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Virtual Closed Loop Identification: A generalized tool for identification in closed loop

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Abstract— **In this paper we propose a virtual closed loop model parameterization to perform system identification. This parameterization is designed to achieve specific goals. We show that the method includes, as special cases, known methods for closed loop identification and also offers additional flexibility. We analyze the ramifications of the new tailor-made parameterization for systems operating in closed loop. The approach exploits a property of Box-Jenkins models in order to minimize the bias arising from feedback and noise model mismatch.**

I. Introduction

Identification of systems operating in closed loop has received considerable attention in the System Identification literature [8], [14], [15], [4], [12].

There are safety and economic reasons to perform identification experiments in closed loop. Also, it is known that the optimal experiment is usually performed in closed loop [17], [11], [15], [5], [10]. Indeed, recent research has established that, for a general class of systems, and when there is a constraint on the output power, the optimal experiment is necessarily closed loop [3].

Unfortunately, the identification of systems operating under the presence of feedback presents several difficulties [15], [4]. For example, correlation between the input signal and the noise is problematic in the context of several identification techniques. In fact, it is well known that the Prediction Error Method (PEM) provides a non-consistent estimate in the presence of under-modeling of the noise transfer function [4].

Several attempts to overcome this difficulty have been made. In particular, *indirect identification* is a popular approach to mitigate this difficulty. Traditional indirect identification is a two step procedure where the identification of a *plant object* is first obtained and then the open loop system is unraveled from this preliminary estimate. Here, and in the sequel, we use the term "*plant object*" to refer to a transfer function that depends on the system. In traditional indirect identification, the plant object to be identified is usually the complementary sensitivity transfer function relating the reference signal to the output [14]. However, several difficulties are known to exist with this approach. For example, it is common that the estimate of the open loop process is not necessarily stabilized by the controller used in the identification experiment, even though it is known that the real system is stabilized by this controller. This difficulty can be overcome by using a particular parameterization of the system, the so called Dual-Youla parameterization [9], [13], [16]. Also, it is often true in practice that the controller can have certain non-linearities e.g. anti-windup schemes [7]. This difficulty renders the usual indirect identification approaches unusable on many problems. Recent research reported in [2], [1], [6] has proposed an alternative method which tackles the difficulty of non-linear or partially known controllers. The method is based on a "virtual controller" which approximates the true one. The method is, in general, non-consistent when PEM is utilized. However, the asymptotic bias is small in the frequency range where the virtual controller approximately matches the true one. This method leads to a new class of estimators for systems operating in closed loop.

In this paper, we generalize the virtual closed loop approach. We show that, by suitable choice of parameters, the method specializes to known closed loop identification schemes. We propose a parameterization of the process which is designed in order to achieve different goals. In particular, we focus on the minimization of the asymptotic bias due to feedback and noise model mismatch.

The remainder of the paper is organized as follows: In Section II we describe the scheme of interest in a general non-linear setting. In Section III we specialize to linear systems. In Section IV we show that the virtual closed loop method generalizes known schemes for closed loop identification. In Section V we show how the choice of parameters in the virtual closed loop affects the asymptotic bias in the identification of systems operating in closed loop. In Section VI we present a numerical example. Finally in Section VII we draw conclusions.

II. General System of interest

We consider the following general non-linear system (see Figure 1):

$$y_t = \mathcal{G}_o(u_t, u_{t-1}, \cdots, w_t, w_{t-1}, \cdots) \tag{1}$$

where u_t, y_t are the input and output signals and w_t is zero mean Gaussian white noise of variance σ_w^2. We assume that the input and output signals are bounded. In this section, we do not make assumptions on the experimental conditions. However, the system may have been operating in closed loop in order to ensure bounded input-output signals for and open loop unstable process.

In order to develop the identification procedure proposed

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in this paper, we first define the following signals:

$$x_t = F_1 u_t + F_2 y_t \tag{2}$$
$$z_t = F_3 u_t + F_4 y_t \tag{3}$$

where F_1, F_2, F_3 and F_4 are stable filters. We assume that F_1 is bi-proper. Notice that the signals x and z are bounded because they are generating by passing bounded signals trough stable filters.

We assume that the filters F_i are written in terms of the following polynomials:

$$F_i = N_i D_i^{-1} \tag{4}$$

where D_i roots are inside the stability boundary.

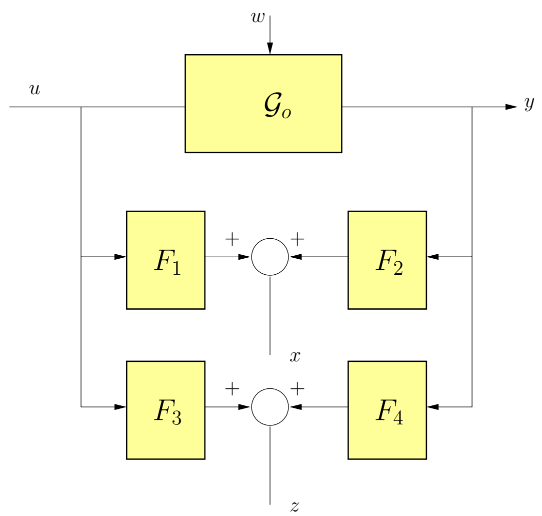

Fig. 1. Signal Generation in the Virtual Closed Loop Method.

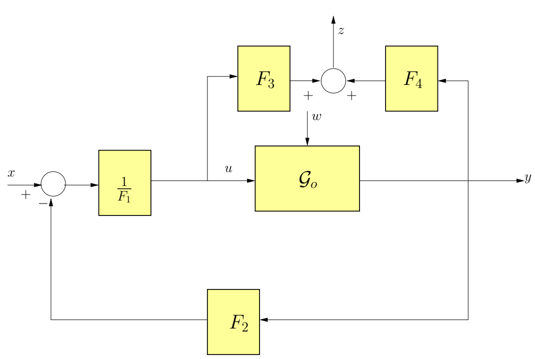

Fig. 2. Generalized Virtual Closed Loop.

We note that the signals x_t and z_t are readily obtained from knowledge of u_t and y_t. We then have the following core result:

Theorem 1: The signal transformations (2) and (3) induce the "virtual closed loop" shown in Figure 2.

Proof: The result is obtained by re-arranging equations (2) and (3) and using some block algebra. □

Notice that the feedback loop in Figure 2 has nothing to do with the existence of otherwise of a real closed loop system. This is the reason for the term "Virtual Closed Loop" (VCL). Also, stability is not an issue for the system of Figure 2 since we already know that all signals are bounded. In the remainder of the paper we will explore the implications of this configuration in System Identification. We propose to identify a model relating the signals x_t and z_t shown in Figure 2.

Towards this goal, we conceive of a model of the same structure as that shown in Figure 2 but parameterized by a vector θ. This is shown in Figure 3. The basic idea of virtual closed loop identification is to estimate θ by minimizing some function of the error between the measured signal z and the model output $\hat{z}$. Note that stability is an issue and this model as we are treating x_t as an external signal.

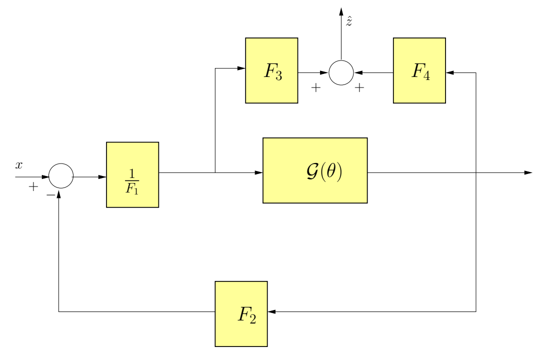

Fig. 3. Predictor Model for the Generalized Virtual Closed Loop.

III. Specialization to Linear Systems

For the case of linear systems, the model (1) can be written as:

$$y_t = G_o(q^{-1}) u_t + v_t \tag{5}$$
$$v_t = H_o(q^{-1}) w_t \tag{6}$$

where q^{-1} is the back-shift operator, and the transfer functions G_o and H_o are defined as follows:

$$G_o = B_o A_o^{-1} \tag{7}$$
$$H_o = P_o Q_o^{-1} \tag{8}$$

where B_o, A_o, P_o, Q_o are polynomials and P_o and Q_o are monic and have roots inside the stability boundary.

Using equations (5), (2) and (3) we obtain the following set of equations describing the virtual closed loop system:

$$\begin{bmatrix} 1 & -F_3 & -F_4 \\ 0 & F_1 & F_2 \\ 0 & -G_o & 1 \end{bmatrix} \begin{bmatrix} z_t \\ u_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0 \\ x_t \\ v_t \end{bmatrix} \tag{9}$$

Solving for z, u and y we have the following:

$$z_t = \frac{F_3 + F_4 G_o}{F_1 + F_2 G_o} x_t + \frac{F_1 F_4 - F_2 F_3}{F_1 + F_2 G_o} H_o w_t \tag{10}$$
$$u_t = \frac{1}{F_1 + F_2 G_o} x_t - \frac{F_2}{F_1 + F_2 G_o} H_o w_t \tag{11}$$
$$y_t = \frac{G_o}{F_1 + F_2 G_o} x_t + \frac{F_1}{F_1 + F_2 G_o} H_o w_t \tag{12}$$

The core idea of the approach proposed in the current paper is to identify the virtual closed loop system in (10). Accordingly, we define the following transfer functions

$$R_o = \frac{F_3 + F_4 G_o}{F_1 + F_2 G_o}, \quad K_o = \frac{F_1 F_4 - F_2 F_3}{F_1 + F_2 G_o} H_o \tag{13}$$

We will use a Box-Jenkins type of model for the virtual closed loop system in which we treat $\frac{F_3+F_4G(\rho)}{F_1+F_2G(\rho)}$ as a Tailor-made parameterization and we independently parameterize K in terms of a parameter η. Hence, the parameters ρ, and η are estimated by minimizing a criterion of the form:

$$J = \sum_{t=1}^{N} \epsilon_t^2 \tag{14}$$

where

$$\epsilon_t = K(\eta)^{-1} [z_t - R(\rho) x_t] \tag{15}$$

$$R(\rho) = \frac{F_3 + F_4 G(\rho)}{F_1 + F_2 G(\rho)} \tag{16}$$

In the subsequent analysis we will analyze the impact of the following two issues:

1) x_t is not, in general, an exogenous signal but is potentially correlated with the noise w_t.
2) The class of models used for $K(\eta)$ may not include the true noise model K_o e.g. we might decide to use a fixed noise model $K \neq K_o$.

IV. Specialization to Direct and Indirect Closed Loop Identification Methods

Here, we show that the Virtual Closed Loop method generalizes known methods for closed loop identification. In particular, it is readily seen that:

- Direct identification (see e.g. [12]) is obtained by the choice $F_1 = F_4 = 1$, $F_2 = F_3 = 0$. This results in
$$x_t = u_t, z_t = y_t, R_o = G_o, K_o = H_o$$
- Traditional Indirect identification ([14]) is obtained by the choice $F_1 = C_o^{-1}$, $F_2 = F_4 = 1$, $F_3 = 0$ where C_o is the (assumed known and linear) true controller. This results in
$$x_t = r_t, z_t = y_t, R_o = \frac{G_o C_o}{1 + G_o C_o}, K_o = \frac{1}{1 + G_o C_o} H_o$$
.
- The Dual Youla method ([9], [13], [16]) results from the choice $F_1 = D_c$, $F_2 = N_c$, $F_3 = -N_x$, $F_4 = D_x$ where $M = N_c N_x + D_c D_x$ is stable, minimum phase, and bi-proper (same number of poles and zeros) and where $N_c D_c^{-1}$ is a co-prime representation of the (assumed known and linear) true controller C_o and where $G_x = NxDx^{-1}$ is a co-prime representation of an a-priori given estimate for G_o.
- The "whitening procedure" (see e.g. [12]) is obtained by the choice $F_1 = F_4 = F$ and $F_2 = F_3 = 0$. In this case we have
$$x_t = u_t, z_t = y_t, R_o = G_o, K_0 = F H_o$$
Note that if $F \approx H_o^{-1}$, then we might consider using a fixed filter $K = 1$ in the estimates.

V. Analysis of Virtual Closed Loop Identification

Here we revert to the general scheme given in (2), (3). We will hypothesize that the true system operates in closed loop with either a non-linear controller or a linear controller which are only partially known.

Remark 1: As a preliminary observation, we see that, when x_t is considered as an exogenous signal, then the virtual closed loop of Figure 3 will be stable if and only if the polynomial $N_1 D_2 \hat{A} + N_2 D_1 \hat{B}$ has its roots inside the stability boundary where $G = \hat{B}/\hat{A}$. Thus, if the virtual controller $\bar{C} = F_2 F_1^{-1}$ is known to stabilize the true system when x_t is exogenous, then it suffices to search for estimated models such that the tailor-made parameterization of R is stable. ▽▽▽

The key tool that we will utilize to analyze the estimates provided by the virtual closed loop schemes is the following:

Lemma 1: Consider the parameter estimation scheme described in (14) to (16) where z_t is related to x_t as in (10).

- For general, possibly non-linear, feedback of the form
$$x_t = \Gamma(x_{t-1}, x_{t-2}, \cdots, z_t, z_{t-1}, \cdots, r_t, r_{t-1}, \cdots) \tag{17}$$
where r_t is a given exogenous reference signal, then the asymptotic bias in the resulting estimate of R_o is
$$B_R = R - R_o = [K_o - K] \left[\frac{\Phi_{wx}}{\Phi_x} \right]_+ \tag{18}$$
where $[\Phi]_+$ represents the causal part of Φ, and where Φ_{xw} and Φ_x respectively denote the cross spectrum between x_t and w_t and the spectrum of x_t.
- For the case of linear feedback, where the feedback takes the form
$$x_t = \gamma_o(q^{-1})(r_t - z_t) \tag{19}$$
then the asymptotic bias can be evaluated explicitly as
$$B_R = [K - K_o] \left[\frac{(\gamma_o K_o S_o)^* \sigma_w^2}{|S_o|^2 \Phi_r + |\gamma_o K_o S_o|^2 \sigma_w^2} \right]_+ \tag{20}$$
where $*$ denotes complex conjugate and S_o represents the sensitivity function given by
$$S_o = \frac{1}{1 + R_o \gamma_o} \tag{21}$$

Proof: Essentially as in [4], [12] with a change of notation. □

Remark 2: If we apply Lemma 1 to direct identification, then we see that the estimates will be biased when K differs from $K_o = H_o$. This is a well known problem with direct identification when the noise model is ill-defined (e.g. time varying). ▽▽▽

We can apply Lemma 1 to the Virtual Closed Loop scheme. To do this, we need to evaluate Φ_{wx} and Φ_x in terms of other signals. To do this we will assume either that

- The true system operates under non-linear feedback of the form

$$u_t = \mathcal{K}\{u_{t-1}, u_{t-2}, \cdots, , y_t, y_{t-1}, \cdots, r_t, r_{t-1}, \cdots\} \tag{22}$$

or

- linear feedback of the form

$$u_t = C_o(q^{-1})(r_t - y_t) \tag{23}$$

Note that the controllers described above are not the controllers given (17) and (19). Of course, there is a relationship between the feedback laws found by solving the various system of equations. Indeed, this underlies the methodology used to prove the following result:

Theorem 2: Consider the virtual closed loop estimates described in (14) to (16). Also, assume that

- G_o lies in the model class $G(\rho)$ for some $\rho = \rho_o$.
- H is the implicit equivalent noise model induced by the relationship

$$H := \frac{F_1 + F_2 G_o}{M} K \tag{24}$$

$$M := F_4 F_1 - F_2 F_3 \tag{25}$$

Then, the asymptotic bias in the estimate of G_o induced by solving $R = \frac{F_3 + F_4 G}{F_1 + F_2 G}$ for G is

$$G_o - G \approx (H_o - H)(F_1 + F_2 G_o) \left[\frac{1}{F_1 + F_2 G_o} \frac{\beta}{\alpha} \right]_+ \tag{26}$$

where

$$\beta := \Phi_{wu} + (\bar{C} H_o \bar{S}_o)^* \sigma_w^2 \tag{27}$$

$$\alpha := \Phi_u + |\bar{C} H_o \bar{S}_o|^2 \sigma_w^2 + 2\text{Re}\ \{(\bar{C} H_o \bar{S}_o)^* \Phi_{wu}\} \tag{28}$$

$$\bar{C} := \frac{F_2}{F_1}, \quad \bar{S}_o := \frac{1}{1 + G_o \bar{C}} \tag{29}$$

Moreover, when the true controller is linear and with transfer function is C_o, we have that:

$$\beta = [(\bar{C} - C_o) H_o S_o \bar{S}_o]^* \sigma_w^2, \quad S_o = \frac{1}{1 + G_o C_o} \tag{30}$$

Proof: See the Appendix. □

Theorem 2 provides a basis for choosing suitable values for F_1, F_2, F_3, F_4. In particular, we see from (26) and (30) that the asymptotic bias is small under either of the following two conditions

- $H_o - H$ is small
- $\bar{C} - C_o$ is small

Note that this holds on a frequency by frequency basis so it suffices for $\bar{C}$ to be near the true controller when $H_o - H$ is large or for $H_o - H$ to be small when $\bar{C}$ is a poor representation of the true controller.

Hence, it makes sense to choose F_1, F_2 such that $\bar{C} = F_2 F_1^{-1}$ is close to the true controller. For example, if the true controller is a linear controller with anti-windup protection, then $\bar{C}$ could be chosen as the linear controller without anti-windup.

From (10), it may be tempting to think that a good to choice for F_3, F_4 would be such that $F_1 F_4 = F_2 F_3$ since this removes all noise from (10). However, in this case, $R_o = F_4 F_3^{-1}$ i.e. we learn nothing about G_o. Thus, it is necessary to design the filters F_i such that M is different from zero in the frequency range of interest.

An alternative choice of F_3, F_4 would be to use a-priori estimates G_x, H_x for G_o, H_o to render $K_o \approx 1$. In this case, we might try using a fixed value for K (namely 1) in (15). The virtual closed loop scheme then reduces to an output error method linking the measured variable z_t to the model output $\hat{z}_t$. Of course, based on Theorem 2, bias may result if $\frac{F_1 + F_2 G_o}{F_1 F_4 - F_2 F_3}$ is significantly different form H_o in frequency ranges where $\bar{C}$ is a poor approximation to the true controller.

VI. A Numerical Example

Consider the following system:

$$G_o = \frac{b_1 q^{-1}}{1 - a_1 q^{-1}} \tag{31}$$

$$H_o = \frac{1 + c_1 q^{-1} + c_2 q^{-2} + c_3 q^{-4} + c_4 q^{-4}}{1 + d_1 q^{-1} + d_2 q^{-2} + d_3 q^{-4} + d_4 q^{-4}} \tag{32}$$

with $a_1 = 0.6$, $b_1 = 0.4$, $c_1 = 1.851$, $c_2 = -1.976$, $c_3 = -0.7605$, $c_4 = 0$, $d_1 = -1.2$, $d_2 = 0.3309$, $d_3 = -0.6484$, $d_4 = 0.605$. The true control law is given by

$$u_t = C_o(q^{-1})(r_t - y_t) \tag{33}$$

$$C_o(q^{-1}) = \frac{0.5q^{-1}}{1 - 0.5q^{-1}} \tag{34}$$

where r_t zero mean Gaussian noise of variance $\sigma^2 = 10$ passing trough the filter $\frac{0.2q^{-1}}{1 - 0.95q^{-1}}$. We use $N = 10000$ data points. The ratio between the variance of the output noise ($v_t = H_o w_t$) and the variance of the output is $\sigma_v^2 / \sigma_y^2 \approx 0.4$ for all the experiments.

For the Virtual Closed Loop identification method we choose $F_1 = 1$, $F_2 = 1$, $F_3 = 0$, $F_4 = 1$. This implies that $\bar{C} = F_2 / F_1 = 1$. We use an output error model for the virtual closed loop, i.e. $K(q^{-1}) = 1$.

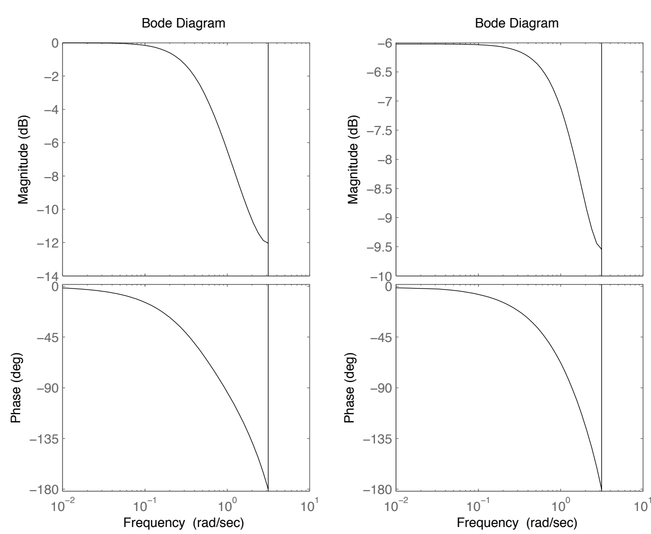

Fig. 4. Bode diagrams for G_o (left) and R_o (right).

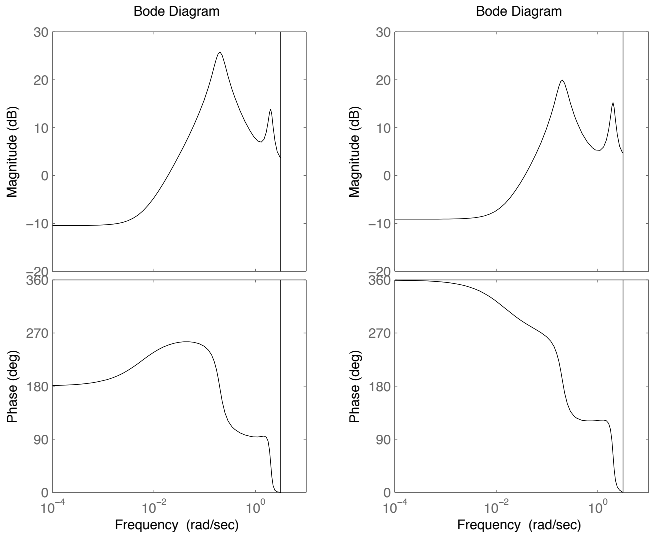

Fig. 5. Bode diagrams for $1 - H_o$ (left) and $1 - K_o$ (right).

Figure 4 shows the Bode diagrams for $G_o(q^{-1})$ and $R_o(q^{-1})$. The Bode diagram for $1 - H_o(q^{-1})$ and for $1-K_o(q^{-1})$ are shown in Figure 5. This figure shows that H_o and R_o are different from 1 in the range of frequencies where the magnitude of G_o and R_o are significant. This implies a difficulty for using an output error model to identify G_o and R_o.

We identify the system by using direct identification with the following model for the transfer function $H_o(q^{-1})$:

$$H(q^{-1}) = \frac{1 + c_1 q^{-1} + \cdots + c_n q^{-n}}{1 + d_1 q^{-1} + \cdots + d_n q^{-n}} \tag{35}$$

for different values of n.

Figure 6 shows the parameter estimates for 300 Monte-Carlo Experiments. We see that, even though we use an output error model for the virtual closed loop, the bias in the estimated model is small. This is due to the fact that the model for the controller is correct in the frequency region of interest. This, actually shows an advantage of using the VCL method. On the other hand, the bias of the models obtained by using direct identification for a Box-Jenkins model is only reduced when the noise model order (n) is increased. Moreover, we see in Figure 6 that the parameters estimated with direct identification are unbiased only when there is no under-modelling ($n = 4$).

It is important to note that direct identification is also covered by the VCL method ($F_2 = F_3 = 0$, $F_1 = F_4 = 1$). However, the VCL method provides additional flexibility which is useful to reduced the bias in the estimates.

VII. CONCLUSIONS

In this paper we have generalized the virtual closed loop (VCL) approach to System Identification. We have focused on systems operating in closed loop and we have analyzed the asymptotic bias due to feedback and noise model mismatching. We have shown that the new parameterization generalizes known methods for closed loop identification and also offers additional flexibility. A numerical example has confirmed the claimed merits of the approach.

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APPENDIX

A. *Proof of Theorem 2:*

Using the model in equation (10) and equation (18), we have that:

$$R_o - R = (K_o - K)\left[\kappa\right]_+ \tag{36}$$

where

$$\kappa = \frac{1}{F_1 + F_2 G_o}\frac{\beta}{\alpha} \tag{37}$$

$$\beta := \Phi_{wu} + (\bar{C}H_o\bar{S}_o)^* \sigma_w^2 \tag{38}$$

$$\alpha := \Phi_u + |\bar{C}H_o\bar{S}_o|^2 \sigma_w^2 + 2\text{Re}\ \{(\bar{C}H_o\bar{S}_o)^* \Phi_{wu}\} \tag{39}$$

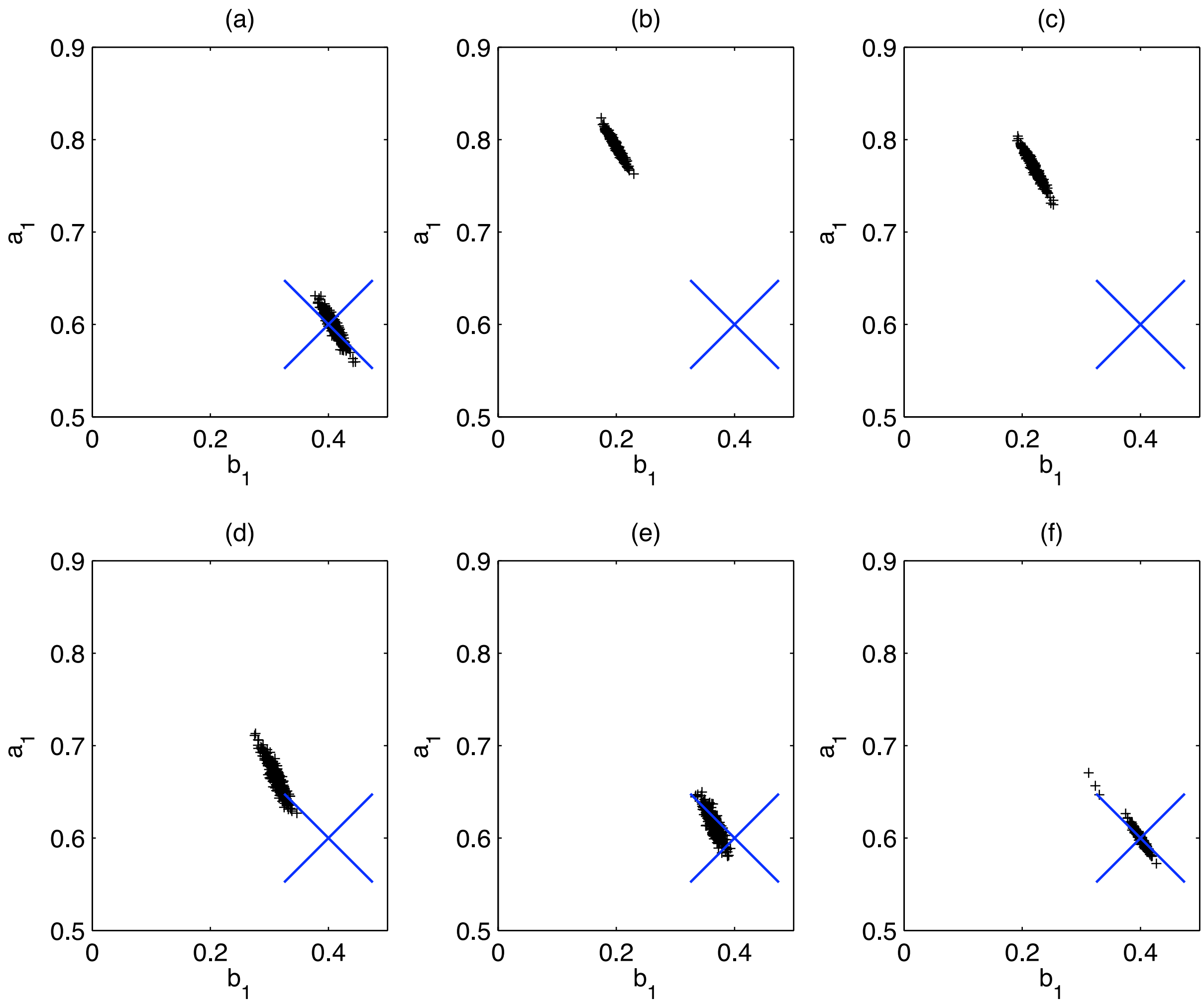

Fig. 6. Results for 300 Monte-Carlo experiments. True value (big-blue-cross). (a) VCL with an output error model ($K = 1$), (b) Direct identification with an output error model ($H = 1$), (c) Direct identification using a noise transfer model with $n = 1$, (d) Direct identification using a noise transfer model with $n = 2$, (e) Direct identification using a noise transfer model with $n = 3$, (f) Direct identification using a noise transfer model with $n = 4$.

We have that the difference between R_o and R is given by:

$$R_o - R = (H_o - H)\frac{M}{F_1 + F_2 G_o}[\kappa]_+ \tag{40}$$

On the other hand, the difference between R_o and its estimate R is also given by:

$$R_o - R = \frac{F_3 + F_4 G_o}{F_1 + F_2 G_o} - \frac{F_3 + F_4 G}{F_1 + F_2 G} \tag{41}$$

$$= \frac{(F_1 F_4 - F_2 F_3)(G_o - G)}{(F_1 + F_2 G_o)(F_1 + F_2 G)} \tag{42}$$

Solving for G we have:

$$G = \frac{M G_o - F_1(F_1 + F_2 G_o)(R_o - R)}{M + F_2(F_1 + F_2 G_o)(R_o - R)} \tag{43}$$

and then calculating the difference between G_o and G we have that:

$$G_o - G = \frac{(F_1 + G_o F_2)^2 (R_o - R)}{M + F_2(F_1 + F_2 G_o)(R_o - R)} \tag{44}$$

Then, using (40) we have that:

$$G_o - G = \frac{(F_1 + G_o F_2)(H_o - H)[\kappa]_+}{1 + F_2(H_o - H)[\kappa]_+} \tag{45}$$

We have that the asymptotic bias on the estimate of G_o is given by:

$$G_o - G = (F_1 + F_2 G_o)\frac{(H_o - H)[\kappa]_+}{1 + F_2(H_o - H)[\kappa]_+} \tag{46}$$

Using a Taylor expansion of first order we have that the asymptotic bias on the estimate of G_o can be approximated as follows:

$$G_o - G \approx (H_o - H)(F_1 + F_2 G_o)\left[\frac{1}{F_1 + F_2 G_o}\frac{\beta}{\alpha}\right]_+ \tag{47}$$

which finishes the proof.