System Identification
Lecture 2

Signals and Systems

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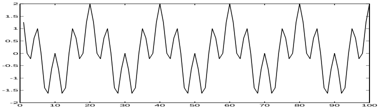
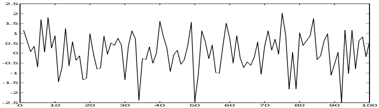
Signals encountered in system identification

Input $u(t)$:

multisine

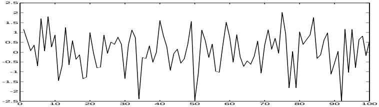


(filtered)
white noise



Disturbance $v(t)$:

stochastic
signal



Output $y(t)$:

$$y = G_0 u + v$$

For deterministic signals

$$u_d(k) = u(kT_s)$$

(discrete-time signal is sampled from continuous signal)

Signal Power:

$$\mathcal{P}_u := \frac{1}{N} \sum_{k=0}^{N-1} u_d(k)^2$$

Decomposition of signals in harmonics (sines and cosines) allows to write $\mathcal{P}_u$ as a summation/integral of frequency-dependent contributions.

$\Rightarrow$ allows f-dependent characterization of signals.

Periodic signals with period N_0

Fourier Series:

$$u_d(k) = \sum_{\ell=0}^{N_0-1} a_\ell e^{j\frac{2\pi}{N_0}\ell k}$$

with the Fourier coefficients

$$a_\ell = \frac{1}{N_0} \sum_{k=0}^{N_0-1} u_d(k) e^{-j\frac{2\pi}{N_0}\ell k}$$

The power of the signal then becomes:

$$\mathcal{P}_u = \sum_{\ell=0}^{N_0-1} |a_\ell|^2.$$

Each term a_ℓ contributes independently to the power of the signal.

Can we recognize the harmonic terms (sines and cosines)?

Tip: by combining the terms a_ℓ and $a_{N_0-\ell}$ with corresponding exponentials, and using the properties that

$$e^{j\phi} = \cos(\phi) + j\sin(\phi)$$

$$\cos(\phi) = \frac{1}{2}[e^{j\phi} + e^{-j\phi}]$$

$$\sin(\phi) = \frac{1}{2j}[e^{j\phi} - e^{-j\phi}]$$

Non-Periodic infinite-time signals (finite power)

Discrete-time Fourier transform (DTFT) (infinite time):

$$U_s(\omega) = \sum_{k=-\infty}^{\infty} u_d(k) e^{-i\omega k T_s}$$

$$u_d(k) = \frac{T_s}{2\pi} \int_{2\pi/T_s} U_s(\omega) e^{i\omega k T_s} d\omega,$$

with $\omega_s := \frac{2\pi}{T_s}$ the sample frequency (rad/sec).
$U_s(\omega)$ is:

- continuous function in ω
- periodic with period ω_s

Non-Periodic finite-time signals

Discrete Fourier Transform (DFT) (finite-time):

$$U_N(\omega) = \sum_{k=0}^{N-1} u_d(k) e^{-i\omega k T_s}.$$

$$u_d(k) = \frac{1}{N} \sum_{\ell=0}^{N-1} U_N\left(\frac{\ell\omega_s}{N}\right) e^{i\frac{2\pi\ell}{N}k}.$$

DFT: $\{U_N(\omega),\ \omega = \frac{\ell}{N}\omega_s,\ \ell = 1, \cdots N\}$, N points in ω-domain.
Because of symmetry: $U_N(-\omega) = U_N(\omega)^*$.

Signal power

$$\mathcal{P}_u = \frac{1}{N}\sum_{k=0}^{N-1} u_d^2(k) =_{N\to\infty} \frac{1}{\omega_s}\int_{\omega_s} \Phi_u(\omega)d\omega$$

with power spectrum $\Phi_u(\omega) = \frac{1}{N}|U_N(\omega)|^2$

For finite-time signals this can be equivalently expressed as:

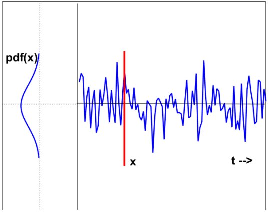
$$\mathcal{P}_u = \frac{1}{N}\sum_{k=0}^{N-1} \frac{1}{N}|U_N(\frac{2\pi k}{N})|^2$$

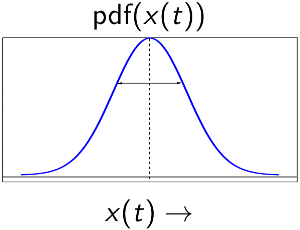
Periodogram:

$$\frac{1}{N}|U_N(\omega)|^2$$

power spectrum for finite-time signals.

Random Variables and Stochastic Processes





Consider $x(t)$ for a fixed t as a random variable

A random variable $\mathbf{x}$ is a real-valued outcome of a random experiment, and satisfies the property that we can assign a probability to the event

$$\mathbf{x} \leq x$$

for every value $x \in \mathbb{R}$

This allows the definition of a distribution function:

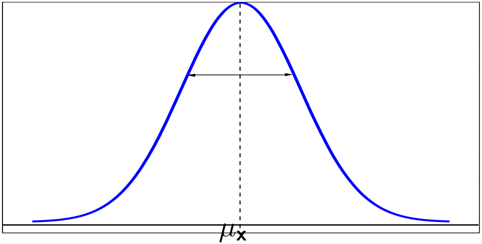
$$F_{\mathbf{x}}(x) := P(\mathbf{x} \leq x)$$

The distribution function can be written as an integral:

$$F_{\mathbf{x}}(x) = \int_{-\infty}^{x} f_{\mathbf{x}}(\mu) d\mu$$

with the probability density function (pdf) $f_{\mathbf{x}}(x)$, that for Gaussian random variables is given by

$$f_{\mathbf{x}}(x) = \frac{1}{\sqrt{2\pi\sigma_{\mathbf{x}}^2}} \, e^{-\frac{(x-\mu_{\mathbf{x}})^2}{2\sigma_{\mathbf{x}}^2}}$$



Distribution function: $F_{\mathbf{x}}(x) := P(\mathbf{x} \leq x)$

For continuous-valued random variables:
Probability density function (pdf):

$$f_{\mathbf{x}}(x) := \frac{dF_{\mathbf{x}}(x)}{dx} \qquad \geq 0 \ \forall \ x$$

$$P(a \leq \mathbf{x} \leq b) = \int_a^b f_{\mathbf{x}}(x)dx$$

For two variables:

$$f_{\mathbf{x},\mathbf{y}}(x,y) := \frac{\partial^2 F_{\mathbf{x},\mathbf{y}}(x,y)}{\partial x \partial y}$$

Note: $\mathbf{x}$ is the random variable, and x is a realization.

Moments

Mean: $\mu_{\mathbf{x}} := \mathbb{E}\{\mathbf{x}\} = \int_{-\infty}^{\infty} x f_{\mathbf{x}}(x) dx$

Variance:

$$\sigma_{\mathbf{x}}^2 = \mathbb{E}\{(\mathbf{x} - \mu_{\mathbf{x}})^2\} = \int_{-\infty}^{\infty} (x - \mu_{\mathbf{x}})^2 f_{\mathbf{x}}(x) dx$$

Correlation:

$$r_{\mathbf{xy}} = \mathbb{E}\{\mathbf{xy}\}$$

Covariance:

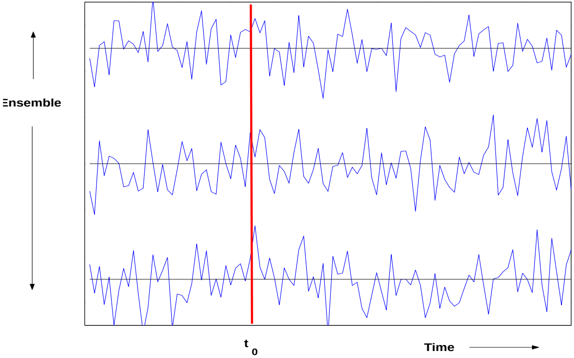
$$\begin{aligned} \sigma_{\mathbf{xy}} &= \mathbb{E}\{(\mathbf{x} - \mu_{\mathbf{x}})(\mathbf{y} - \mu_{\mathbf{y}})\} \\ &= \int_{-\infty}^{\infty} (x - \mu_{\mathbf{x}})(y - \mu_{\mathbf{y}}) f_{\mathbf{x},\mathbf{y}}(x, y) dx dy \end{aligned}$$

Random variables are called

- independent if $f_{\mathbf{x},\mathbf{y}}(x,y) = f_{\mathbf{x}}(x) \cdot f_{\mathbf{y}}(y)$
- orthogonal if $r_{\mathbf{xy}} = 0$ or equivalently $\mathbb{E}\{\mathbf{xy}\} = 0$
- uncorrelated if $\sigma_{\mathbf{xy}} = 0$ or equivalently $\mathbb{E}\{\mathbf{xy}\} = \mu_{\mathbf{x}} \cdot \mu_{\mathbf{y}}$

Stationary stochastic processes

3 Realizations of a process:



Ensemble of time sequences

Stationarity $\rightarrow$ statistical properties do not change over time

Dynamics is not captured in the description of a single random variable (fixed time), but in the statistical correlation between different time moments:

Consider a process with zero mean $\mu_{\mathbf{x}} = 0$, then

- (Auto)-correlation function

$$R_x(\tau) = \mathbb{E}[x(k)][x(k-\tau)]$$

describes the statistical relation between two samples that are τ steps apart.
It reflects the "memory" or dynamics in the signal.

Power of a (zero-mean) process x: $\mathbb{E}x^2(k)$

- (Auto)-correlation function

$$R_x(\tau) = \mathbb{E}[x(k)][x(k-\tau)]$$

- Power spectrum

$$\Phi_x(\omega) = \sum_{k=-\infty}^{\infty} R_x(k) e^{-i\omega k T_s}$$

$$\Rightarrow \quad R_x(0) = \mathbb{E}x^2(k) = \frac{1}{\omega_s} \int_{\omega_s} \Phi_x(\omega) d\omega$$

Power spectrum is Fourier transform of auto-correlation function

Cross- and auto-correlation function

The cross-correlation $R_{yu}(\tau)$ between zero-mean processes y and u is a function which allows to verify whether two stochastic signals $y(k)$ and $u(k)$ are correlated with each other

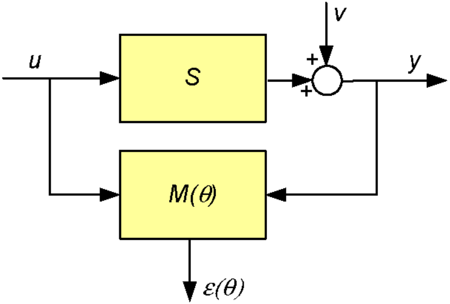
$$R_{yu}(\tau) := \mathbb{E}\left(y(k)u(k-\tau)\right)$$

Properties:

- the value of $y(k)$ at time k is not (cor)related in any way to the value of $u(k-\tau) \Longrightarrow R_{yu}(\tau) = 0$
- the signals $y(k)$ and $u(k)$ are independent $\Longrightarrow R_{yu}(\tau) = 0 \ \forall\tau$

NB. $R_u(\tau) = R_{uu}(\tau)$

Back to the simple paradigm of Lecture 1:



Would we expect ε and u to be correlated?

Quasi-stationarity

$$y(k) = w(k) + v(k)$$

w deterministic signal
v stationary stochastic process

With repetition of experiments: w unchanged, v not.
Consequence: y *non-stationary* stochastic process!

Introduction of $\bar{\mathbb{E}}$:

$$\bar{\mathbb{E}}y(k) := \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbb{E}y(k)$$

- time-averaging operates on deterministic part
- $\mathbb{E}$ operates on stochastic part

Resulting notions

- *(Auto-)correlation function*:

$$R_y(\tau) := \bar{\mathbb{E}}[y(k)y(k-\tau)]$$

- *Cross-correlation function*:

$$R_{yu}(\tau) := \bar{\mathbb{E}}[y(k)u(k-\tau)]$$

- *Power spectrum*:

$$\Phi_y(\omega) := \sum_{\tau=-\infty}^{\infty} R_y(\tau)e^{-i\omega\tau}$$

- *Cross power spectrum*:

$$\Phi_{yu}(\omega) := \sum_{\tau=-\infty}^{\infty} R_{yu}(\tau)e^{-i\omega\tau}$$

Signals for which these expressions exist: *quasi-stationary*.

Consequence:

$$\bar{\mathbb{E}}y^2(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_y(\omega)d\omega$$

Power spectrum of quasi-stationary signals

Let

$$y(k) = w(k) + v(k)$$

with w quasi-stationary deterministic, and v stationary stochastic with $\mathbb{E}v(k) = 0 \ \forall k$, then

$$\Phi_y(\omega) = \Phi_w(\omega) + \Phi_v(\omega)$$

Proof follows from the observation that $\bar{\mathbb{E}}[w(k)v(k-\tau)] = 0$

In this course we will further use notation t also to indicate (discrete) time,
So: $t = 0, 1, 2, 3, 4, ...$ etc.

Discrete-time systems

$$y(t) = \sum_{k=0}^{\infty} g(k)u(t-k) \qquad t = 0, 1, 2, \cdots$$

$$\{g(k)\}_{k=0,1,\cdots} \quad \text{puls-response}$$

- Linearity $\Rightarrow$ convolution
- Time-invariance $\Rightarrow$ $g(k)$ time-independent
- Causality $\Rightarrow$ $g(k) = 0,\ k < 0$
- discrete-time

Output signal is determined through

- puls-response
- initial conditions; often chosen $u(t) = 0,\ t < 0$

Shift operators

$$\begin{aligned} q\ u(t) &= u(t+1) \\ q^{-1}\ u(t) &= u(t-1) \end{aligned}$$

$$y(t) = \sum_{k=0}^{\infty} g(k) q^{-k}\ u(t) = G(q)\ u(t)$$

Transfer(-operator): $G(q) = \sum_{k=0}^{\infty} g(k)\ q^{-k}$

Transfer function (formal) $\mathbb{C} \to \mathbb{C}$: $G(z) = \sum_{k=0}^{\infty} g(k)\ z^{-k}$.

System is finite dimensional $\leftrightarrow$ G is rational function in z

Example

$$y(t) + a_1 y(t-1) = b_1 u(t-1)$$

For u a discrete pulse, i.e. $u(0) = 1$, $u(t) = 0$, $t \neq 0$, and $y(t) = 0, t < 0$, it follows:

$$y(0) = 0; \quad y(1) = b_1; \quad y(2) = -a_1 b_1; \quad y(3) = a_1^2 b_1; \quad \text{etcetera}$$

So: $g(k) = b_1(-a_1)^{k-1}$, $k \geq 1$.

$$G(z) = \sum_{k=1}^{\infty} b_1(-a_1)^{k-1} z^{-k} = \sum_{k=1}^{\infty} -\frac{b_1}{a_1}(-a_1 z^{-1})^k$$

$$= \frac{b_1 z^{-1}}{1 - (-a_1 z^{-1})} = \frac{b_1}{z + a_1}$$

Direct through z-transform (for zero initial conditions):

$$y(t) + a_1 y(t-1) = b_1 u(t-1)$$

$$Y(z) + a_1 z^{-1} Y(z) = b_1 z^{-1} U(z)$$

$$Y(z) = \frac{b_1 z^{-1}}{1 + a_1 z^{-1}} U(z)$$

$$G(z) = \frac{b_1}{z + a_1}$$

Stability

G BIBO stable if

$$\sum_{k=0}^{\infty} |g(k)| < \infty$$

- $|u(t)| \leq c_1 \quad \Rightarrow \quad |y(t)| \leq c_2$
- $G(z)$ analytic (no poles) on/outside unit circle $z = e^{i\omega}$.

Frequency response $G(e^{i\omega})$

If $u(t) = cos\ \omega t$, then

$$y(t) = |G(e^{i\omega})| \cdot cos(\omega t + \phi)$$

$$\phi = arg[G(e^{i\omega})]$$

$G(e^{i\omega}),\ 0 \leq \omega \leq \pi$ delivers discrete Bode plot.

Processing of signals:

If $y(t) = G(q)u(t)$,
$u(t)$ quasi-stationary, then

$$\begin{aligned} \Phi_{yu}(\omega) &= G(e^{i\omega})\Phi_u(\omega) \\ \Phi_y(\omega) &= |G(e^{i\omega})|^2 \cdot \Phi_u(\omega) \end{aligned}$$

Estimation

Estimator $\hat{\theta}_N$ of θ_0, is a mapping from $\{u(t), y(t)\}_{t=1,\cdots N}$ to $\hat{\theta}_N$. If data contains random variables, then $\hat{\theta}_N$ is a random variable.

Concepts:

a. *Unbiased* (zuiver): $\mathbb{E}\hat{\theta}_N = \theta_0$

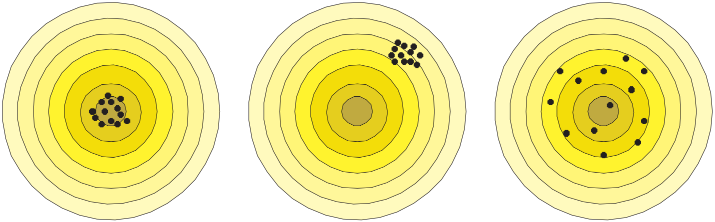
b. *Consistent*. $\hat{\theta}_N$ is consistent if:

- $Pr[\lim_{N\to\infty} \hat{\theta}_N = \theta_0] = 1$
- $\hat{\theta}_N \to \theta_0$ with probability 1 voor $N \to \infty$.

c. *Variance*: $cov(\hat{\theta}_N) = \mathbb{E}(\hat{\theta}_N - \mathbb{E}\hat{\theta}_N)(\hat{\theta}_N - \mathbb{E}\hat{\theta}_N)^T$.

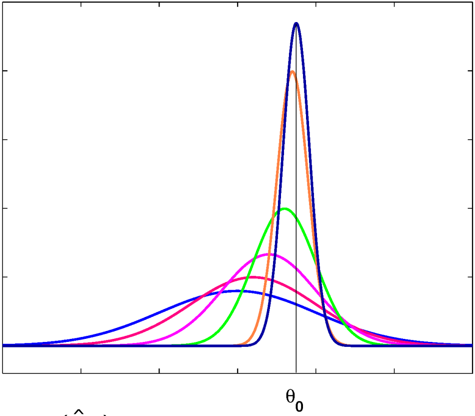
d. *Asymptotic normal:* $\hat{\theta}_N \in As\mathcal{N}(\theta^*, P_\theta)$

Consistency = Asymptotic unbiased and variance tends to 0



Bull's eye represents θ_0;
left: unbiased estimate with small variance
middle: biased estimate with small variance
right: unbiased estimate with large variance

Probability density function (pdf) of a consistent estimate



$f(\hat{\theta}_N)$ for increasing values of N

For a consistent estimate the parameter variance will become 0 (for increasing N)

Summary

- Stationary stochastic process to model random signals used for describing disturbance signals
- Quasi-stationary signals to combine det+stoch signals
- Estimation concepts: bias and variance
- Consult material of Chapter 2 (and background texts) if necessary