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Asymptotic variance expressions for closed-loop identification☆

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Abstract

Asymptotic variance expressions are analyzed for models that are identified on the basis of closed-loop data. The considered methods comprise the classical 'direct' method, as well as the more recently developed indirect methods, employing coprime factorized models, dual Youla/Kucera parametrizations and the two-stage approach. The variance expressions are compared with the open-loop situation, and evaluated in terms of their relevance for subsequent model-based control design.

Keywords: System identification; Closed-loop identification; Asymptotic variance expressions, Prediction error methods; Model-based control design

1. Introduction

When identifying dynamic models for the specific purpose of subsequent model-based control design, it is argued that a closed-loop experimental setup during the identification experiments supports the construction of an identified model that is particularly accurate in that frequency region that is relevant for the control design. This mechanism which plays a major role in many contributions in the area of "identification for control", originally was motivated mainly on the basis of bias considerations in the form of a "control-relevant" distribution of the bias over frequency (Schrama, 1992; Gevers, 1993; Lee, Anderson, Kosut, & Mareels, 1993; Van den Hof & Schrama, 1995). Later it has been shown in Hjalmarsson, Gevers, and De Bruyne (1996) that, for a particular class of control design methods, also from a variance point of view closed-loop experiments are preferred over open-loop ones.

In this technical note, asymptotic variance expressions will be presented for identified models based on several different closed-loop identification methods, including the recently introduced indirect methods using a coprime factor model representation (Schrama, 1992; Van den Hof, Schrama, de Callafon, & Bosgra, 1995), the method employing the so-called dual Youla/Kucera parametrization (Hansen, Franklin, & Kosut, 1989; Schrama, 1992; Lee et al., 1993) and the two-stage method (Van den Hof & Schrama, 1993). The results for the classical 'direct' method (Ljung, 1993) are extended to also include variance expressions for the estimated noise model, while they are shown to remain the same for the mentioned alternative indirect methods.

These variance expressions are compared to related expressions for the open-loop situation, and consequences are discussed for subsequent model-based control design.

The paper relates to the general framework of the survey Forssell and Ljung (1999), in particular to Section 7 of that paper. In the current paper, though, we focus on explicit results for the specific methods mentioned above.

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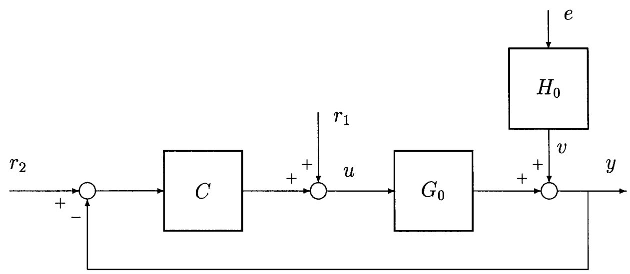

Fig. 1. Closed-loop configuration.

2. Preliminaries

Consider the closed-loop configuration as depicted in Fig. 1, where G_0 and C are linear time-invariant, possibly unstable, finite-dimensional systems, with G_0 strictly proper, while C is a stabilizing controller for G_0; e is a white noise process with variance λ_0, and H_0 a stable and stably invertible monic transfer function. Signals r_1 and r_2 are external reference signals. For purpose of efficient notation, we will often deal with the signal $r(t) := r_1(t) + C(q)r_2(t)$ being the result of external excitation through either r_1 or r_2. Additionally, we will denote: $u(t) = u^r(t) + u^e(t)$ with

$$u^r(t) := S_0(q)r(t),$$

$$u^e(t) := -C(q)S_0(q)H_0(q)e(t) = C(q)S_0(q)v(t),$$

where the sensitivity function S_0 is given by $S_0(q) := [1 + C(q)G_0(q)]^{-1}$. The signals $u^r(t)$ and $u^e(t)$ refer to those parts of the input signal that originate from the independent signals r and e, respectively. For the corresponding spectra it follows that $\Phi_u = \Phi_u^r + \Phi_u^e$ with

$$\Phi_u^r = |S_0|^2\Phi_r \quad \text{and} \quad \Phi_u^e = |CS_0|^2\Phi_v. \tag{1}$$

The arguments q and $e^{i\omega}$ will be omitted when appropriate. We will consider parametrized models $G(q,\theta)$ for G_0 and $H(q,\theta)$ for H_0 with $\theta \in \Theta$, and in accordance with Ljung (1987) we will use expressions $\mathcal{S} \in \mathcal{M}$ and $G_0 \in \mathcal{G}$ to indicate the situations that both G_0 and H_0 or only G_0 can be modelled exactly within the model set. The variance expressions that are considered in this paper are asymptotic in both n (model order) and N (number of data), while n^2/N is supposed to tend to 0, as in the standard framework of Ljung (1987).

3. Direct identification

The direct method of closed-loop identification is characterized by $\hat{\theta}_N = \arg\min_\theta \frac{1}{N}\sum_{t=0}^{N-1}\varepsilon(t,\theta)^2$ with

$$\varepsilon(t,\theta) = H(q,\theta)^{-1}[y(t) - G(q,\theta)u(t)]. \tag{2}$$

An expression for the asymptotic variance of the transfer function estimate can be given for the situation that $\mathcal{S} \in \mathcal{M}$, and both plant model and noise are estimated. In this case (Ljung, 1987)

$$cov\begin{pmatrix}\hat{G}(e^{i\omega})\\ \hat{H}(e^{i\omega})\end{pmatrix} \sim \frac{n}{N}\Phi_v(\omega)\begin{bmatrix}\Phi_u(\omega) & \Phi_{eu}(\omega)\\ \Phi_{ue}(\omega) & \lambda_0\end{bmatrix}^{-1},$$

where $\Phi_{ue}(\omega)$ is the cross-spectrum between u and e. With the relation $\Phi_{ue} = -CS_0H_0\lambda_0$ and using the fact that $\Phi_u\lambda_0 - |\Phi_{ue}|^2 = \lambda_0\Phi_u^r$ it follows that

$$cov\begin{pmatrix}\hat{G}\\ \hat{H}\end{pmatrix} \sim \frac{n}{N}\frac{\Phi_v}{\Phi_u^r}\begin{bmatrix}1 & (CS_0H_0)^*\\ CS_0H_0 & \dfrac{\Phi_u}{\lambda_0}\end{bmatrix}. \tag{3}$$

The variance expressions for $\hat{G}$ and $\hat{H}$ then become

$$cov(\hat{G}) \sim \frac{n}{N}\frac{\Phi_v}{\Phi_u^r} = \frac{n}{N}\frac{\Phi_v}{\Phi_u}\left[1 + \frac{\Phi_u^e}{\Phi_u^r}\right], \tag{4}$$

$$cov(\hat{H}) \sim \frac{n}{N}\frac{\Phi_v}{\lambda_0}\frac{\Phi_u}{\Phi_u^r} = \frac{n}{N}\frac{\Phi_v}{\lambda_0}\left[1 + \frac{\Phi_u^e}{\Phi_u^r}\right]. \tag{5}$$

The case of an open-loop experimental situation now appears as a special situation in which $\Phi_u^e = 0$, $\Phi_u^r = \Phi_u$, and $C = 0$, leading to the well-known (open-loop) expressions

$$cov(\hat{G}) \sim \frac{n}{N}\frac{\Phi_v}{\Phi_u}, \quad cov(\hat{H}) \sim \frac{n}{N}\frac{\Phi_v}{\lambda_0}.$$

As indicated in Ljung (1993), the closed-loop expressions show that only the noise-free part u^r of the input signal contributes to variance reduction of the estimates. In other words, increasing the input power only leads to a smaller variance if the increase in power is achieved by increasing the reference signal power. The given expressions are restricted to the situation that $\mathcal{S} \in \mathcal{M}$ and that both $G(\theta)$ and $H(\theta)$ are identified.

Remark 1. The situation of estimating a plant model in the situation $G_0 \in \mathcal{G}$ and having a fixed and correct noise model $H_* = H_0$ is considered in Ljung (1993). Using the fact that

$$cov\,\hat{\theta}_N = \frac{\lambda_0}{N}[E\psi(t)\psi^T(t)]^{-1}, \tag{6}$$

where $\psi(t)$ is the negative gradient of the prediction error (2), this leads to

$$cov(\hat{G}) \sim \frac{n}{N}\frac{\Phi_v}{\Phi_u} \tag{7}$$

for closed-loop identification, as it is immaterial whether the input spectrum is a result of open-loop or closed-loop operation. Note that this expression gives a smaller variance than the situation in which both G and H are estimated, and that in this (unrealistic) case the total input power contributes to a reduction of the estimate variance.

4. Indirect identification

4.1. Introduction

Over the last decade several different indirect approaches to closed-loop identification have been presented, see e.g. Gevers (1993), Van den Hof and Schrama (1995) and Forssell and Ljung (1999). These methods have been introduced mainly from considerations related to the bias on the estimate of $\hat{G}$ that occurs in direct closed-loop identification of approximate models. Here, we will briefly illustrate their properties with respect to the variance of the estimates.

4.2. Coprime factor identification

Coprime factor identification is treated in detail in Schrama (1992) and Van den Hof et al. (1995). It is a scheme that relates to (and generalizes) the classical joint input/output method of closed-loop identification as, e.g. described in Gustavsson et al. (1977). The basic principle is that the (two-times-two) transfer function $(r,e)^{\mathrm{T}} \to (y,u)^{\mathrm{T}}$ is identified, while the plant models $(\hat{G},\hat{H})$ are retrieved from these closed-loop estimates. Consider the system's relations:

$$y(t) = G_0 S_0 r(t) + S_0 H_0 e(t), \tag{8}$$

$$u(t) = S_0 r(t) - CS_0 H_0 e(t). \tag{9}$$

They are rewritten, by using a filtered signal $x(t) := F(q)r(t)$, into the form

$$y(t) = N_{0,F} x(t) + S_0 H_0 e(t), \tag{10}$$

$$u(t) = D_{0,F} x(t) - CS_0 H_0 e(t) \tag{11}$$

with $N_{0,F} := G_0 S_0 F^{-1}$ and $D_{0,F} := S_0 F^{-1}$, constituting a coprime factor representation of G_0 as $G_0 = N_{0,F} D_{0,F}^{-1}$. The linear and stable filter F can be chosen by the user to serve several purposes, like minimal-order properties or normalization of the coprime factorization as discussed in Van den Hof et al. (1995); this will not be pursued here any further as it is immaterial for the variance analysis. The important observation here is that the signals x and e are uncorrelated. Identification of the four transfer functions in (10) and (11) from the signals $x(t)$, $y(t)$, $u(t)$ therefore corresponds to a one-input–two-output open-loop identification problem. Denote

$$\varepsilon_y(t,\theta) = W_y(q,\theta)^{-1}[y(t) - N(q,\theta)x(t)],$$

$$\varepsilon_u(t,\theta) = W_u(q,\theta)^{-1}[y(t) - D(q,\theta)x(t)].$$

Least-squares minimization of $(\varepsilon_y, \varepsilon_u)^{\mathrm{T}}$ provides estimated models $\hat{N}, \hat{D}, \hat{W}_y, \hat{W}_u$.

Plant and noise model $\hat{G}$ and $\hat{H}$ are then retrieved by

$$\hat{G} = \hat{N}(\hat{D})^{-1},$$

$$\hat{H} = (1 + C\hat{G})\hat{W}_y.$$

For the variance of $\hat{G}$ and $\hat{H}$, use can be made of first-order approximations: $\hat{G} = G_0 + \Delta G$, $\hat{N} = N_{0,F} + \Delta N$, $\hat{D} = D_{0,F} + \Delta D$, etc. leading to

$$\Delta G = \frac{\Delta N}{D_{0,F}} - \frac{N_{0,F}\Delta D}{D_{0,F}^2},$$

$$\Delta H = (1 + CG_0)\Delta W_y + C(\Delta G)W_y. \tag{12}$$

This leads to the result

$$cov\begin{pmatrix}\hat{G}\\ \hat{H}\end{pmatrix} \sim \frac{n}{N}\frac{\Phi_v}{\Phi_u^r}\begin{bmatrix} 1 & (CS_0H_0)^* \\ CS_0H_0 & \dfrac{\Phi_u}{\lambda_0}\end{bmatrix}. \tag{13}$$

A sketch of the derivation of this result is given in the appendix. Note that (13) is identical to expression (3) for direct identification.

4.3. Identification in a dual Youla–Kucera parametrization

The dual Youla–Kucera parametrization utilizes a particular parametrization of the plant G_0. As C stabilizes the plant, G_0 can be parametrized within the class of all plants that are stabilized by C. This parametrization involves the relation

$$G(\theta) = \frac{N_x + D_c R(\theta)}{D_x - N_c R(\theta)}, \tag{14}$$

where $N_x/D_x =: G_x$ is any (auxiliary) system that is stabilized by C; $N_c/D_c = C$, and $R(\theta)$ ranges over the class of all stable proper transfer functions. The different factors that build up the quotient expressions G_x and C are required to be stable and coprime.

Using an expression like (14) for the plant G_0 with a Youla–Kucera parameter R_0, and substituting this in the system's relations, shows — after some manipulations — that these can be rewritten as

$$z(t) = R_0 x(t) + W_0 e(t)$$

with $R_0 = D_x S_0 (G_0 - G_x)/D_c$, $W_0 = H_0 S_0 / D_c$, and

$$z = (D_c + G_x N_c)^{-1}(y - G_x u),$$

$$x = (D_x + CN_x)^{-1} r.$$

Since x is not correlated with e, the identification of R_0 and W_0 can again be considered as an open-loop identification problem. The signals z and x can be constructed by the user, as they are dependent on known quantities and measured signals. Least-squares identification is performed on the basis of the prediction error

$$\varepsilon_z(t,\theta) = W(q,\theta)^{-1}[z(t) - R(q,\theta)x(t)]$$

and the estimated transfers are denoted by $\hat{W}$ and $\hat{R}$. The plant and noise model can then be reconstructed from these estimates according to

$$\hat{G} = \frac{N_x + D_c\hat{R}}{D_x - N_c\hat{R}}, \tag{15}$$

$$\hat{H} = \hat{W}D_c\hat{S}^{-1} = \hat{W}D_c[1 + C\hat{G}]. \tag{16}$$

In order to guarantee that $\hat{H}$ is monic it will assumed that D_c is monic. Variance expressions for $\hat{R}$ and $\hat{W}$ are available through the standard (open-loop) expressions:

$$cov(\hat{R}) \sim \frac{n}{N}\frac{|W_0|^2\lambda_0}{\Phi_x} \quad \text{and} \quad cov(\hat{W}) \sim \frac{n}{N}|W_0|^2$$

while $cov(\hat{R},\hat{W}) = 0$. In a similar way, as in Section 4.2, the variance of $(\hat{G},\hat{H})$ can be obtained, relying on first-order approximating expressions. Not surprisingly (see the appendix) the resulting expressions are again given by (13).

Further details on this identification method can be found in Lee et al. (1993) and Van den Hof and Schrama (1995). It can be shown that it is a direct generalization of the classical indirect method of closed-loop identification, see Van den Hof and de Callafon (1996).

4.4. Two-stage method

A two-stage method for closed-loop identification has been introduced in Van den Hof and Schrama (1993). It operates directly on reference, input and output data, and does not require knowledge of the implemented controller. It can best be explained by considering the system's relations:

$$y(t) = G_0u^r(t) + S_0H_0e(t),$$

$$u(t) = S_0r(t) - CS_0H_0e(t).$$

In the first step, measured signals r and u are used to estimate a model $\hat{S}$ of the sensitivity function S_0. Next, this model is used to construct (by simulation) an estimate $\hat{u}^r$ of u^r according to $\hat{u}^r(t) = \hat{S}(q)r(t)$. In the second stage, the signals $\hat{u}^r$ and y are used as a basis for the identification of a plant model $\hat{G}$. The procedure is very much like the coprime factor identification scheme, albeit that the final plant model is not calculated through division of two identified models; this division is circumvented by constructing the auxiliary simulated signal $\hat{u}^r = S(q,\hat{\gamma})r$.

Consider the prediction errors

$$\varepsilon_y(t,\theta,\gamma) = W_y^{-1}[y(t) - G(q,\theta)S(q,\gamma)r(t)],$$

$$\varepsilon_u(t,\gamma) = W_u^{-1}[u(t) - S(q,\gamma)r(t)],$$

then the parameter estimate $\hat{\theta}_N$ of this method can be written as the minimizing θ argument of $V_N(\theta,\lambda)$ for $\lambda \to \infty$, with

$$V_N(\theta,\lambda) = \frac{1}{N}\sum_{t=1}^{N}\left[\frac{1}{\lambda}\varepsilon_y^2(t,\theta,\gamma) + \varepsilon_u^2(t,\gamma)\right].$$

(Note that for $\lambda \to \infty$, $\hat{\gamma}$ will be determined fully on the basis of r and u.) Applying the coprime factor results from Section 4.2 to this situation then shows that the variance becomes independent of λ and equal to (13).[2]

4.5. Summarizing comments

For the considered indirect methods, the asymptotic variance expressions for plant and noise model are exactly the same as the expressions for direct identification. This may not be too surprising, as similar results for the classical indirect and joint i/o methods were already available (Gustavsson, Ljung, & Söderström, 1977). However what has to be stressed here, is that for the indirect-type methods the variance expressions for $\hat{G}$ are valid also in the situation that $G_0 \in \mathcal{G}$ but $\mathcal{S} \notin \mathcal{M}$, while for the direct method the results are only achieved under the stronger condition that $\mathcal{S} \in \mathcal{M}$. With indirect identification we can thus, e.g. fix the noise model to a predetermined choice, only identifying the plant model $\hat{G}$, and obtain the same asymptotic variance as would be obtained when indeed estimating a noise model.

5. Open-loop versus closed-loop experiments

Considering that the variance expressions are identical for all closed-loop identification methods, we can now make a comparison between the variances obtained from open-loop and closed-loop experimental conditions. The appropriate expressions are summarized in Table 1. The results show that, if the input spectrum is similarly constrained in both cases, then the variance of $\hat{G}$ and $\hat{H}$ obtained under closed-loop identification is larger than for open-loop identification. If the input power is not constrained, and if the reference power is chosen such that $\Phi_u^r \gg \Phi_u^e$, then the closed-loop expressions converge to the open-loop expressions. Observe also that the input signal plays no role in the variance of $\hat{H}$ in the open-loop situation, while in the closed-loop situation an increase in the reference signal power results in a decrease of $Var(\hat{H}_N)$.

The results suggest that in terms of variance of the model estimates $\hat{G}_N$ and $\hat{H}_N$, open-loop identification

[2] The authors acknowledge the contribution of Urban Forssell (Univ. Linköping) to the proof of this result.

Table 1
Variance expressions under open-loop and closed-loop conditions

	Open-loop		Closed-loop
$Var(\hat{G}_N)$	$\dfrac{n}{N}\dfrac{\Phi_v}{\Phi_u}$	$<$	$\dfrac{n}{N}\dfrac{\Phi_v}{\Phi_u^r} = \dfrac{n}{N}\dfrac{\Phi_v}{\Phi_u}\left(1+\dfrac{\Phi_u^e}{\Phi_u^r}\right)$
$Var(\hat{H}_N)$	$\dfrac{n}{N}\dfrac{\Phi_v}{\lambda_0}$	$<$	$\dfrac{n}{N}\dfrac{\Phi_v}{\lambda_0}\left(1+\dfrac{\Phi_u^e}{\Phi_u^r}\right)$

always has to be preferred over closed-loop identification. However, perhaps surprisingly, this is not the case if the objective of the identification is model-based control design. When the model estimates $\hat{G}_N$ and $\hat{H}_N$ are used for the design of a controller $\hat{C}_N = C(\hat{G}_N, \hat{H}_N)$, then this controller is a random variable, and one can consider the problem of selecting an identification experiment that minimizes the variance of the error $\hat{C}_N - C(G_0, H_0)$ between the controller estimated from the model and the controller that would be obtained from the true system. Somewhat surprisingly perhaps, the minimization of this controller variance is obtained by a closed-loop identification experiment, as soon as the control design depends on both G and H. The apparent contradiction with the results of Table 1 comes from the fact that the cross terms of the covariance matrix of $(\hat{G}_N \hat{H}_N)^{\mathrm{T}}$ also play a role in the expression of the controller variance. In the case of a control design that only depends on the input–output dynamics G, open-loop identification is optimal. We refer the reader to (Hjalmarsson et al., 1996) for details.

6. Concluding remarks

Asymptotic variance expressions have been derived for several closed-loop identification schemes, showing that the several approaches lead to the same asymptotic variance. Although asymptotic variance of plant model and noise model generally will increase when performing closed-loop identification, in comparison with open-loop identification, closed-loop identification can still be preferred when the identified model is used as a basis for control design, provided that a controller is designed on the basis of both plant model and noise model.

The doubly asymptotic nature of the presented analysis (asymptotic in both model order and number of data) apparently diminishes possible differences between the several methods presented. A further analysis of parameter variance expressions for the considered closed-loop identification methods, is recently provided in Ljung and Forssell (1997), while in Codrons, Anderson, and Gevers (2000) it has been shown that significant differences between the closed-loop identification methods occur when the controller contains an unstable pole (e.g. an integrator) or a nonminimum phase zero.

Additionally, it has to be remarked that refinements of the considered general asymptotic high-order variance analysis have recently been discussed in Ninness, Hjalmarsson, and Gustafsson (1999).

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Appendix

Proof of (13). Applying the standard variance expressions to the multivariable situation of (10) and (11) it follows that

$$cov\begin{pmatrix}\hat{N}\\ \hat{D}\end{pmatrix} \sim \frac{n}{N}\frac{|S_0|^2\Phi_v}{\Phi_x}\begin{bmatrix}1 & -C^*\\ C & |C|^2\end{bmatrix}, \tag{A.1}$$

$$cov\begin{pmatrix}\hat{W}_y\\ \hat{W}_u\end{pmatrix} \sim \frac{n}{N}\frac{|S_0|^2\Phi_v}{\lambda_0}\begin{bmatrix}1 & -C^*\\ C & |C|^2\end{bmatrix}. \tag{A.2}$$

Since (10) and (11) reflect an open-loop situation (as x and e are uncorrelated) this implies that the cross-covariance terms between $(\hat{N}, \hat{D})^{\mathrm{T}}$ and $(\hat{W}_y, \hat{W}_u)$ are zero. From the first-order approximations in (12), it follows that

$$|\Delta G|^2 = \frac{|\Delta N|^2}{|D_{0,F}|^2} + \frac{|G_0|^2}{|D_{0,F}|^2}|\Delta D|^2 - 2\mathrm{Re}\left\{\frac{G_0(\Delta D)(\Delta N)^*}{|D_{0,F}|^2}\right\}.$$

Substitution of (A.1) then provides the result for $cov(\hat{G})$.

For $\hat{H}$ one can similarly write (when neglecting terms that have expectation 0):

$$|\Delta H|^2 = |1 + CG_0|^2|\Delta W_y|^2 + |CW_y|^2|\Delta G|^2 \tag{A.3}$$

and the result for $cov(\hat{H})$ follows after substitution of (A.2). The expression for $cov(\hat{G}, \hat{H})$ follows from $cov(\hat{G}, \hat{H}) = -(CW_y)^*\, cov(\hat{G})$.

Variance result for dual Youla–Kucera method: Using (15) and (16), the related expressions for the first-order approximation errors become

$$\Delta G = \frac{(D_x - N_c R_0) D_c(\Delta R) + (N_x + D_c R_0) N_c(\Delta R)}{(D_x - N_c R_0)^2},$$

$$\Delta H = \frac{D_c(\Delta W)}{S_0} + W_0 N_c(\Delta G). \tag{A.4}$$

For ΔG this leads to

$$\Delta G = \frac{D_c + G_0 N_c}{D_x - N_c R_0} \Delta R = \frac{D_c(\Delta R)}{D_x S_0^2 (1 + C G_x)}$$

and so

$$cov(\hat{G}) = \left| \frac{D_c}{D_x S_0^2 (1 + C G_x)} \right|^2 cov(\hat{R}).$$

Substituting the expression for $cov(\hat{R})$ and using the property that $\Phi_x = |D_x(1 + CG_x)|^2 \Phi_r$ it follows after some manipulation that $cov(\hat{G}) \sim n/N \cdot \Phi_v / \Phi_u^r$.

For $cov(\hat{H})$ it follows from (A.4) that

$$cov(\hat{H}) = \frac{|D_c|^2 \, cov\, \hat{W}}{|S_0|^2} + |N_c W_0|^2 \, cov\, \hat{G}.$$

Substituting the known expressions in the right-hand side, will show that $cov(\hat{H}) \sim n/N \; |H_0|^2 [1 + \Phi_u^e / \Phi_u^r]$.

For $cov(\hat{G}, \hat{H})$ it follows from (A.4) that $cov(\hat{G}, \hat{H}) = (W_0 N_c)^* cov(\hat{G})$ which leads to the appropriate result.

References

Codrons, B., Anderson, B. D. O., & Gevers, M. (2000). Closed-loop identification with an unstable or nonminimum phase controller. *Proceedings of IFAC symposium on system identification* (*SYSID 2000*), Santa Barbara, CA, USA, to appear.

Forssell, U., & Ljung, L. (1999). Closed-loop identification revisited. *Automatica*, *35*, 1215–1241.

Gevers, M. (1993). Towards a joint design of identification and control? In H. L. Trentelman, & J. C. Willems (Eds.), *Essays on control: Perspectives in the theory and its applications* (pp. 111–151). Boston: Birkhäuser.

Gustavsson, I., Ljung, L., & Söderström, T. (1977). Identification of processes in closed loop — identifiability and accuracy aspects. *Automatica*, *13*, 59–75.

Hansen, F. R., Franklin, G. F., & Kosut, R. L. (1989). Closed-loop identification via the fractional representation: Experiment design. *Proceedings of American control conference*, Pittsburgh, PA, USA (pp. 1422–1427).

Hjalmarsson, H., Gevers, M., & De Bruyne, F. (1996). For model-based control design, closed-loop identification gives better performance. *Automatica*, *32*, 1659–1673.

Lee, W. S., Anderson, B. D. O., Kosut, R. L., & Mareels, I. M. Y. (1993). A new approach to adaptive robust control. *International Journal of Adaptive Control and Signal Processing*, *7*, 183–211.

Ljung, L. (1987). *System identification: Theory for the user*. Englewood Cliffs, NJ: Prentice-Hall.

Ljung, L. (1993). Information contents in identification data from closed-loop operation. *Proceedings of 32nd IEEE conference on decision and control*, San Antonio, TX (pp. 2248–2252).

Ljung, L., & Forssell, U. (1997). Variance results for closed-loop identification methods. *Proceedings of 36th IEEE conference on decision and control*, San Diego, CA (pp. 2435–2440).

Ninness, B. M., Hjalmarsson, H., & Gustafsson, F. (1999). The fundamental role of generalized orthonormal bases in system identification. *IEEE Transactions on Automatic Control*, *44*, 1384–1406.

Schrama, R. J. P. (1992). *Approximate identification and control design.* Dr. dissertation, Delft University of Technology.

Van den Hof, P. M. J., & de Callafon, R. A. (1996). Multivariable closed-loop identification: From indirect identification to dual-Youla parametrization. *Proceedings of 35th IEEE conference decision and control*, Kobe, Japan (pp. 1397–1402).

Van den Hof, P. M. J., & Schrama, R. J. P. (1993). An indirect method for transfer function estimation from closed-loop data. *Automatica*, *29*, 1523–1527.

Van den Hof, P. M. J., & Schrama, R. J. P. (1995). Identification for control — closed-loop issues. *Automatica*, *31*, 1751–1770.

Van den Hof, P. M. J., Schrama, R. J. P., de Callafon, R. A., & Bosgra, O. H. (1995). Identification of normalised coprime plant factors from closed-loop experimental data. *European Journal of Control*, *1*, 62–74.