorem 1 is a result of vertex-wise analysis starting from the top layer to the lower ones. It is worth noting that (10) is also a necessary condition for measured vertices, while for every unmeasured vertex, condition (2) in Theorem 1 needs to be satisfied to guarantee that each module with this vertex as an output is identifiable in $\mathcal{M}$. In contrast to Theorem 1, we can also apply a dual reasoning that starts from the measured bottom layer in an acyclic network. Then, we would find an alternative condition for identifiability as follows.

Corollary 3: Consider the network model set $\mathcal{M}$ in (2). Let $\mathcal{R} \subseteq \mathcal{V}$ and $\mathcal{C} \subseteq \mathcal{V}$ be the sets of excited and measured vertices such that $\mathcal{R} \cup \mathcal{C} = \mathcal{V}$. The model set $\mathcal{M}$ is identifiable from $[T]_{\mathcal{C},\mathcal{R}}$ if the following conditions hold:

1) Every excited vertex i in the network satisfies
$$\text{Rank}\left([T]_{\mathcal{C},\mathcal{N}_i^+}\right) = |\mathcal{N}_i^+|;$$
2) For each unexcited vertex i which is measured, there exist sets of measured vertices $\mathcal{C}_i$ and excited vertices $\mathcal{R}_i$ such that $\widehat{\mathcal{N}}_i^+ \subset \mathcal{R}_i$, $i \in \mathcal{C}_j$, and
$$\text{Rank}\left([T]_{\mathcal{C}_i,\mathcal{R}_i}\right) = |\mathcal{C}_i|,$$
$$\text{Rank}\left([T]_{(\mathcal{C}_i \cup \mathcal{S}_i)\setminus i,\mathcal{R}_i}\right) = |\mathcal{C}_i| - 1,$$
where $\widehat{\mathcal{N}}_i^+ \subseteq \mathcal{N}_i^+$ collects all the unmeasured out-neighbors of i, and $\mathcal{S}_i := \widehat{\mathcal{N}}_i^+ \cup \left(\bigcup_{j \in \mathcal{N}_i^+} \mathcal{N}_j^-\right)$.

The results can be proved by directly applying Lemma 1, considering the transpose network of $\mathcal{M}$, and thus is omitted here. Similarly, we can extend the rank conditions in Corollary 3 to path-based conditions for generic identifiability of $\mathcal{M}$ as in Corollary 2, where we have $b_{\mathcal{N}_i^+ \to \mathcal{C}} = |\mathcal{C}_i|$ and $b_{\mathcal{R}_i \to \mathcal{C}_i} = b_{\mathcal{R}_i \to (\mathcal{C}_i \cup \mathcal{S}_i)\setminus i} + 1 = |\mathcal{C}_i|$ instead for the two conditions in Corollary 3.

Remark 2: All the rank conditions in Theorem 1 and Corollary 3 can be regarded as a generalization of the conditions in the full measurement case [30] or full excitation case [12], [27]. If all the vertices are measured, only (10) is required. When all the vertices are excited, we turn to verify condition (2) for each vertex i. Moreover, we can reformulate (11) and (12) by considering the identifiability of the modules $G_{\mathcal{N}_i^+,i}$ in the MISO setting. To this end, j in (12) is replaced by $\mathcal{N}_i^+$ that directly leads to (12). Now, we have $\text{Rank}([T]_{\mathcal{C}_j,\mathcal{R}_j}) = |\mathcal{R}_j|$ for some sets $\mathcal{C}_j \subseteq \mathcal{C}$ and $\mathcal{R}_j \subseteq \mathcal{R}$, where $\mathcal{N}_i^+ \subseteq \mathcal{R}_j$. Clearly, it is equivalent to $\text{Rank}([T]_{\mathcal{C},\mathcal{N}_i^+}) = |\mathcal{N}_i^+|$, that is necessary and sufficient condition for network identifiability in the full excitation case.

B. Allocation of Actuators and Sensors in Acyclic Networks

In this part, we present a synthesis scheme to allocate excitation and measurement signals for identifiability of an acyclic network model set. The rank condition in Theorem 1 and Corollary 3 have provided an instrumental tool for verifying identifiability of an acyclic network model set. However, these conditions are established based on a vertex-wise analysis, which is less effective for a synthesis problem. Thus, it is desired to have a more compact identifiability condition for an acyclic network.

To this end, we resort to a graph covering approach as in [5]. In [5], all the vertices in a directed network are measured, and the allocation scheme therein is to decompose the underlying graph of the network into edge-disjoint *pseudotrees*, whose roots are supposed to be excited for generic identifiability. In this work, we extend the work in [5] to the partial excitation and measurement case. To cover an acyclic network, we simply use a set of disjoint directed trees.

A *directed tree*, denoted by $\mathcal{T}$, is a special acyclic graph containing a unique *root* vertex, from which there is exactly one directed path to every other vertex in $\mathcal{T}$. The sinks in $\mathcal{T}$ are also called *leaves*, and the vertices that are neither the root nor leaves of $\mathcal{T}$ are *internal vertices*. Analogously, an anti-tree $\breve{\mathcal{T}}$ is defined as such each vertex in $\breve{\mathcal{T}}$ has exactly one directed path to a unique *root* vertex, while all the sources in $\breve{\mathcal{T}}$ are called *leaves*.

Then, we define the concept of disjoint trees.

Definition 3 (Disjoint trees): In a directed acyclic graph $\mathcal{G}$, two trees $\mathcal{T}_1$ and $\mathcal{T}_2$ are called **disjoint** if

1) $\mathcal{T}_1$ and $\mathcal{T}_2$ do not share edges, and
2) the edges incident from a common vertex in $\mathcal{T}_1$ or $\mathcal{T}_2$ are included in the same tree.

In parallel with disjoint trees, we can define disjoint anti-trees. Two anti-trees $\mathcal{T}_1$ and $\mathcal{T}_2$ are *disjoint* if $\mathcal{T}_1$ and $\mathcal{T}_2$ do

not share edges, the edges incident to a common vertex in $\mathcal{T}_1$ or $\mathcal{T}_2$ are included in the same anti-tree.

It should be emphasized that any acyclic graph can be decomposed into a set of disjoint trees or disjoint anti-trees, namely, there always exist a disjoint tree/anti-tree covering all the edges of an acyclic graph, and such covering is not unique. Based on that, we present a new identifiability condition for acyclic networks as follows.

Theorem 2 (Tree/Anti-Tree Covering): Consider a network model set $\mathcal{M}$ associating with an acyclic graph $\mathcal{G}$ with $\mathcal{V} = \mathcal{R} \cup \mathcal{C}$. Then, $\mathcal{M}$ is generically identifiable if either of the following two conditions holds:

1) $\mathcal{G}$ can be decomposed into a set of disjoint trees, and for each tree, its root is excited and all the leaves are measured.
2) $\mathcal{G}$ can be decomposed into a set of disjoint anti-trees, and for each anti-tree, its root is measured, and all the leaves are excited.

The detailed proof is presented in Appendix F. Note that the second condition in Theorem 2 can be proven using the results in Section III on transpose networks, but it is worth noting that the two conditions in Theorem 2 are dual but not equivalent. Even if there does not exist a disjoint tree covering obeying condition (1), we may still find a set of disjoint anti-trees that covers all the edges of $\mathcal{G}$ and satisfies condition (1). Thus, the two conditions are actually complementary to each other. The following example is given to demonstrate this point.

Example 4: Consider a model set $\mathcal{M}$ governed by the dynamic network in Fig. 5a. It can be found that there are two disjoint trees highlighted by the blue and red colors, covering all the edges of the graph, while their roots are excited, and every leaf is measured. Therefore, it follows from condition (1) in Theorem 2 that $\mathcal{M}$ is generically identifiable. Note that we can also measure the vertices $2, 3, 4$, then they are not necessarily excited. However, we cannot apply condition (2) in this scenario as there is no set of disjoint anti-trees with measured roots and excited leaves. If we change the network to Fig. 5b where vertex 4 becomes measured, then two disjoint anti-trees are found indicated by the blue and red colors where both anti-trees have measured roots and excited leaves, so that condition (2) is applied to yield generic identifiability of $\mathcal{M}$ in this case.

Compared to the sufficient conditions in Theorem 1, the ones in Theorem 2 are more conservative in the sense of identifiability analysis. This conservativeness is embodied in the following facts:

1) If condition (1) of Theorem 2 holds, then $|\mathcal{N}_j^-| \leqslant 1$ for every unmeasured vertex j in the network;
2) If condition (2) of Theorem 2 holds, then $|\mathcal{N}_i^+| \leqslant 1$ for every unexcited vertex i in the network.

These necessary conditions can be verified by using the disjointness property of trees and anti-trees, and they are implicitly required in the theorem. Despite the conservativeness compared to Theorem 1, Theorem 2 provides more compact conditions on the level of a full graph rather than a vertex-wise analysis. This will facilitate a synthesis procedure towards the signal allocation problem in acyclic networks.

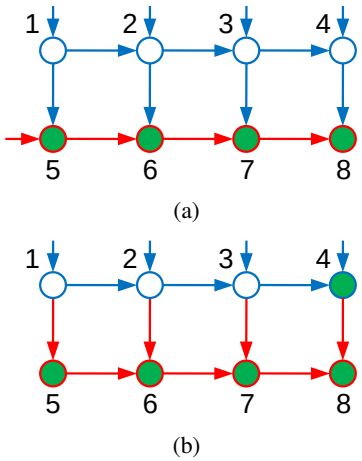


Fig. 5: Generic identifiability analysis of an acyclic network using Theorem 2. (a) The network is decomposed into two disjoint trees, indicated by different colors, and each tree has a excited root and measured leaves. (b) An anti-tree covering is found in a network with the same topology as in (a), where each anti-tree has a measured root and excited leaves.

Algorithm 1 Tree Covering of Acyclic Graphs

Input: An acyclic graph $\mathcal{G}$

1: Decompose $\mathcal{G}$ into a set of minimal trees.
2: **repeat**
3: Merge two trees into a single tree if their union remains a tree.
4: **until** There is no mergeable trees.

C. Algorithm

Given an acyclic network without any excitations and measurements, the synthesis problem in this paper is to *allocating a minimum number of actuators and sensors, i.e.,* $|\mathcal{R}| + |\mathcal{C}|$, *to achieve generic identifiability* of the network.

To tackle this problem, we devise a graph-theoretical algorithm to allocate actuators and sensors on the basis of Theorem 2. The main steps follows similarly as in [5], where we have presented a heuristic algorithm to find a set of disjoint pseudotrees to cover all the edges in a cyclic graph. In the following, we devise a two-step scheme for signal allocation in acyclic networks. The detailed procedure of is now elaborated, with an illustration of an example in Fig. 6.

(1) *Tree Covering.* For any acyclic graph, we can always find a set of disjoint trees covering all its edges. Due to the identifiability condition in Theorem 2, it is desirable to have a covering with a minimum number of trees. However, the this minimal covering problem is a combinatorial optimization problem, whose solution is not unique. Therefore, we adopt a heuristic algorithm to find a local optimal solution to the tree covering problem, see Algorithm 1.

In Algorithm 1, a minimal tree is a tree that consists of a vertex and all its out-neighbors with all the edges incident from the vertex to the out-neighbors. Note that any acyclic graph admits such a decomposition, and any two minimal trees which share common vertices are disjoint [5]. In Fig. 6a,

a graph of 20 vertices is partitioned into 16 minimal trees indicated by different colors. Starting from the minimal tree covering, the merging step will reduce the total number of trees required to cover the graph. For the graph in Fig. 6a, a tree covering is generated, shown in Fig. 6b, which contains only 7 disjoint trees. It can be verified that for this specific example, the resulting tree covering has a minimum cardinality.

Remark 3: Note that the result of the tree covering for a given graph is dependent on the order of merging. The minimum covering is not always guaranteed, while a local minimum can be achieved. In [5], a particular order of merging is proposed to yield an effective solution to the minimum covering problem. The details therefore is omitted in this paper.

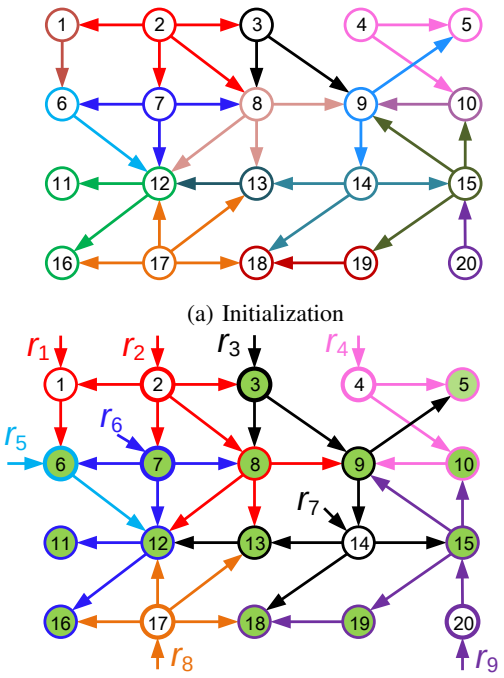
(2) *Signal Allocation.* A direct application of condition (1) in Theorem 2 implies that we can allocate excitation signals at the roots of the trees in the resulting covering and measure all the leaves of these trees, while the rest of vertices in the graph can be either excited or measured. Condition (1) in Theorem 2 then guarantees the generic identifiability of the model set of this acyclic network.

Note that if two or more trees have a vertex in common, this shared vertex is required to be measured, which may lead to a set of measured roots in the trees, coinciding with $\mathcal{R} \cap \mathcal{C}$, while the other vertices in $\mathcal{G}$ are either excited or measured. For the network in Fig. 6b, the roots $\mathscr{R}_t = \{2, 3, 4, 6, 7, 17, 20\}$ are to be excited, and the leaves $\mathscr{L}_t = \{3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 18\}$ are to be measured. The remaining vertices 1, 14 and 19 are internal vertices in the trees that can be either excited or measured. In this example, we choose to excite vertices 1 and 14, and measure vertex 19. Then, the network model set is generically identifiable due to Theorem 2.

Observe that when $|\mathcal{R}| + |\mathcal{C}|$ is minimized, it equivalently yields a minimum number of vertices that are excited and measured simultaneously, i.e. $|\mathcal{R} \cap \mathcal{C}|$ is minimized. To further reduce the number of required signals, a subsequent operation is taken to check the redundant excitation signals assigned to the vertices in the set $\mathcal{R} \cap \mathcal{C}$ by using Corollary 2. Removing these unnecessary excitation signals further gives a smaller cardinality of $\mathcal{R} \cap \mathcal{C}$.

Specifically, we inspect each tree $\mathcal{T}_k$ whose root is an element in $\mathcal{R} \cap \mathcal{C}$ that is excited and measured simultaneously. If the network model set remains generic identifiable after removing the excitation signal on the root of $\mathcal{T}_k$, i.e. all measured vertices in $\mathcal{T}_k$ still satisfy (14), and all the excited vertices in $\mathcal{T}_k$ fulfill the conditions in (15) and (16), then we can take out the excitation signal allocated at the root of $\mathcal{T}_k$.

We consider the network in Fig. 6b as an example. Observe that $\mathcal{R} \cap \mathcal{C} = \{3, 6, 7\}$, which are all measured root vertices. If we remove r_5 on vertex 6, the resulting network will remains generic identifiable since $b_{\mathcal{R}\setminus\{6\} \to \mathcal{N}_k^-} = |\mathcal{N}_k^-|$ holds for both measured vertex $k = 7$ and 8 in the cyan tree. Following a similar reasoning, we can also remove r_6 by analyzing vertex 7 as the root of the blue tree. Therefore, we can delete 6 and 7 in $\mathcal{R}$. Note that we cannot remove r_3 in the black tree, since vertex 9 in this tree has four in-neighbors and thus requires four vertex-disjoint paths from the excited vertices to its in-neighbors, which will not happen if r_3 is removed.



(a) Initialization

(b) Tree Covering and Signal Allocation

Fig. 6: Illustration of the tree covering method for actuator/sensor allocation. (a) An acyclic graph is covered by a set of disjoint minimal trees, which are indicated by different colors. (b) A tree covering with a minimum cardinality. The roots and leaves of the trees are to be excited and measured, respectively. Furthermore, r_5 and r_6 are redundant for generic identifiability by applying Corollary 2 and thus can be removed.

Remark 4: The above procedure utilizes the first condition in Theorem 2 to find the two potential sets $\mathcal{R}$ and $\mathcal{C}$ for allocating actuators and sensors, respectively. Alternatively, we can also design a different scheme according to the second condition in Theorem 2, where an anti-tree covering of $\mathcal{G}$ is considered. Particularly, we first seek for a minimum set of disjoint anti-trees that cover all the edges of $\mathcal{G}$ and then allocate sensors and actuators at the roots and leaves of the anti-trees, respectively. To pursue a minimum number of measured vertices, we apply Corollary 3 to check if the vertices in $\mathcal{R} \cap \mathcal{C}$ are necessary to be measured. We apply the anti-tree covering scheme to the network in Fig. 6 and obtain a set of anti-trees as shown in Fig. 7, where the roots of the anti-trees, i.e. $\mathscr{R}_t = \{5, 6, 10, 11, 13, 16, 18, 19\}$, are measured, and the all the leaves in $\mathscr{L}_t = \{2, 4, 7, 8, 10, 12, 13, 14, 15, 17, 20\}$ are excited. The internal vertices include 1, 9, 19 can be either excited or measured. Thereby, we have $\mathcal{R} \cap \mathcal{C} = \{6, 10, 13\}$. Notice that $b_{\mathcal{N}_k^+ \to \mathcal{C}\setminus\{6\}} = |\mathcal{N}_k^+|$ holds for all the vertices in the tree rooted at vertex 6. Therefore, we can further remove 6 from $\mathcal{C}$, and the network model set will remain generic identifiability.

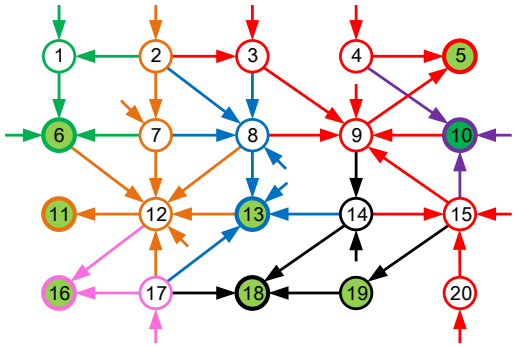


Fig. 7: Illustration of the anti-tree covering method for actuator/sensor allocation. The acyclic graph in Fig. 6 is now covered by a set of disjoint anti trees, which are indicated by different colors. In each anti-tree, the root is measured, and the leaves are excited. Furthermore, it can be verified that the measurement at vertex 6 is redundant for generic identifiability and thus can be removed.

V. Conclusions

In this paper, we have analyzed identifiability in acyclic dynamic networks where only partial excitation and measurement signals are available. The concept of transpose networks is introduced, whose identifiability is shown to be equivalent to identifiability of its corresponding original network. We have also presented a necessary condition for identifiability of general networks, where the identifiability problem is interpreted as solving a system of nonlinear equations with the parameterized modules as indeterminate variables.

The other major contribution of this paper is to present two sufficient conditions for identifiability of acyclic networks. The first condition can be implemented to analyze identifiability of a given acyclic network based on a vertex-wise check. The second condition, on the other hand, is established using the concept of disjoint tree/anti-tree covering and is instrumental to develop an algorithmic procedure that selects excitation and measured signals such that an acyclic network is generically identifiable.

Appendix

A. Proof of Lemma 1

From Definition 1, a module G_{ij} is identifiable in the model set $\mathcal{M}$ if and only if the following implication holds:

$$CT(q,\theta_0)R = CT(q,\theta_1)R \Rightarrow G_{ij}(q,\theta_0) = G_{ij}(q,\theta_1), \quad (18)$$

for all $\theta_0, \theta_1 \in \Theta$. Analogously, the network identifiability of a module $\tilde{G}_{ji}$ in the model set $\mathcal{M}(\mathcal{G}')$ is equivalent to the implication

$$\begin{aligned} R^\top(I-\tilde{G}(q,\theta_0))^{-1}C^\top &= R^\top(I-\tilde{G}(q,\theta_1))^{-1}C^\top \\ &\Rightarrow \tilde{G}_{ji}(q,\theta_0) = \tilde{G}_{ji}(q,\theta_1), \end{aligned} \quad (19)$$

Observe that $(CTR)^\top = R^\top(I-\tilde{G})^{-1}C^\top$, which is the transfer matrix of the transpose network. Therefore, the equations on the left hand side of equations (18) and (19) are equivalent, from which the first claim immediately follows. The second statement on the overall model set is obtained by applying the above reasoning to all the modules in the network (1).

B. Proof of Lemma 2

The analytic function T can be expanded in the Taylor series as

$$T = (I-G)^{-1} = I + \sum_{k=1}^{\infty} G^k. \quad (20)$$

To prove the two statements in the lemma, it is sufficient to show that, for any $k \geqslant 1$, $[G^k]_{ii} = 0$, and $[G^k]_{ij}$ is the sum of all directed paths from j to i of length k.

We proceed the proof by induction on k. For $k = 1$, the result holds immediately, since $G^1 = G$, where the main diagonal entries are all 0, while each nonzero off-diagonal entry G_{ij} is the path of length 1 from j to i. Assume now that the inductive hypothesis holds for some $k \geqslant 1$. Note that any path of length $k+1$ from j to i, with $i \neq j$, consists of an edge G_{iz} for some vertex $z \in \mathcal{N}_i^-$ and a path of length k from j and z. It then gives

$$[G^{k+1}]_{ij} = \sum_{z \in \mathcal{N}_i^-} G_{iz}[G^k]_{zj}, \quad (21)$$

where $[G^k]_{zj}$ represents the sum of all directed paths of length k from j to z. Therefore, the expression of $[G^{k+1}]_{ij}$ in (21) implies the sum of all directed paths of length $k+1$ from j to i. For the case that $i = j$ in (21), $[G^k]_{zi} = 0$ holds since $\mathcal{G}$ is acyclic, e.g., there is no directed circle from vertex i to itself, which leads to $[G^{k+1}]_{ii} = 0$. That completes the proof.

C. Proof of Proposition 2

If the condition in Proposition 2 holds, the mapping T_{ji}, without loss of generality, can be represented as

$$T_{ji} = T_{j\mu}G_{\mu\nu}T_{\nu i} + T_{\text{rem}}, \quad (22)$$

where $T_{j\mu}$ and $T_{\nu i}$ include all the paths from μ to j and from i to ν, respectively; T_{rem} represents all the paths from i to j that exclude the ones in $\mathcal{P}$ and do not contain the edge (ν, μ). It implies that all the elements in T_{rem} are known from the previous $k-1$ iterations. Note that the set $\mathcal{P}$ collects all the unknown paths corresponding to the transfer matrix $T_{j\mu}G_{\mu\nu}T_{\nu i}$, which is known and computed as $T_{ji} - T_{\text{rem}}$.

Then, the identifiability of $G_{\mu\nu}$ in $\mathcal{M}$ is analyzed under four situations. First, if vertex ν is measured, and μ is excited, then $T_{j\mu}$ and $T_{\nu i}$ can be obtained so that $G_{\mu\nu} = T_{j\mu}^{-1}(T_{ji} - T_{\text{rem}})T_{\nu i}^{-1}$. Second, if vertex ν is excited, and μ is measured, then $G_{\mu\nu}$ can be directly obtained from the mapping $T_{\mu\nu}$, yielding that all the paths from ν to μ other than $G_{\mu\nu}$ are known from T_{rem}. Third, if both vertices μ and ν are excited, the two terms $T_{j\mu}$ and $T_{j\nu} = T_{j\mu}G_{\mu\nu}$ in (22) can be obtained due to excited vertex i and measured vertex j. From the two terms, $G_{\mu\nu}$ can be obtained. For the fourth situation where μ and ν are both measured, we simply compute G_{ji} from the two terms $T_{\mu i} = G_{\mu\nu}T_{\nu i}$ and $T_{\mu i}$ in (22), which shows the identifiability of $G_{\nu\mu}$ in $\mathcal{M}$.

D. Proof of Theorem 1

In our previous works [23], [24], the concept of disconnecting sets has been used as a key enabler for the identifiability analysis and signal allocation problem in a single module setting. In this paper, we also capitalize on the properties of disconnecting sets to prove the identifiability conditions for a full network in Theorem 1.

Before proving Theorem 1, two important properties of disconnecting sets are recapped from [20], [23], [24]. First, the concept of disconnecting sets is closely related to vertex disjoint paths, as presented in the well-known Menger's theorem.

Theorem 3: [20] Let $\mathcal{X}$ and $\mathcal{Y}$ be two vertex subsets in a directed graph $\mathcal{G}$. Let $\mathcal{D}$ be a minimum disconnecting set from $\mathcal{X}$ to $\mathcal{Y}$. Then,

$$|\mathcal{D}| = b_{\mathcal{X}\to\mathcal{Y}}, \tag{23}$$

where $b_{\mathcal{X}\to\mathcal{Y}}$ is the maximum number of vertex disjoint paths from $\mathcal{X}$ to $\mathcal{Y}$.

It is further shown in [23] that we can obtain a factorization of the mapping from the vertex signals from $\mathcal{X}$ to $\mathcal{Y}$ in a dynamic network by means of a disconnecting set $\mathcal{D}$ from $\mathcal{X}$ to $\mathcal{Y}$ in the associated graph. This factorization is instrumental for the proof of Theorem 1.

Lemma 3: [23], [24] Consider a network model in (1) associated with a directed graph $\mathcal{G}$, and the transfer matrix T in (3). Let $\mathcal{D}$ be a disconnecting set from $\mathcal{X}$ to $\mathcal{Y}$ in $\mathcal{G}$. Then there exist proper transfer matrices K and Q such that

$$[T]_{\mathcal{Y},\mathcal{X}} = K[T]_{\mathcal{D},\mathcal{X}} = [T]_{\mathcal{Y},\mathcal{D}}Q, \tag{24}$$

where K is left invertible if $\mathrm{Rank}([T]_{\mathcal{Y},\mathcal{D}}) = |\mathcal{D}|$, and Q is right invertible if $\mathrm{Rank}([T]_{\mathcal{D},\mathcal{X}}) = |\mathcal{D}|$.

Thanks to the properties of disconnecting sets in Theorem 3 and Lemma 3, we are ready to prove Theorem 1.

The proof relies on a particular property of acyclic graphs, that is a hierarchical structure in terms of vertex reachability. Specifically, for any acyclic graph $\mathcal{G}$, we can partition its vertex set $\mathcal{V}$ into different layers (subsets), denoted by $\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_N$ with $N \leqslant L$, such that the following conditions hold:

1) The vertices in the same layer cannot reach each other;
2) For any two layers $\mathcal{L}_\mu$, $\mathcal{L}_\nu$ with $\mu < \nu$, there is a vertex in $\mathcal{L}_\mu$ that can reach a vertex in $\mathcal{L}_\nu$, but not vise versa.

Note that there may be multiple options to partition the vertices in an acyclic graph $\mathcal{G}$ to form a hierarchical structure. In this proof, we consider a specific one where the top layer $\mathcal{L}_1$ is the set of source vertices, while the bottom layer $\mathcal{L}_N$ collects all the sink vertices. Moreover, we can prove that $\mathcal{L}_1$ is excited and $\mathcal{L}_N$ is measured as follows. By contradiction, assume a vertex $j \in \mathcal{L}_N$ unmeasured, then j is excited due to $\mathcal{R}\cup\mathcal{C} = \mathcal{V}$. From condition (2) in Theorem 1, it is required that there exists a set of measured vertices $\mathcal{C}_j$ such that (11) holds. Since j has no out-neighbors and $j \in \mathcal{R}_j$, the j-th column of the matrix $[T]_{\mathcal{C}_j,\mathcal{R}_j}$ is zero, which contradicts (11) that requires full column rank of $[T]_{\mathcal{C}_j,\mathcal{R}_j}$. Therefore, all the vertices in $\mathcal{L}_N$ are measured. Next, we can show that all the vertices in $k \in \mathcal{L}_1$ are excited. If the conditions in Theorem 1 hold, then for each vertex j in $\mathcal{G}$,

$$b_{\mathcal{R}\to\mathcal{N}_j^-} = |\mathcal{N}_j^-|. \tag{25}$$

We refer to (14) and Proposition 1 for this statement, which are implied by Theorem 1. Therefore, any $k \in \mathcal{L}_1$ should be excited. Otherwise, there will be a out-neighbor j of k that does not satisfy (25).

We can apply an analogous reasoning to the transpose network with graph $\mathcal{G}'$ of the original network, which gives that all the sinks in $\mathcal{G}'$ are measured if the model set of the transpose network is identifiable. With Lemma 1, it is necessary to have all the sources of $\mathcal{G}$, i.e. the vertices in $\mathcal{L}_1$, are excited for identifiability of $\mathcal{M}$. Then, identifiability of the network model set $\mathcal{M}$ is equivalent to that all the modules G_{ji} is identifiable in $\mathcal{M}$, for all $i \in \mathcal{N}_j^-$ and j in all the layers. We, therefore, proceed the proof by induction.

First, we analyze the vertices in $\mathcal{L}_2$. For any measured vertex $j \in \mathcal{L}_2$, all its in-neighbors $\mathcal{N}_j^-$ are sources in $\mathcal{L}_1$, which are excited, and $\hat{\mathcal{N}}_j^- = \varnothing$. From each $k \in \mathcal{N}_j^- \subseteq \mathcal{L}_1$, the directed edge (i, j) is the unique path from i to j, thus we obtain from Corollary 1 that G_{ji} is identifiable in $\mathcal{M}$ for all $i \in \mathcal{N}_j^-$. For the other vertices in $\mathcal{L}_2$ that are not measured, they are supposed to be excited due to $\mathcal{C} \cup \mathcal{R} = \mathcal{V}$. Consider the modules $G_{j,\mathcal{N}_j^-}$ with $j \in \mathcal{L}_2$ and $\mathcal{N}_j^- \subseteq \mathcal{L}_1$ excited. Then, we show identifiability of $G_{j,\mathcal{N}_j^-}$ with both $\mathcal{N}_j^-$ and j excited whenever the condition (2) in Theorem 1 is satisfied. Here, we extend our result in [24] for single modules to treat the MISO problem regarding the unmeasured vertex j.

It is implied from (11) and (12) that

$$\mathrm{Rank}\left([T]_{\mathcal{C}_j,\mathcal{R}_j\backslash j}\right) = |\mathcal{R}_j| - 1 = \mathrm{Rank}\left([T]_{\mathcal{C}_j,(\mathcal{R}_j\cup\mathcal{S}_j)\backslash j}\right).$$

Let $\mathcal{D}$ be a minimum disconnecting set from $(\mathcal{R}_j \cup \mathcal{S}_j)\backslash j$ to $\mathcal{C}_j$, where in this case, $\mathcal{S}_j = \bigcup_{i\in\mathcal{N}_j^-} \mathcal{N}_i^+$ contains all the out-neighbors of each in-neighbor of j. With the relation between transfer matrix rank and vertex disjoint paths in [12], [26], the rank equality in the preceding equation leads to

$$b_{\mathcal{R}_j\backslash j\to\mathcal{C}_j} = b_{(\mathcal{R}_j\cup\mathcal{S}_j)\backslash j\to\mathcal{C}_j} = |\mathcal{D}|. \tag{26}$$

It implies that $\mathcal{D}$ is also a disconnecting set from $\mathcal{R}_j\backslash j$ to $\mathcal{C}_j$ as well as a disconnecting set from $\mathcal{N}_i^+\backslash j$ to $\mathcal{C}_j$ for each $i \in \mathcal{N}_j^-$. Then, we follow Lemma 3 to obtain that there exist two transfer matrices Q_1 and Q_{2i} satisfying

$$[T]_{\mathcal{C}_j,\mathcal{R}_j\backslash j} = [T]_{\mathcal{C}_j,\mathcal{D}}Q_1, \tag{27a}$$
$$[T]_{\mathcal{C}_j,\mathcal{N}_i^+\backslash j} = [T]_{\mathcal{C}_j,\mathcal{D}}Q_{2i}. \tag{27b}$$

Observe that $[T]_{\mathcal{C}_j,\mathcal{R}_j}$ is full column rank due to (11), which implies that $[T]_{\mathcal{C}_j,\mathcal{R}_j\backslash j}$ is also full column rank, and it is obtained from (26) that $\mathrm{Rank}([T]_{\mathcal{C}_j,\mathcal{R}_j\backslash j}) = |\mathcal{D}|$. Therefore, Q_1 is right invertible from Lemma 3 and its right inverse is denoted by $Q_1^\dagger$. Then, it is obtained from (27) that

$$[T]_{\mathcal{C}_j,\mathcal{N}_i^+\backslash j} = [T]_{\mathcal{C}_j,\mathcal{R}_j\backslash j}Q_1^\dagger Q_{2i}. \tag{28}$$

Furthermore, we take the i-th column of the equation $CT(I-G)=C$ that can be permuted to

$$\begin{bmatrix}[T]_{\mathcal{C}_j,j} & [T]_{\mathcal{C}_j,\mathcal{N}_i^+\backslash j} & [T]_{\mathcal{C}_j,i} & \star\end{bmatrix}\begin{bmatrix}-G_{ji}\\ -G_{\mathcal{N}_i^+\backslash j,i}\\ 1\\ 0\end{bmatrix}=C_{\star i}, \tag{29}$$

where $C_{\star i}$ denotes the i-th column of the matrix C, and $C_{\star i}$ is nonzero if vertex i is measured, and $C_{\star i}=0$ otherwise. Substituting (28) into (29) then yields

$$[T]_{\mathcal{C}_j,i}-C_{\star i}=[T]_{\mathcal{C}_j,j}G_{ji}+[T]_{\mathcal{C}_j,\mathcal{R}_j\backslash j}Q_1^\dagger Q_{2i}G_{\mathcal{N}_i^+\backslash j,i}. \tag{30}$$

Let $[C]_{\star\mathcal{N}_j^-}$ be the submatrix of C that consists of the columns in C indexed by $\mathcal{N}_j^-$. By stacking all the equations in (30) for all $i\in\mathcal{N}_j^-$, we then obtain

$$\begin{aligned}&[T]_{\mathcal{C}_j,\mathcal{N}_j^-}-[C]_{\star\mathcal{N}_j^-} \\ &=[T]_{\mathcal{C}_j,j}G_{j,\mathcal{N}_j^-}+[T]_{\mathcal{C}_j,\mathcal{R}_j\backslash j}Q_1^\dagger\sum_{i\in\mathcal{N}_j^-}Q_{2i}G_{\mathcal{N}_i^+\backslash j,\mathcal{N}_j^-}\cdot \\ &=\underbrace{\begin{bmatrix}[T]_{\mathcal{C}_j,j} & [T]_{\mathcal{C}_j,\mathcal{R}_j\backslash j}\end{bmatrix}}_{[T]_{\mathcal{C}_j,\mathcal{R}_j}}\begin{bmatrix}G_{j,\mathcal{N}_j^-}\\ Q_1^\dagger\sum_{i\in\mathcal{N}_j^-}Q_{2i}G_{\mathcal{N}_i^+\backslash j,\mathcal{N}_j^-}\end{bmatrix},\end{aligned} \tag{31}$$

from which $G_{j,\mathcal{N}_j^-}$ can be uniquely determined from the known terms $[T]_{\mathcal{C}_j,\mathcal{R}_j}$ and $[T]_{\mathcal{C}_j,\mathcal{N}_j^-}-[C]_{\star\mathcal{N}_j^-}$, since $[T]_{\mathcal{C}_j,\mathcal{R}_j}$ is full column rank as in (11), and the vertices in $\mathcal{N}_j^-\subseteq\mathcal{L}_1$ are excited. So far, it has been manifested that for any vertex $j\in\mathcal{L}_2$ that are measured or excited, all the modules in $G_{j,\mathcal{N}_j^-}$ are identifiable in the network model set $\mathcal{M}$ if the two conditions in Theorem 1 hold.

To proceed with induction on the layers, we assume that the transfer vector $G_{\ell,\mathcal{N}_\ell^-}$ is identifiable in $\mathcal{M}$ for each $\ell\in\hat{\mathcal{L}}:=\mathcal{L}_1\cup\mathcal{L}_1\cup\cdots\cup\mathcal{L}_{k-1}$. We aim to show that $G_{j,\mathcal{N}_j^-}$ is also identifiable, for all $j\in\mathcal{L}_k$. Notice that the hierarchical structure of the acyclic graph allows for a permutation matrix P such that

$$G=P\begin{bmatrix}G_{\hat{\mathcal{L}},\hat{\mathcal{L}}} & 0\\ G_{\mathcal{V}\backslash\hat{\mathcal{L}},\hat{\mathcal{L}}} & G_{\mathcal{V}\backslash\hat{\mathcal{L}},\mathcal{V}\backslash\hat{\mathcal{L}}}\end{bmatrix}P^\top,$$

in which all the nonzero entries in $G_{\hat{\mathcal{L}},\hat{\mathcal{L}}}$ are identifiable in $\mathcal{M}$. Based on that, we have

$$T=P^\top(I-PGP^\top)^{-1}P=P^\top\begin{bmatrix}(I-G_{\hat{\mathcal{L}},\hat{\mathcal{L}}})^{-1} & 0\\ \star & \star\end{bmatrix}P,$$

and thus the transfer matrix $[T]_{\hat{\mathcal{L}},\hat{\mathcal{L}}}=(I-G_{\hat{\mathcal{L}},\hat{\mathcal{L}}})^{-1}$ can be identified.

Now we consider a measured vertex $j\in\mathcal{L}_k$, with $\mathcal{N}_j^-\subseteq\hat{\mathcal{L}}$. Then, there exists a set of excited vertices $\mathcal{R}_j\subseteq\mathcal{R}\cap\hat{\mathcal{L}}$ such that the following relation holds:

$$[T]_{j,\mathcal{R}_j}=G_{j,\mathcal{N}_j^-}[T]_{\mathcal{N}_j^-,\mathcal{R}_j}, \tag{32}$$

in which $[T]_{\mathcal{N}_j^-,\mathcal{R}_j}$ is a submatrix of $[T]_{\hat{\mathcal{L}},\hat{\mathcal{L}}}$, and it is full row rank as vertex j satisfies (7). Then, the entries in $G_{j,\mathcal{N}_j^-}$ are uniquely solved from (32), given the fact that both $[T]_{j,\mathcal{R}_j}$ and $[T]_{\mathcal{N}_j^-,\mathcal{R}_j}$ are identified from the data.

In the case that vertex j is unmeasured, j has to be excited. Then, we follow a similar reasoning as the analysis for $j\in\mathcal{L}_2$ and thereby obtain (31). If all the in-neighbors of j are excited, then identifiability analysis of $G_{j,\mathcal{N}_j^-}$ follows the same argument as $j\in\mathcal{L}_2$. While $\widehat{\mathcal{N}}_j^-\neq\varnothing$, i.e. there exist unexcited but measured in-neighbors of j, we cannot acquire all the elements in the transfer matrix $[T]_{\mathcal{C}_j,\mathcal{N}_j^-}$ in (31). However, we note that $\mathcal{S}_i$ in this case is the union of $\widehat{\mathcal{N}}_j^-$ and $\bigcup_{i\in\mathcal{N}_j^-}\mathcal{N}_i^+$ that satisfies (12). A minimum disconnecting set $\mathcal{D}$ from $(\mathcal{R}_j\cup\mathcal{S}_i)\backslash j$ to $\mathcal{C}_j$ is also a disconnecting set from $\widehat{\mathcal{N}}_j^-$ to $\mathcal{C}_j$, owing to $\widehat{\mathcal{N}}_j^-\subset\mathcal{C}_j$. Lemma 3 is thereby applied to have

$$[T]_{\mathcal{C}_j,\widehat{\mathcal{N}}_j^-}=[T]_{\mathcal{C}_j,\mathcal{D}}Q_3=[T]_{\mathcal{C}_j,\mathcal{R}_j\backslash j}Q_1^\dagger Q_3, \tag{33}$$

and thus $[T]_{\mathcal{C}_j,\mathcal{N}_j^-}=\begin{bmatrix}[T]_{\mathcal{C}_j,\mathcal{N}_j^-\backslash\widehat{\mathcal{N}}_j^-} & [T]_{\mathcal{C}_j,\mathcal{R}_j\backslash j}Q_1^\dagger Q_3\end{bmatrix}$, where the vertices in $\mathcal{N}_j^-\backslash\widehat{\mathcal{N}}_j^-$ are excited, and $[T]_{\mathcal{C}_j,\mathcal{N}_j^-\backslash\widehat{\mathcal{N}}_j^-}$ is known from data. Then, (31) can be written as

$$\begin{aligned}&\begin{bmatrix}[T]_{\mathcal{C}_j,\mathcal{N}_j^-\backslash\widehat{\mathcal{N}}_j^-} & 0\end{bmatrix}-[C]_{\star\mathcal{N}_j^-} \\ &=\underbrace{\begin{bmatrix}[T]_{\mathcal{C}_j,j} & [T]_{\mathcal{C}_j,\mathcal{R}_j\backslash j}\end{bmatrix}}_{[T]_{\mathcal{C}_j,\mathcal{R}_j}}\begin{bmatrix}G_{j,\mathcal{N}_j^-}\\ Q_1^\dagger\sum_{i\in\mathcal{N}_j^-}Q_{2i}G_{\mathcal{N}_i^+\backslash j,i}-Q_1^\dagger Q_3\end{bmatrix},\end{aligned} \tag{34}$$

in which $[T]_{\mathcal{C}_j,\mathcal{R}_j}$ and the two matrices on the left-hand side can be obtained from the measurement data (y,r). Therefore, $G_{j,\mathcal{N}_j^-}$ can be uniquely obtained due to the left invertible transfer matrix $[T]_{\mathcal{C}_j,\mathcal{R}_j}$.

Consequently, by induction, we have verified that $G_{j,\mathcal{N}_j^-}$ is identifiable, for all $j\in\mathcal{L}_k$ under the conditions in Theorem 1. This finalizes the proof.

E. Proof of Proposition 3

The in-neighbors of the unmeasured vertex j can be divided into two categories: the unexcited but measured vertices $\widehat{\mathcal{N}}_j^-$ and excited vertices $\check{\mathcal{N}}_j^-:=\mathcal{N}_j^-\backslash\widehat{\mathcal{N}}_j^-$, where the latter may have overlap with $\mathcal{R}_j\backslash j$. We thereby denote $\mathcal{R}_N:=(\mathcal{R}_j\backslash j)\cap\check{\mathcal{N}}_j^-$. It is implied from (15) that there are $|\mathcal{R}_j|-1$ vertex-disjoint paths from $\mathcal{R}_j\backslash j$ to $\mathcal{C}_j$, where $\mathcal{C}_j$ includes all the measured in-neighbors of j, i.e. $\widehat{\mathcal{N}}_j^-\subseteq\mathcal{C}_j$. Let $\mathcal{P}$ be a set of the vertex-disjoint paths from $\mathcal{R}_j\backslash j$ to $\mathcal{C}_j$. To show (17), we require $\mathcal{P}$ capable to be divided into two subsets, namely, $|\check{\mathcal{N}}_j^-|$ vertex-disjoint paths from the excited in-neighbors $\check{\mathcal{N}}_j^-$ to $\mathcal{C}_j\backslash\widehat{\mathcal{N}}_j^-$, and $|\mathcal{R}_j|-|\check{\mathcal{N}}_j^-|-1$ vertex-disjoint paths from $(\mathcal{R}_j\backslash j)\backslash\check{\mathcal{N}}_j^-$ to $\widehat{\mathcal{N}}_j^-$. This is then equivalent to prove that $\mathcal{P}$ does not contain paths from $\mathcal{R}_N$ to $\widehat{\mathcal{N}}_j^-$.

From (17), there exists a measured successor k of j such that a path from j to k is vertex-disjoint with $\mathcal{P}$. Besides, since $\widehat{\mathcal{N}}_j^-\subseteq\mathcal{C}_j$ and $\widehat{\mathcal{N}}_j^-\subseteq\mathcal{S}_j$, there are always $|\widehat{\mathcal{N}}_j^-|$ vertex-disjoint paths from $\mathcal{S}_j\backslash j$ to $\widehat{\mathcal{N}}_j^-$. From (15) and (16), we have the equality

$$b_{\mathcal{R}_j\backslash j\to\mathcal{C}_j}=b_{(\mathcal{R}_j\cup\mathcal{S}_i)\backslash j\to\mathcal{C}_j}=|\mathcal{R}_j|-1.$$

Therefore, $|\widehat{\mathcal{N}}_j^-|$ vertex-disjoint paths can be found in $\mathcal{P}$ from $\mathcal{R}_j\backslash j$ to $\widehat{\mathcal{N}}_j^-$. Furthermore, none of these paths starts from $\mathcal{R}_N$. Otherwise, we would have $b_{\mathcal{R}_j\backslash j\to\mathcal{C}_j} < b_{(\mathcal{R}_j\cup\mathcal{S}_i)\backslash j\to\mathcal{C}_j}$, where the set of vertex-disjoint paths from $(\mathcal{R}_j\cup\mathcal{S}_i)\backslash j$ to $\mathcal{C}_j$ includes a path from $\mathcal{R}_N$ to k via j.

F. Proof of Theorem 2

We prove that $\mathcal{M}$ is generically identifiable if condition (1) holds. Let $\mathcal{G}$ be a composition of disjoint trees $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_\kappa$, where all the roots are excited, all the leaves are measured, and the rest of vertices are either excited or measured. Therefore, all the sources in $\mathcal{G}$ are excited and all the sinks in $\mathcal{G}$ are measured. Furthermore, it implies that a vertex has more than one in-neighbor *only if* it is measured. To show generic identifiability of $\mathcal{M}$, we consider the graph-based conditions in Corollary 2. More specifically, (14) is satisfied for every measured vertex in $\mathcal{G}$, while for each excited but unmeasured vertex, there exist a measured vertex set $\mathcal{C}_j$ and excited vertex set $\mathcal{R}_j$ including j such that (15) and (16) hold.

First, we consider all the measured vertices in the acyclic graph $\mathcal{G}$. It is implied in Definition 3 that each vertex and its out-neighbors are included in the same tree. Thus, for any vertex $j\in\mathcal{C}$, all the edges incident from the vertices in $\mathcal{N}_j^-$ to j should belong to distinct trees. Therefore, there exist at least $|\mathcal{N}_j^-|$ vertex-disjoint paths from the roots of the disjoint trees $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_\kappa$ to $\mathcal{N}_j^-$. As in condition (1), the root of each tree is excited, we have $b_{\mathcal{R}_j\to\mathcal{N}_j^-} = |\mathcal{N}_j^-|$, where $\mathcal{R}_j$ is the set of all the roots in the trees. As a result, (14) holds because of $\mathcal{R}_j\subseteq\mathcal{R}$.

Next, all the unmeasured vertices in $\mathcal{G}$ are analyzed. Let j be an unmeasured vertex, which is excited due to $\mathcal{R}\cup\mathcal{C}=\mathcal{V}$. If condition (1) holds, j has only one in-neighbor in $\mathcal{G}$, which is denoted by i. We then show that G_{ji} is generically identifiable in $\mathcal{M}$. Suppose that both vertices i and j are in the same tree $\mathcal{T}_a$, and we can find the first measured descendant of j in $\mathcal{T}_a$, denoted by k, such that (j,k) is an edge in $\mathcal{T}_a$, or there are no measured vertices on the directed path from j to k. Hereafter, three cases are discussed.

Case I: If the in-neighbor i is excited, and there is a unique directed path from i to k via j, then this path is in $\mathcal{T}_a$, namely, it only consists of edges in $\mathcal{T}_a$. As a result, we directly obtain G_{ji} identifiable from Corollary 1.

Case II: If the in-neighbor i is excited, and there are multiple paths from i to k, i.e., there is one path in $\mathcal{T}_a$, but the other paths pass through edges in the other trees. Then we can find a minimum number of measured leaves in $\mathcal{T}_a$, denoted by $\bar{\mathcal{C}}_j$, whose removal will lead to a single path from i to k via j in the entire graph $\mathcal{G}$. Correspondingly, there exits a set of trees $\mathcal{N}_\mathcal{T}$, with $|\bar{\mathcal{C}}_j| = |\mathcal{N}_\mathcal{T}|$, such that each tree $\mathcal{T}_b\in\mathcal{N}_\mathcal{T}$ shares with $\mathcal{T}_a$ a distinct vertex in $\bar{\mathcal{C}}_j$ and the measured vertex k.

Let $\bar{\mathcal{R}}_j$ collect all the roots of the trees in $\mathcal{N}_\mathcal{T}$, where $|\bar{\mathcal{R}}_j| = |\bar{\mathcal{C}}_j|$, and the vertices in $\bar{\mathcal{R}}_j$ are excited by condition (1). It is clear that there is a one-to-one correspondence between $\bar{\mathcal{R}}_j$ and $\bar{\mathcal{C}}_j$, that is, the root of $\mathcal{T}_b\in\mathcal{N}_\mathcal{T}$ has a unique path in $\mathcal{T}_b$ to k via an only measured vertex in $\bar{\mathcal{C}}_j$.

When vertex i is not reachable from $\bar{\mathcal{R}}_j$, the set $\bar{\mathcal{C}}_j$ is a minimum disconnecting set from $\bar{\mathcal{R}}_j$ to $\bar{\mathcal{C}}_j\cup k$ and from $\mathcal{N}_i^+\backslash j$ to $\bar{\mathcal{C}}_j\cup k$, simultaneously. Consequently, we have

$$b_{\mathcal{R}_j\to\mathcal{C}_j} = b_{\bar{\mathcal{R}}_j\to\bar{\mathcal{C}}_j} + b_{j\to k} = |\bar{\mathcal{R}}_j| + 1 = |\mathcal{R}_j|,$$
$$b_{\mathcal{R}_j\backslash j\to\mathcal{C}_j} = |\bar{\mathcal{R}}_j| = |\bar{\mathcal{C}}_j| = b_{(\mathcal{R}_j\cup\mathcal{N}_i^+)\backslash j},$$

where $\mathcal{R}_j = \bar{\mathcal{R}}_j\cup j$, and $\mathcal{C}_j = \bar{\mathcal{C}}_j\cup k$.

When vertex i is reachable from $\bar{\mathcal{R}}_j$, then either i is measured or there exists a measured vertex ℓ in $\mathcal{T}_a$, which can be either i or an ascendant of i such that (ℓ,i) is an edge in $\mathcal{G}$, or every vertex along the path from ℓ to i is excited. Thereby, ℓ disconnects all the paths from $\bar{\mathcal{R}}_j\cup r_a$ to i, where r_a is the root of $\mathcal{T}_a$. Note that $\bar{\mathcal{C}}_j\cap\ell=\varnothing$ as ℓ is not a leaf. Denote $\mathcal{C}_j := \bar{\mathcal{C}}_j\cup\ell\cup k$, and $\mathcal{R}_j := \bar{\mathcal{R}}_j\cup r_a\cup j$, which satisfies

$$b_{\mathcal{R}_j\to\mathcal{C}_j} = b_{\bar{\mathcal{R}}_j\to\bar{\mathcal{C}}_j} + b_{r_a\to\ell} + b_{j\to k} = |\mathcal{R}_j|. \tag{35}$$

Moreover, it is not hard to verify that $\bar{\mathcal{C}}_j\cup\ell$ is a minimum disconnecting set from $\mathcal{R}_j\backslash j$ to $\mathcal{C}_j$ as well as a disconnecting set from $(\mathcal{R}_j\cup\mathcal{N}_i^+)\backslash j$ to $\mathcal{C}_j$. Therefore, (16) is also satisfied.

Case III: If the in-neighbor i is measured but unexcited, then we can still find a minimum number of measured leaves $\bar{\mathcal{C}}_j$ in the tree $\mathcal{T}_a$, and removing the vertices in $\bar{\mathcal{C}}_j$ will lead to a single path from i to k via j in $\mathcal{G}$. Moreover, a set of trees $\mathcal{N}_\mathcal{T}$ exists as in Case II, whose root set $\bar{\mathcal{R}}_j$ has a one-to-one relation with $\bar{\mathcal{C}}_j$, thus $|\bar{\mathcal{C}}_j| = |\mathcal{N}_\mathcal{T}| = |\bar{\mathcal{R}}_j|$.

Let r_a be the root of $\mathcal{T}_a$. Now i disconnects all the paths from $\bar{\mathcal{R}}_j\cup r_a$ to j, as i is the only in-neighbor of j. Consider $\mathcal{C}_j := \bar{\mathcal{C}}_j\cup i\cup k$, and $\mathcal{R}_j := \bar{\mathcal{R}}_j\cup r_a\cup j$. Therefore, we have

$$b_{\mathcal{R}_j\to\mathcal{C}_j} = b_{\bar{\mathcal{R}}_j\to\bar{\mathcal{C}}_j} + b_{r_a\to i} + b_{j\to k} = |\mathcal{R}_j|. \tag{36}$$

Furthermore, similar to the analysis in Case II, we can verify that $\bar{\mathcal{C}}_j\cup i$ is a minimum disconnecting set from $\mathcal{R}_j\backslash j$ to $\mathcal{C}_j$ as well as a disconnecting set from $(\mathcal{R}_j\cup\mathcal{N}_i^+\cup i)\backslash j$ to $\mathcal{C}_j$, leading to (16).

According to the above analysis, for every edge (i,j) in $\mathcal{G}$ with j an unmeasured vertex, the path-based conditions in (15) and (16) are fulfilled. Therefore, the network model set $\mathcal{M}$ is generically identifiable.

Then the proof regarding the condition (1) has been completed, and the proof for the second condition directly follows by considering the transpose graph of $\mathcal{G}$. If condition (2) holds for $\mathcal{G}$, then condition (1) is satisfied in the transpose graph $\tilde{\mathcal{G}}$. Therefore, the model set $\tilde{\mathcal{M}}$ associated with $\tilde{\mathcal{G}}$ is generically identifiable. Then, it follows from Lemma 1 that the original model set $\mathcal{M}$ is generically identifiable. That completes the proof.

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